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THE RAY TRANSFORM OF SYMMETRIC TENSOR FIELDS WITH INCOMPLETE PROJECTION DATA, I: THE KERNEL OF THE RAY TRANSFORM

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Abstract. We consider the ray transform I_Γ that integrates symmetric rank m tensor fields on $\mathbb{R}^n$ supported in a bounded convex domain $D \subset \mathbb{R}^n$ over lines. The integrals are known for the family Γ of lines l such that endpoints of the segment $l \cap D$ belong to a given part $\gamma = \partial D \cap \mathbb{R}^n_+$ of the boundary, for some half-space $\mathbb{R}^n_+ \subset \mathbb{R}^n$. We prove that the kernel of the operator I_Γ coincides with the space of γ-potential tensor fields.

Keywords: tomography with incomplete data, ray transform, tensor analysis.

1. Introduction

The ray transform I integrates rank m symmetric tensor fields on $\mathbb{R}^n$ over lines. For $m = 0$, the operator I is the main mathematical tool of Computer Tomography. In the case of $m = 1$, the operator I is called the Doppler transform and is the main mathematical tool of Doppler Tomography. The case of $m = 2$ is of a particular importance since the problem of inverting I is the lianerization of the inverse problem of recovering a Riemannian metric on a bounded domain from known distances between boundary points, when the linearization is going around the Euclidean metric [13, Chapter 1]. The operator I for $m = 4$ appears in the inverse problem of determining elasticity parameters of anisotropic media from travel times of compressional waves [13, Chapter 7].

The main difference between tensor tomography and scalar one is caused by the following fact. For $m > 0$, the operator I has a big kernel (= the null space)

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consisting of so called potential tensor fields. A precise definition of a potential tensor field will be presented later, and now we give the following informal explanation. A symmetric rank m tensor field f is potential if it can be represented in the form $f = dv$ for some rank $m-1$ symmetric tensor field v, where $d = \sigma\nabla$ is the symmetrized covariant derivative. The definition also involves an important boundary condition (or decay condition at infinity): the potential v must vanish either on the whole boundary of the domain under consideration or on a part of the boundary (or must certainly decay at infinity). The latter condition varies in various versions of the problem.

For many tomographic problems, the two-dimensional version of the problem is the main one since the case of $n \geq 3$ can be reduced to the case of $n = 2$. For instance, let us consider the problem of recovering a function $f \in C(D)$ defined in a closed bounded domain $D \subset \mathbb{R}^n$ from the family of its integrals If known for all lines of the space $\mathbb{R}^n$ (assuming f to be extended by zero outside D). Restricting the data If to the family of lines lying in some affine 2D plane $P \subset \mathbb{R}^n$, we arrive to the two-dimensional version of the same problem for the function $f|_{P\cap D}$. If $f|_{P\cap D}$ were recovered for a sufficiently large family of planes P (for example, for all planes parallel to a subspace $\mathbb{R}^2 \subset \mathbb{R}^n$), the function f would be recovered too.

The problem becomes more complicated if, instead of a function, we consider a symmetric tensor field f of a positive rank. Here we do not discuss the problem of recovering f from the data If which cannot be uniquely solved and needs some additional definitions. Let us only consider the question of determining the kernel of the operator I. Let f be a continuous rank $m > 0$ symmetric tensor field defined on a closed bounded domain $D \subset \mathbb{R}^n$ and satisfying $If = 0$ for all lines of the space $\mathbb{R}^n$. We would like to prove that f is a potential tensor field, i.e., there exists a rank $m-1$ symmetric tensor field v of class C^1 on D such that $dv = f$ and $v|_{\partial D} = 0$. Assume the validity of the statement for $n = 2$. For $n \geq 3$, we again restrict the data $If = 0$ to lines lying in 2D affine planes, apply the two-dimensional version and obtain, for any 2D affine plane $P \subset \mathbb{R}^n$, a symmetric tensor field v^P defined on $P \cap D$ and satisfying $dv^P = f|_P$ and $v^P|_{\partial(P\cap D)} = 0$. It remains to check that the tensor fields v^P, being defined for all planes P, are agreed to each other, i.e., they together determine a symmetric tensor field v of class C^1 on the domain D satisfying the equation $dv = f$. The pretty tedious check takes the most part of the proof of Theorem 2 below. It is technically more convenient to reduce the n-dimensional problem to a family of $(n-1)$-dimensional similar problems, i.e., the proof of Theorem 2 is going by induction in n.

In practical tomography, problems with incomplete projection data are very common when the ray transform If is known not for all lines of $\mathbb{R}^n$ but for some restricted family of lines. Such a situation appears if the domain of interest involves a non-transparent inclusion such that the sounding radiation does not transmit the inclusion. As an example the Doppler tomography problem can be mentioned when we want to determine a velocity field of air, flowing around an object in a wind tunnel, by testing the field with ultrasound radiation. The object itself is a non-transparent inclusion for ultrasound rays.

Investigation of scalar incomplete data problems is a classical subject of mathematical tomography. There is a big variety of publications on the subject, we mention the most popular of them [1, 2, 4, 5, 7, 18]. As far as studying incomplete data problems of tensor tomography is concerned, this direction is in its very beginning. To author's knowledge, there are very few mathematical papers in the direction [9, 11, 12, 14, 15, 16, 19]. In the latter list, the article [3] by A. Denisjuk should be distinguished where some new ideas have arisen, and many new relations on the ray transform have been discovered.

The paper is arranged as follows. In Section 2, we consider the ray transform with complete projection data on the space of compactly supported symmetric tensor fields. This section does not contain new results and is included for the reader convenience. Here we present main definitions (the ray transform, a potential tensor field) and show that the kernel of the operator I coincides with the space of potential tensor fields in the case of complete projection data. There is no proof in Section 2 since all statements are slight modifications of corresponding results of [13, Chapter 2].

In Section 3, we consider the two-dimensional ray transform with incomplete projection data on the space of symmetric tensor fields. After the definition of a potential tensor field is appropriately modified, we prove Theorem 1 on the coincidence of the kernel of the operator I with the space of potential tensor fields in the case of incomplete projection data. In Section 4, the same result is proved in the multi-dimensional case. Theorems 1 and 2 are main results of the current work.

Let us mention several partial cases when Theorems 1 and 2 were known before. In the case of $m = 0$, these theorems state that a function $f \in C(D)$ can be uniquely recovered on the "transparent part" of the domain D from incomplete projection data If. This statement actually coincides with the well known *support theorem* for the Radon transform [6, Chapter 1, Theoren 2.6 and Corollary 2.8]. For $m = 1$, Theorem 1 is easily proved with the help of the Green formula; the proof was known before to the author and, probably, to other experts. Our proof of Theorem 1 is going by induction in m, the cases $m = 0$ and $m = 1$ serve as the induction basis.

In the case of $m = 2$, Theorems 1 and 2 constitute a very partial case of the *support theorem for the geodesic ray transform* proved by V. Krishnan and P. Stefanov [8]. They consider the ray transform that integrates second rank symmetric tensor fields over geodesics of a given Riemannian metric. In the case of a really analytic metric, they prove a statement similar to our Theorem 2. Since the Euclidean metric is really analytic, Theorem 2 follows from [8]. In their proof, Krishnan and Stefanov use the pretty complicated microlocal analysis technics including the so called *analytic wave front set of a distribution*. Even more heavy microlocal analysis is used in the paper [17] by S. Stefanov, G. Uhlmann and A. Vasy. They prove an analog of Theorem 2 in the case of $n \geq 3, m \leq 2$ for the geodesic ray transform in the case of a smooth Riemannian metric satisfying some additional convexity condition. As compared with [8] and [17], our proof of Theorem 1 is elementary. We hope our approach can serve as the basis of numerical algorithms for solving tomographic problems with incomplete projection data.

The present work is restricted to the study of the kernel of the ray transform. Our main result is formulates as follows. A symmetric tensor field f is determined by the data If uniquely up to a potential tensor field. But the definition of a "potential tensor field" slightly varies in various version of the problem. What information on a tensor field f is contained in the data If? The question remains open so far. We are going to answer the question in our forthcoming work. This plan is shortly discussed in the last section.

The author is grateful to K.V. Storozhuk for his proof of Lemma 1 presented below.

2. The ray transform with complete projection data

For an integer $m \geq 0$, let $S^m\mathbb{R}^n$ be the complex vector space of rank m symmetric tensors on $\mathbb{R}^n$. Its dimension is $\binom{n+m-1}{m}$. In particular, $S^0\mathbb{R}^n = \mathbb{C}$ and $S^1\mathbb{R}^n = \mathbb{C}^n$. It is also convenient to assume that $S^{-1}\mathbb{R}^n = 0$.

For an integer $k \geq 0$, the space of k times continuously differentiable $S^m\mathbb{R}^n$-valued functions on $\mathbb{R}^n$ is denoted by $C^k(\mathbb{R}^n; S^m\mathbb{R}^n)$. Its elements are called k times continuously differentiable symmetric tensor field of rank m on $\mathbb{R}^n$. The notations $C^k(\mathbb{R}^n; S^0\mathbb{R}^n)$ and $C^k(\mathbb{R}^n; S^1\mathbb{R}^n)$ are abbreviated to $C^k(\mathbb{R}^n)$ and $C^k(\mathbb{R}^n; \mathbb{C}^n)$ respectively. Let $C^k_c(\mathbb{R}^n; S^m\mathbb{R}^n)$ be the subspace of $C^k(\mathbb{R}^n; S^m\mathbb{R}^n)$ consisting of compactly supported tensor fields. With respect to Cartesian coordinates (only such coordinates on $\mathbb{R}^n$ are used in what follows), a tensor field $f \in C^k(\mathbb{R}^n; S^m\mathbb{R}^n)$ is determined by its coordinates $f_{i_1\dots i_m} \in C^k(\mathbb{R}^n)$ $(1 \leq i_1, \dots, i_m \leq n)$ which are symmetric in indices $(i_1, \dots, i_m)$, i.e., do not change under any permutation of the indices. This fact is denoted by $f = (f_{i_1\dots i_m})$ on assuming the choice of coordinates to be clear from the context.

We identify the family of all oriented lines in $\mathbb{R}^n$ with the tangent bundle of the unit sphere $\mathbb{S}^{n-1}$

$$T\mathbb{S}^{n-1} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid |\xi| = 1, \langle x, \xi\rangle = 0\}$$

by identifying $(x, \xi) \in T\mathbb{S}^{n-1}$ with the line $l_{x,\xi} = \{x + t\xi \mid t \in \mathbb{R}\}$ through the point x in the direction ξ. Hereinafter $\langle\cdot,\cdot\rangle$ is the standard dot product on $\mathbb{R}^n$ and $|\cdot|$, the corresponding norm. Since $T\mathbb{S}^{n-1}$ is a smooth manifold, the spaces of (complex-valued) functions $C^k(T\mathbb{S}^{n-1})$ and $C^k_c(T\mathbb{S}^{n-1})$ are defined.

The *ray transform* is initially defined as the linear continuous operator

$$I : C^k_c(\mathbb{R}^n; S^m\mathbb{R}^n) \to C^k(T\mathbb{S}^{n-1}) \tag{1}$$

by

(2)
$$If(x,\xi) = \int\limits_{-\infty}^{\infty} f_{i_1\dots i_m}(x+t\xi)\,\xi^{i_1}\dots\xi^{i_m}\,dt = \int\limits_{-\infty}^{\infty} \langle f(x+t\xi), \xi^m\rangle\,dt \quad \big((x,\xi) \in T\mathbb{S}^{n-1}\big).$$

We use the Einstein summation rule: the summation from 1 to n is assumed over any index repeated in a monomial in lower and upper positions. To adopt our formulas to the summation rule, we use both lower and upper indices for coordinates of vectors and tensors. For instance, $\xi^i = \xi_i$ in (2). There is no difference between covariant and contravariant tensors since we use Cartesian coordinates only. The integration in (2) is actually going over a finite segment since f is compactly supported.

The first order differential operator

$$d : C^{k+1}(\mathbb{R}^n; S^m\mathbb{R}^n) \to C^k(\mathbb{R}^n; S^{m+1}\mathbb{R}^n)$$

defined in coordinates by the formula

$$(df)_{i_1\dots i_{m+1}} = \sigma\frac{\partial f_{i_1\dots i_m}}{\partial x_{i_{m+1}}},$$

where σ is the symmetrization in all indices of a tensor, is called the *inner derivative*. We say that $f \in C^k_c(\mathbb{R}^n; S^m\mathbb{R}^n)$ is a *potential tensor field* if there exists $v \in C^{k+1}_c(\mathbb{R}^n; S^{m-1}\mathbb{R}^n)$ such that

$$dv = f. \tag{3}$$

Potential tensor fields belong to the kernel of the ray transform, i.e., $If \equiv 0$ for any potential $f \in C^k_c(\mathbb{R}^n; S^m\mathbb{R}^n)$. Indeed,

$$\begin{aligned} If(x,\xi) &= \int\limits_{-\infty}^{\infty} f_{i_1\dots i_m}(x+t\xi)\,\xi^{i_1}\dots\xi^{i_m}\,dt = \int\limits_{-\infty}^{\infty} \frac{\partial v_{i_1\dots i_{m-1}}}{\partial x^{i_m}}(x+t\xi)\,\xi^{i_1}\dots\xi^{i_m}\,dt \\ &= \int\limits_{-\infty}^{\infty} \frac{d}{dt}\Big(v_{i_1\dots i_{m-1}}(x+t\xi)\,\xi^{i_1}\dots\xi^{i_{m-1}}\Big)\,dt = 0. \end{aligned}$$

The last equality of the chain holds since v is compactly supported. The converse statement is also true although its proof is more complicated:

If a tensor field $f \in C^k_c(\mathbb{R}^n; S^m\mathbb{R}^n)$ belongs to the kernel of the ray transform, then it is potential. More precisely: there exists a field $v \in C^{k+1}_c(\mathbb{R}^n; S^{m-1}\mathbb{R}^n)$ supported in the convex hull of the support of f such that (3) *holds.*

See Theorems 2.2.1 and 2.5.1 of [13] for the proof of the statement.

The operator d obeys the following ellipticity. If the right-hand side of the equation (3) is of the class C^k in an open domain $U \subset \mathbb{R}^n$, then any solution to the equation is of the class C^{k+1} in U. This follows from [13, Theorem 2.2.2].

3. The two-dimensional ray transform with incomplete data

Let $D \subset \mathbb{R}^2$ be a closed convex domain bounded by the closed C^1-smooth curve ∂D. We additionally assume that D is a *strictly convex* domain, i.e., there is no straight line segment of positive length in ∂D. In such a case we say that D is a *closed bounded strictly convex domain of the class* C^1.

Let us recall the definition of differentiability on a closed set. We say that a function $f \in C(D)$ belongs to $C^k(D)$ if, for every point of D, there exist an open neighborhood $U \subset \mathbb{R}^2$ and function $g \in C^k(U)$ such that $g|_{D\cap U} = f|_{D\cap U}$. Now the space $C^k(D; S^m\mathbb{R}^2)$ of symmetric tensor fields of the class C^k is defined as the set of continuous maps $D \to S^m\mathbb{R}^2$ whose all coordinates belong to $C^k(D)$.

For a closed bounded strictly convex domain $D \subset \mathbb{R}^2$ of the class C^1, the *ray transform*

$$I : C^k(D; S^m\mathbb{R}^2) \to C(T\mathbb{S}^1) \tag{4}$$

is defined by the same formula (2) where f is extended by zero outside D. Unlike (1), the function $If \in C(T\mathbb{S}^1)$ can be not differentiable for $f \in C^k(D; S^m\mathbb{R}^2)$ with any $k \ge 1$; this is seen from the example when D is a disk, $m = 0$, and f is a constant non-zero function. The strict convexity is assumed in order $If(x,\xi)$ be a continuous function on $T\mathbb{S}^1$. Indeed, if the intersection of the line $l_{x,\xi} = \{x + t\xi \mid t \in \mathbb{R}\}$ with the boundary ∂D is a segment of positive length, then the function If can have a discontinuity at the point (x,ξ).

Let $D \subset \mathbb{R}^2$ be a closed bounded strictly convex domain of the class C^1. Choose a line $l_0 \subset \mathbb{R}^2$ through an inner point of D. By $\mathbb{R}^2_+$ we denote one of two closed half planes bounded by l_0 and by $\mathbb{R}^2_-$, the second one. Set $D_\pm = D \cap \mathbb{R}^2_\pm$ and $\gamma = \partial D \cap \mathbb{R}^2_+$. Let Γ be the closed set in $T\mathbb{S}^1$ consisting of $(x,\xi) \in T\mathbb{S}^1$ such that the intersection of the line $l_{x,\xi} = \{x + t\xi \mid t \in \mathbb{R}\}$ with D is a non-empty segment with both endpoints belonging to the curve γ (the segment can consist of one point). We introduce the linear continuous operator

$$I_\Gamma : C^k(D; S^m\mathbb{R}^2) \to C(\Gamma) \tag{5}$$

by $I_\Gamma f = (If)|_\Gamma$ where If is the value of the operator (4) on the tensor field $f \in C^k(D; S^m\mathbb{R}^2)$. The operator (5) is called the *ray transform with incomplete projection data determined on the domain* $\Gamma \subset T\mathbb{S}^1$.

The function $I_\Gamma f$ contains no information on the restriction $f|_{D_-}$ of the field f to the domain D_-. Given $I_\Gamma f$, we can hope to recover some information on the field $f|_{D_+}$ only. However, the letter field is also not uniquely determined since, as before, the operator (5) has a big kernel for $m > 0$.

We say that $f \in C^k(D; S^m\mathbb{R}^2)$ is a *γ-potential tensor field* if there exists a field $v \in C^{k+1}(D_+; S^{m-1}\mathbb{R}^2)$ satisfying the boundary condition

$$v|_\gamma = 0 \tag{6}$$

and the equation

$$dv = f \quad \text{in the domain} \quad D_+. \tag{7}$$

A γ-potential tensor field $f \in C^k(D; S^m\mathbb{R}^2)$ belongs to the kernel of the operator I_Γ, this fact is checked in the same way as has been done in the previous section for the operator (1). The converse statement is the main result of the current work.

Theorem 1. *Let $k \geq m \geq 0$, $D \subset \mathbb{R}^2$ be a closed bounded strictly convex domain of the class C^1, domains $D_\pm$ and the curve $\gamma \subset \partial D$ be chosen as above. The kernel of the operator* (5) *coincides with the space of γ-potential tensor fields.*

Starting the proof, first of all we choose Cartesian coordinates (x_1, x_2) on $\mathbb{R}^2$ so that the line l_0 participating in above-presented definitions coincides with the X-axis $\{x_2 = 0\}$. Then $D_\pm = \{(x_1, x_2) \in D \mid \pm x_2 \geq 0\}$ and $\gamma = \{(x_1, x_2) \in \partial D \mid x_2 \geq 0\}$.

The proof is going by induction in m. We first prove Theorem 1 for $m = 0$ and $m = 1$.

Proposition 1. *Theorem 1 is true for $m = 0$.*

Proof. Let a function $f \in C^k(D)$ belong to the kernel of the operator (5). We have to prove that $f|_{D_+} = 0$. First of all we will prove that

$$f|_\gamma = 0. \tag{8}$$

We prove (8) by contradiction. Assuming the existence of a point $x \in \gamma$ such that$f(x) > 0$, we choose a convex neighborhood $U \subset \mathbb{R}^2$ of the point such that $f(y) > 0$ for all $y \in D \cap U$. For every point $y \in \gamma \cap U,\ y \neq x$, the segment $[x, y]$ lies in U and therefore

$$\int\limits_0^1 f(x + ty) > 0.$$

By the strict convexity hypothesis, $[x, y] = l_{x',\xi} \cap D$ for some point $(x', \xi) \in \Gamma$ and the latter inequality can be written in the form $I_\Gamma f(x', \xi) > 0$; this contradicts to the hypotheses $I_\Gamma f = 0$. Thus, (8) is proved.

Let g be the extension of the function f to the domain $D \cup \mathbb{R}^n_+$ defined by $g|_{\mathbb{R}^n_+ \setminus D} = 0$. By (8), $g \in C(D \cup \mathbb{R}^n_+)$. This function still satisfies

$$Ig(x, \xi) = 0 \quad \text{for} \quad (x, \xi) \in \Gamma. \tag{9}$$

Three possible positions of the domain D with respect to the X-axis are shown on Fig. 1, where the curve γ is drawn together with two tangent lines at endpoints. As is seen from the figure, for every $(x,\xi)\in\Gamma$, the line $l_{x,\xi}$ does not intersect the interior of the shaded domain D'. It is also obvious from the figure that g can be extended to a function $F\in C_c(\mathbb{R}^2)$ supported in $D\cup D'$. Now (9) implies the same statement for F:

$$IF(x,\xi)=0\quad\text{for}\quad(x,\xi)\in\Gamma. \tag{10}$$

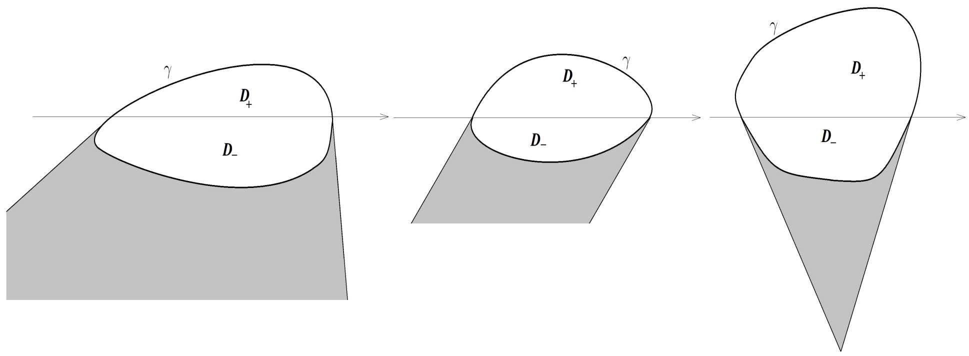

Fig. 1

In the 2D case for $m=0$, the ray transform I coincides up to notations with the Radon transform R. Let us use the support theorem for the Radon transform [6, Chapter 1, Theorem 2.6], more precisely its corollary [6, Chapte 1, Corollary 2.8] which states:

If a function $F\in C(\mathbb{R}^2)$ sufficiently fast decays at infinity and $RF(l)=0$ for all lines l which do not intersect a compact convex set $K\subset\mathbb{R}^2$, then $\operatorname{supp}F\subset K$.

By (10), for the above-defined function $F\in C_c(\mathbb{R}^2)$, we can choose K to be some compact convex subset of $D_-\cup D'$. Applying the latter statement, we obtain $F|_{D_+}=0$. Whence $f|_{D_+}=F|_{D_+}=0$. □

Proposition 2. *Theorem 1 is true for $m=1$.*

Proof. Let a vector field $f=(f_1,f_2)\in C^k(D;\mathbb{C}^2)$ $(k\ge 1)$ belong to the kernel of the operator I_Γ. We are going to prove that the 1-form

$$\omega=f_1dx_1+f_2dx_2 \tag{11}$$

is exact on the domain D_+.

We parameterize the curve γ by the arc length

$$\gamma(t)=\big(\gamma_1(t),\gamma_2(t)\big)\quad(0\le t\le T),$$

where γ_1 and γ_1 are C^1-functions on $[0,T]$ satisfying $\dot\gamma_1^2+\dot\gamma_2^2=1$. For $t_1,t_2\in[0,T]$, let $[\gamma(t_1),\gamma(t_2)]$ be the straight line segment joining the points $\gamma(t_1)$ and $\gamma(t_2)$. By the strict convexity of D, the hypothesis $I_\Gamma f=0$ can be written in the form

$$\int\limits_{[\gamma(t_1),\gamma(t_2)]}\omega=0\quad(t_1,t_2\in[0,T]). \tag{12}$$

First of all we will prove that

(13) $$f_1(\gamma(t))\dot\gamma_1(t) + f_2(\gamma(t))\dot\gamma_2(t) = 0 \quad (t \in [0,T]).$$

To this end let us consider two close points $\gamma(t)$ and $\gamma(t+\Delta t)$ of the curve γ. By (12),

$$\int\limits_{[\gamma(t),\gamma(t+\Delta t)]} \omega = 0.$$

We rewrite this equation in the form

(14) $$\int\limits_0^1 \Big[f_1\big((1-s)\gamma(t) + s\gamma(t+\Delta t)\big)\eta_1 + f_2\big((1-s)\gamma(t) + s\gamma(t+\Delta t)\big)\eta_2\Big]ds = 0,$$

where

$$\eta = \eta(t,\Delta t) = \frac{\gamma(t+\Delta t) - \gamma(t)}{\Delta t}.$$

By the Lagrange theorem on the mean value,

$$\int\limits_0^1 \Big[f_1\big((1-s)\gamma(t) + s\gamma(t+\Delta t)\big)\eta_1 + f_2\big((1-s)\gamma(t) + s\gamma(t+\Delta t)\big)\eta_2\Big]ds =$$

$$= f_1\big((1-\theta)\gamma(t) + \theta\gamma(t+\Delta t)\big)\eta_1 + f_2\big((1-\theta)\gamma(t) + \theta\gamma(t+\Delta t)\big)\eta_2$$

for some $\theta \in [0,1]$. Together with (14), this gives

$$f_1\big((1-\theta)\gamma(t) + \theta\gamma(t+\Delta t)\big)\eta_1 + f_2\big((1-\theta)\gamma(t) + \theta\gamma(t+\Delta t)\big)\eta_2 = 0 \quad (\theta \in [0,1]).$$

Taking the limit as $\Delta t \to 0$, we arrive to (13).

For $t_1, t_2 \in [0,T]$, $t_1 \le t_2$, let D_{t_1,t_2} be the subdomain of D_+ bounded by the part $\gamma|_{[t_1,t_2]}$ of γ and straight line segment $[\gamma(t_1),\gamma(t_2)]$. By the Green formula,

$$\iint\limits_{D_{t_1,t_2}} d\omega = \int\limits_{\gamma|_{[t_1,t_2]}} \omega - \int\limits_{[\gamma(t_1),\gamma(t_2)]} \omega.$$

Both integrals on the right-hand side are equal to zero by (12) and (13). Thus,

$$\iint\limits_{D_{t_1,t_2}} d\omega = 0.$$

Since $t_1, t_2 \in [0,T]$ are arbitrary, this easily implies with the help of (13) that

(15) $$d\omega = 0 \quad \text{in the domain } D_+,$$

i.e., the form ω is closed in the domain D_+. Since the domain is simply connected, the form ω is exact in D_+, i.e., there exists a function $v \in C^1(D_+)$ satisfying

(16) $$dv = \omega \quad \text{in the domain } D_+.$$

The function v is determined by the equation (16) uniquely up to an additive constant. Choose the constant such that $v(\gamma(0)) = 0$. Again using (13), we obtain

$$v(\gamma(t)) = \int\limits_{\gamma|_{[0,t]}} \omega = 0 \quad (t \in [0,T]),$$

i.e.,

(17) $$v|_\gamma = 0.$$

In equations (15) and (16), d is the exterior derivative on differential forms. But for the form (11), the equation (16) is equivalent to

$$dv = f \quad \text{in the domain } D_+, \tag{18}$$

where d is the inner derivative.

It remains to use the ellipticity of the operator d mentioned in the previous section: a solution v to the equation (18) belongs to $C^{k+1}(D_+)$ if $f \in C^k(D_+;\mathbb{C}^2)$. Equations (17) and (18) mean that f is a γ-potential field. □

In the 2D case, a rank m symmetric tensor f has $m+1$ different coordinate. Introduce the simplified notation for the coordinates by setting

$$\tilde f_i = f_{\underbrace{1\dots1}_{i}\underbrace{2\dots2}_{m-i}} \quad (i = 0,1,\dots,m).$$

It is also convenient to set $\tilde f_{-1} = \tilde f_{m+1} = 0$. The equality $If(x,\xi) = 0$ is written in new notations as follows:

$$\int\limits_{-\infty}^{\infty} \sum_{i=0}^{m} \binom{m}{i} \tilde f_i(x + t\xi)\xi_1^i \xi_2^{m-i}\, dt = 0,$$

and the equation (3) takes the form

$$\frac{1}{m}\Big(i\frac{\partial \tilde v_{i-1}}{\partial x_1} + (m-i)\frac{\partial \tilde v_i}{\partial x_2}\Big) = \tilde f_i \quad (i = 0,1,\dots,m). \tag{19}$$

Proof of Theorem 1. Choose Cartesian coordinates on the plane as mentioned after the statement of the theorem. The proof is going by induction in m. Let a tensor field $f \in C^k(D; S^m\mathbb{R}^2)$ $(m \geq 2)$ belong to the kernel of the operator I_Γ. We are looking for a rank $m-1$ symmetric tensor field v defined on the domain D_+, vanishing on the curve γ, and satisfying the system (19) in the domain D_+.

We first define an auxiliary tensor field $w \in C^k(D_+; S^{m-1}\mathbb{R}^2)$ which has only two non-zero coordinates $\tilde w_0$ and $\tilde w_{m-1}$. We choose the latter coordinates as a solution to the subsystem of (19) consisting of the first and last equations, i.e.,

$$\frac{\partial \tilde w_{m-1}}{\partial x_1} = \tilde f_m, \quad \frac{\partial \tilde w_0}{\partial x_2} = \tilde f_0. \tag{20}$$

These equations are independent of each other. The first equation of the system (20) is solved by integration over horizontal chords of the domain D_+:

$$\tilde w_{m-1}(x_1,x_2) = \int\limits_{-\infty}^{x_1} \tilde f_m(t,x_2)\, dt \quad (x_2 \geq 0). \tag{21}$$

The second equation of system (20) is solved by integration over vertical chords of the domain D_+:

$$\tilde w_0(x_1,x_2) = -\int\limits_{x_2}^{\infty} \tilde f_0(x_1,t)\, dt \quad (x_2 \geq 0). \tag{22}$$

On the right-hand sides of (21) and (22), the field f is assumed to be extended by zero outside the domain D. Other coordinates of w are equal to zero by the definition. The smoothness of w is not less than that of f, i.e., $w \in C^k(D_+; S^{m-1}\mathbb{R}^2)$.

Formulas (21) and (22) imply with the help of the main hypothesis of the theorem $I_\Gamma f=0$ that w satisfies the boundary condition

$$w|_\gamma=0. \tag{23}$$

Whence $dw\in C^{k-1}(D_+;S^m\mathbb{R}^2)$ is a γ-potential tensor field.

Let us now consider the tensor field $g=f-dw\in C^{k-1}(D_+;S^m\mathbb{R}^2)$. It satisfies

$$\tilde g_0=0,\quad \tilde g_m=0 \tag{24}$$

by (20). The field g also belongs to the kernel of the operator I_Γ, i.e.,

$$\int\limits_{-\infty}^{\infty}\sum_{i=0}^{m}\binom{m}{i}\tilde g_i(x+t\xi)\xi_1^i\xi_2^{m-i}\,dt=0\quad\big((x,\xi)\in\Gamma\big).$$

The first and last summands of the sum are identically equal to zero by (24). Therefore the equation can be written in the form

$$\xi_1\xi_2\int\limits_{-\infty}^{\infty}\sum_{i=1}^{m-1}\binom{m}{i}\tilde g_i(x+t\xi)\xi_1^{i-1}\xi_2^{m-i-1}\,dt=0\quad\big((x,\xi)\in\Gamma\big).$$

For $\xi_1\xi_2\neq0$, this implies

$$\int\limits_{-\infty}^{\infty}\sum_{i=1}^{m-1}\binom{m}{i}\tilde g_i(x+t\xi)\xi_1^{i-1}\xi_2^{m-i-1}\,dt=0\quad\big((x,\xi)\in\Gamma\big). \tag{25}$$

In fact the equation (25) holds for $\xi_1\xi_2=0$ too, as is proved by passing to the limit.

In the case of $m=2$, the equation (25) reads

$$\int\limits_{-\infty}^{\infty}\tilde g_1(x+t\xi)\,dt=0\quad\big((x,\xi)\in\Gamma\big).$$

By Proposition 1, this implies $\tilde g_1\equiv0$. Together with (24), this gives $g\equiv0$ in D_+. Recall that g was defined by $g=f-dw\in C^{k-1}(D_+;S^2\mathbb{R}^2)$ $(k\ge2)$. Therefore the statement $g\equiv0$ means that

$$f=dw\quad\text{in the domain } D_+. \tag{26}$$

We use the ellipticity of the operator d: since f is of the class C^k, the equation (26) implies that $w\in C^{k+1}(D_+;\mathbb{C}^n)$. Now, (23) and (26) mean that f is a γ-potential tensor field. This proves Theorem 1 in the case of $m=2$.

Now, we continue the proof for $m\ge3$.

The left-hand side of (25) is very similar to the value of the ray transform of some rank $m-2$ tensor field. Namely, let us introduce a tensor field $u\in C^{k-1}(D_+;S^{m-2}\mathbb{R}^2)$ by

$$\binom{m-2}{i}\tilde u_i=\binom{m}{i+1}\tilde g_{i+1}. \tag{27}$$

Then the equation (25) can be rewritten in the form

$$\int\limits_{-\infty}^{\infty}\sum_{i=0}^{m-2}\binom{m-2}{i}\tilde u_i(x+t\xi)\xi_1^i\xi_2^{m-2-i}\,dt=0\quad\big((x,\xi)\in\Gamma\big).$$

In other words, the rank $m-2$ tensor field u belongs to the kernel of the operator I_Γ. By the induction hypothesis, there exists a tensor field $h \in C^k(D_+; S^{m-3}\mathbb{R}^2)$ satisfying the boundary condition

$$h|_\gamma = 0 \tag{28}$$

and the equation

$$dh = u \quad \text{in the domain } D_+. \tag{29}$$

The equation (29) looks in coordinates as follows:

$$\frac{1}{m-2}\Big(i\frac{\partial \tilde h_{i-1}}{\partial x_1} + (m-i-2)\frac{\partial \tilde h_i}{\partial x_2}\Big) = \tilde u_i \quad (i = 0, 1, \dots, m-2).$$

Substitute the expression (27) for coordinates of u into the latter formula

$$\frac{1}{m-2}\Big(i\frac{\partial \tilde h_{i-1}}{\partial x_1} + (m-i-2)\frac{\partial \tilde h_i}{\partial x_2}\Big) = \binom{m-2}{i}^{-1}\binom{m}{i+1}\tilde g_{i+1}.$$

Decreasing the value of the index i by 1, we easily transform this system to the form

$$\frac{1}{m}\Big(i\frac{(i-1)(m-i)}{(m-1)(m-2)}\frac{\partial \tilde h_{i-2}}{\partial x_1} + (m-i)\frac{i(m-i-1)}{(m-1)(m-2)}\frac{\partial \tilde h_{i-1}}{\partial x_2}\Big) = \tilde g_i \quad (i = 1, \dots, m-1). \tag{30}$$

Define the functions

$$\tilde z_i = \frac{i(m-i-1)}{(m-1)(m-2)}\tilde h_{i-1} \quad (i = 1, \dots, m-2). \tag{31}$$

Then the system (30) takes the form

$$\frac{1}{m}\Big(i\frac{\partial \tilde z_{i-1}}{\partial x_1} + (m-i)\frac{\partial \tilde z_i}{\partial x_2}\Big) = \tilde g_i \quad (i = 1, \dots, m-1). \tag{32}$$

Recall that $\tilde g_0 = \tilde g_m = 0$. Therefore, setting $\tilde z_0 = \tilde z_{m-1} = 0$ by the definition, the equation (32) holds for $i = 0$ and $i = m$, i.e., we have the system

$$\frac{1}{m}\Big(i\frac{\partial \tilde z_{i-1}}{\partial x_1} + (m-i)\frac{\partial \tilde z_i}{\partial x_2}\Big) = \tilde g_i \quad (i = 0, \dots, m).$$

It means that

$$g = dz \quad \text{in the domain } D_+. \tag{33}$$

As is seen from (28) and (33),

$$z|_\gamma = 0. \tag{34}$$

Together with the equality $g = f - dw$, (33) gives

$$f = dv \quad \text{in the domain } D_+, \tag{35}$$

where $v = w + z$. From (23) and (34) we see that

$$v|_\gamma = 0. \tag{36}$$

It remains to use the ellipticity of the operator d mentioned in the previous section: a solution v to the equation (35) belongs to $C^{k+1}(D_+; S^{m-1}\mathbb{R}^2)$ if $f \in C^k(D_+; S^m\mathbb{R}^2)$. Now (35) and (36) mean that f is a γ-potential tensor field. □

Remark 1. The presented proof is also valid for the *ray transform with complete data* when $\gamma = \partial D,\ D_+ = D,\ D_- = \emptyset$. In this case, Theorem 1 states that, for $k \geq m$, the kernel of the operator (4) coincides with the space of tensor fields $f \in C^k(D; S^m\mathbb{R}^2)$ such that there exists $v \in C^{k+1}(D; S^{m-1}\mathbb{R}^2)$ satisfying the equation $dv = f$ in the domain D and boundary condition $v|_{\partial D} = 0$. Such v are called *potential tensor fields* (without the prefix γ-) in [13]. In fact the latter statement is true for any $k \geq 0$ and is a corollary of Theorem 2.5.1 of [13].

Remark 2. Most probably, the strict convexity hypothesis of Theorem 1 can be replaced with the standard convexity of the domain D. Besides this, the hypothesis on C^1-smoothness of D can be omitted. Thus, Theorem 1 is valid for a closed bounded convex domain $D \subset \mathbb{R}^2$ containing an inner point. Let us discuss these possibilities.

If a closed bounded domain $D \subset \mathbb{R}^2$ is convex but not strictly convex, then the function $If(x, \xi)$ can have first kind discontinuities. Therefore instead of (4) we have to use the linear continuous operator

$$I : C^k(D; S^m\mathbb{R}^2) \to L^\infty(T\mathbb{S}^1). \tag{37}$$

Here the space $L^\infty(T\mathbb{S}^1)$ is introduced with the help of the measure $dx\,d\xi$ on $T\mathbb{S}^1$ which is defined by

$$\int\limits_{T\mathbb{S}^1} \varphi(x,\xi)\,dxd\xi = \int\limits_{\mathbb{S}^1}\int\limits_{\xi^\perp} \varphi(x,\xi)\,dxd\xi \quad \big(\varphi \in C_c(T\mathbb{S}^1)\big).$$

On the right-hand side of the latter formula, dx is the length element of the line $\xi^\perp = \{x \in \mathbb{R}^2 \mid \langle x, \xi\rangle = 0\}$ and $d\xi$ is the length element of the circle $\mathbb{S}^1 = \{\xi \in \mathbb{R}^2 \mid |\xi| = 1\}$. Nevertheless, the restriction of the function If to $\Gamma \subset T\mathbb{S}^1$ is continuous and (5) remains true.

The strict convexity of the domain D was used in the proof of Proposition 1 for the definition of a function $F \in C_c(\mathbb{R}^2)$ satisfying (10). Then we applied the support theorem for the Radon transform [6, Chapter 1, Theorem 2.6] to F. The continuity of F is one of hypotheses of the latter Theorem. Instead of that, we could imply the following version of the support theorem to the function $f \in C(D)$:

Let $f \in C(K)$ for some compact convex set $K \subset \mathbb{R}^n$. Extend f by zero outside K. Assume the existence of a constant $A > 0$ such that $Rf(P) = 0$ for any hyperplane $P \subset \mathbb{R}^n$ such that the distance from the origin to P is $> A$. Then $\operatorname{supp} f \subset K \cap \{x \in \mathbb{R}^n \mid |x| \leq A\}$.

This statement can be proved in the same way as the proof of the support theorem presented in [6].

The strict convexity of the domain D was also used in the proof of Proposition 2 to derive the equation (13). If the domain D is convex but not strictly convex, then the equation (13) still holds at strict convexity points of the curve γ but can be wrong at inner points of a straight line segment $[\gamma(t_1), \gamma(t_2)] \subset \gamma$ (there can be a countable family of such segments). But the equation (12) remains true for such segments in virtue of the hypothesis $I_\Gamma f = 0$. Therefore

$$\int\limits_{\gamma|_{[t_1,t_2]}} \omega = 0 \quad (t_1, t_2 \in [0, T],\ t_1 \leq t_2).$$

Only this statement is used in the rest of the proof of Proposition 2.

To omit the hypothesis on the C^1-smoothness of D, we have to use the well known statement: the boundary curve of a convex domain $D \subset \mathbb{R}^2$ is differentiable at almost all its points. In virtue of the latter statement, the integral on the left-hand side of (12) is well defined although the integrand is not continuous.

An accurate implementation of this program would require a significant lengthening of the proof of Theorem 1, making it difficult to understand the simple main idea of the proof. Therefore we restricted ourselves to considering a strictly convex domain of the class C^1.

4. The multi-dimensional ray transform with incomplete projection data

Let $D \subset \mathbb{R}^n$ $(n \geq 2)$ be a closed convex domain bounded by a closed hypersurface ∂D of the class C^1. We additionally assume that D is a *strictly convex domain*, i.e., the hypersurface ∂D does not contain straight line segments of positive length. In such a case we say that D is a *closed bounded strictly convex domain of the class* C^1. For such a domain, the ray transform

$$I : C^k(D; S^m\mathbb{R}^n) \to C(T\mathbb{S}^{n-1}) \tag{38}$$

is defined by the same formula (2), where f is extended by zero outside D. Choose an affine hyperplane $P_0 \subset \mathbb{R}^n$ through an inner point of the domain D. By $\mathbb{R}^n_+$ we denote one of two closed half spaces bounded by P_0 and by $\mathbb{R}^n_-$, the second one. Set $D_\pm = D \cap \mathbb{R}^n_\pm$ and $\gamma = \partial D \cap \mathbb{R}^n_+$. Let Γ be the closed set in $T\mathbb{S}^{n-1}$ consisting of $(x,\xi) \in T\mathbb{S}^{n-1}$ such that the intersection of the line $l_{x,\xi} = \{x + t\xi \mid t \in \mathbb{R}\}$ with D is a non-empty segment with both endpoints belonging to the hypersurface γ (the segment can consist of one point). We introduce the linear continuous operator

$$I_\Gamma : C^k(D; S^m\mathbb{R}^n) \to C(\Gamma) \tag{39}$$

by $I_\Gamma f = (If)|_\Gamma$, where If is the value of the operator (38) on the field$f \in C^k(D; S^m\mathbb{R}^n)$. The operator (39) is called the *ray transform with incomplete projection data determined on the domain* $\Gamma \subset T\mathbb{S}^{n-1}$.

We say that $f \in C^k(D; S^m\mathbb{R}^n)$ is a *γ-potential tensor field* if there exists a tensor field $v \in C^{k+1}(D_+; S^{m-1}\mathbb{R}^n)$ satisfying the boundary condition

$$v|_\gamma = 0 \tag{40}$$

and equation

$$dv = f \quad \text{in} \quad D_+. \tag{41}$$

Theorem 2. *Let $k \geq m \geq 0$ and $D \subset \mathbb{R}^n$ $(n \geq 2)$ be a closed bounded strictly convex domain of the class C^1, the domains $D_\pm$ and hypersurface $\gamma \subset \partial D$ be chosen as above. The kernel of the operator* (39) *coincides with the space of γ-potential tensor fields.*

Proof. We present the proof for $m \geq 1$. For $m = 0$ the proof is the same with many simplifications. The proof is going by induction in n. For $n = 2$, Theorem 2 coincides with Theorem 1. Now assume $n \geq 3$. We are going to restrict all the data of our n-dimensional problem to each affine hyperplane, thereby obtaining a family of similar $(n-1)$-dimensional problems.

For an affine hyperplane $P \subset \mathbb{R}^n$, we set $D^P = D \cap P, D^P_\pm = D_\pm \cap P$, $\gamma^P = \gamma \cap P$. Then D^P is a closed bounded strictly convex domain of the class C^1 in P. Of course, for some hyperplanes, the domain D^P can be either empty or consisting of one point.

Such cases are not interesting and we do not consider them. We also denote by Γ^P the set of $(x,\xi)\in T\mathbb{S}^{n-1}$ such that the intersection of the line $l_{x,\xi}=\{x+t\xi\mid t\in\mathbb{R}\}$ with D^P is a non-empty segment with endpoints belonging γ^P (including the case when the segment consists of one point).

For an affine hyperplane $P\subset\mathbb{R}^n$, by $\mathbb{R}^{n-1}_P$ we denote the vector subspace of dimension $n-1$ of $\mathbb{R}^n$ parallel to the hyperplane P (directing subspace of the hyperplane P). Let us consider a tensor field $f\in C^k(D;S^m\mathbb{R}^n)$. The restriction $f^P\in C^k(D^P;S^m\mathbb{R}^{n-1}_P)$ of f to a hyperplane P is defined as follows. The field f can be identified with the function

$$f(x,\xi)=f_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}\quad(x\in D,\xi\in\mathbb{R}^n) \tag{42}$$

which is a homogeneous polynomial of degree m in ξ with coefficients $f_{i_1\dots i_m}\in C^k(D)$. The tensor field f^P is identified with the restriction of the polynomial to $D^P\times\mathbb{R}^{n-1}_P$.

Let a tensor field $f\in C^k(D;S^m\mathbb{R}^n)$ belong to the kernel of the operator (39). Then, for every affine hyperplane $P\subset\mathbb{R}^n$, the tensor field $f^P\in C^k(D^PP;S^m\mathbb{R}^{n-1}_P)$ belongs to the kernel of the *ray transform on the hyperplane* P

$$I_{\Gamma^P}:C^k(D^P;S^m\mathbb{R}^{n-1}_P)\to C(\Gamma^P). \tag{43}$$

If $D^P_-=D_-\cap P=\emptyset$, then (43) is the ray transform with complete projection data. Otherwise (43) is the ray transform with incomplete data.

Applying the induction hypothesis, we obtain, for every affine hyperplane $P\subset\mathbb{R}^n$, a tensor field $v^P\in C^{k+1}(D^P_+;S^{m-1}\mathbb{R}^{n-1}_P)$ satisfying the boundary condition

$$v^P|_{\gamma^P}=0 \tag{44}$$

and equation

$$d^Pv^P=f^P\quad\text{in the domain}\quad D^P_+, \tag{45}$$

where d^P is the inner derivative on the hyperplane P. Similarly to (42), we identify the tensor field $v^P\in C^{k+1}(D^P_+;S^{m-1}\mathbb{R}^{n-1}_P)$ with the function $v^P(x,\xi)$ of variables $(x,\xi)\in D^P_+\times\mathbb{R}^{n-1}_P$ which is a homogeneous polynomial of degree $m-1$ in ξ. Choosing Cartesian coordinates $(x_1,\dots,x_{n-1})$ on the hyperplane P, the function is written as

$$v^P(x,\xi)=v^P_{\alpha_1\dots\alpha_{m-1}}(x)\xi^{\alpha_1}\dots\xi^{\alpha_{m-1}}\quad(x\in D^P_+,\xi\in\mathbb{R}^{n-1}_P). \tag{46}$$

Hereafter Greek indices vary from 1 to $n-1$ and the summation from 1 to $n-1$ is assumed over repeating Greek indices.

It remains to check that the tensor fields v^P, being defined for all affine hyperplanes P, are agreed to each other, i.e., they together determine some tensor field on the whole of D_+.

For $x\in D$ and $0\neq\xi\in\mathbb{R}^n$, the line $l_{x,\xi}=\{x+t\xi\mid t\in\mathbb{R}\}$ intersects ∂D at $t=\tau_\pm(x,\xi)$ such that $\tau_-(x,\xi)\le 0\le\tau_+(x,\xi)$. Observe that $\tau_-(x,\xi)$ is a C^1-function on the set

$$\big(D\times(\mathbb{R}^n\setminus\{0\})\big)\setminus T(\partial D), \tag{47}$$

where $T(\partial D)=\{(x,\xi)\mid x\in\partial D,\xi\in T_x(\partial D)\}$. Indeed, for (x,ξ) belonging to the set (47), the line $l_{x,\xi}$ intersects the boundary ∂D transversally.

Let a point $x \in D_+$ and vector $0 \neq \xi \in \mathbb{R}^n$ be such that $x + \tau_-(x,\xi)\xi \in \gamma$. Choose a hyperplane $P \subset \mathbb{R}^n$ such that $x \in D_+^P, \xi \in \mathbb{R}_P^{n-1}$. As follows from (45) and (46),

$$\begin{aligned}\frac{dv^P(x+t\xi,\xi)}{dt} &= \frac{\partial v^P_{\alpha_1\dots\alpha_{m-1}}(x+t\xi)}{\partial x_{\alpha_m}}\xi^{\alpha_1}\dots\xi^{\alpha_{m-1}}\\ &= (d^P v^P)_{\alpha_1\dots\alpha_m}(x+t\xi)\xi^{\alpha_1}\dots\xi^{\alpha_m}\\ &= f^P_{\alpha_1\dots\alpha_m}(x+t\xi)\xi^{\alpha_1}\dots\alpha^{i_m}\\ &= f^P(x+t\xi,\xi) = f(x+t\xi,\xi).\end{aligned}$$

The last equality of the chain holds since the function $f^P(x,\xi)$ is the restriction of the function $f(x,\xi)$. Integrating the latter equation with respect to t in the limits $\tau_-(x,\xi) \le t \le 0$, we obtain

$$v^P(x,\xi) - v^P\big(x+\tau_-(x,\xi)\xi,\xi\big) = \int\limits_{\tau_-(x,\xi)}^{0} f(x+t\xi,\xi)\,dt.$$

The second term on the left-hand side is equal to zero by (44) and the formula simplifies to the following one:

$$v^P(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} f(x+t\xi,\xi)\,dt \quad \big(x \in D_+^P, 0 \neq \xi \in \mathbb{R}_P^{n-1}, x+\tau_-(x,\xi)\xi \in \gamma^P\big). \tag{48}$$

The right-hand side of (48) is independent of the affine hyperplane P. Therefore this equality can be written in the form

$$v(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} f(x+t\xi,\xi)\,dt \quad \big((x,\xi) \in A\big), \tag{49}$$

where $v(x,\xi)$ is the function defined on the set

$$A = \{(x,\xi) \mid x \in D_+,\ 0 \neq \xi \in \mathbb{R}^n,\ x+\tau_-(x,\xi)\xi \in \gamma\} \subset D_+ \times (\mathbb{R}^n \setminus \{0\})$$

by

$$v(x,\xi) = v^P(x,\xi) \tag{50}$$

if $x \in D_+^P,\ 0 \neq \xi \in \mathbb{R}_P^{n-1},\ x+\tau_-(x,\xi)\xi \in \gamma^P$.

Observe that, for $x \in D_+$ and $0 \neq \xi \in \mathbb{R}^n$, at least one of two pairs (x,ξ) and $(x,-\xi)$ belongs to A. By (46), the function v^P satisfies $v^P(x,-\xi) = (-1)^m v^P(x,\xi)$. Therefore, setting $v(x,-\xi) = (-1)^m v(x,\xi)$ for $(x,\xi) \in A$, we obtain the function v that is defined on $D_+ \times (\mathbb{R}^n \setminus \{0\})$ and satisfies (50). By (46) and (50), the function v possesses the following property. For every point $x \in D_+$ and every $(n-1)$-dimensional vector subspace $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, the restriction of the function $v(x,\cdot)$ to $\mathbb{R}^{n-1} \setminus \{0\}$ is a homogeneous polynomial of degree $m-1$.

The function v can be extended to $D_+ \times \mathbb{R}^n$ with preserving the latter property. In other words, it is possible to define values $v(x,0)$ for $x \in D_+$. Indeed, for $m \ge 2$, we can set $v(x,0) = 0$ since a homogeneous polynomial of degree $m-1$ vanishes at $\xi = 0$. For $m = 1$, the restriction of the function $v(x,\cdot)$ to $\mathbb{R}^{n-1} \setminus \{0\}$ is a constant function for every point $x \in D_+$ and every $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. These constants coincide since any two $(n-1)$-dimensional subspaces have a non-zero intersection (recall that $n \geq 3$). Therefore we can set $v(x,0) = v(x,\xi)$ for any $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$.

Let us use the following statement.

Lemma 1. *Let V be a finite-dimensional real vector space of dimension $n \geq 3$ and $f : V \to \mathbb{C}$ be a function satisfying the condition: for every $(n-1)$-dimensional vector subspace $V^{n-1} \subset V$, the restriction $f|_{V^{n-1}}$ is a homogeneous polynomial of degree m. Then f itself is a homogeneous polynomial of degree m.*

The proof of the lemma will be presented later. Now we finish the proof of Theorem 2 with the help of the lemma.

For a point $x \in D_+$, we apply Lemma 1 to the function $v(x,\cdot)$ defined by (50). The restriction of this function to $\mathbb{R}^{n-1}$ is a homogeneous polynomial of degree $m-1$ for any $(n-1)$-dimensional vector subspace $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ by the statement before Lemma 1. Thus,

$$v(x,\xi) = v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\dots\xi^{i_{m-1}} \quad (x \in D_+, \xi \in \mathbb{R}^n). \tag{51}$$

The equation (49) takes now the form

$$v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\dots\xi^{i_{m-1}} = \int\limits_{\tau_-(x,\xi)}^{0} f(x+t\xi,\xi)\,dt \quad \big((x,\xi) \in A\big). \tag{52}$$

Let us demonstrate that all functions $v_{i_1\dots i_{m-1}}$ belong to $C^1(D_+)$. Indeed, the integrand in (52) is a function of the class C^k ($k \geq 1$). The lower integration limit in (52) is a function of the class C^1 on the set (47). Therefore the left-hand side of (52) is a function of the class C^1 on the set $A \setminus T(\partial D)$. Observe that, for a point $x \in D_+$, the cone $\{\xi \in \mathbb{R}^n \mid (x,\xi) \in A \setminus T(\partial D)\}$ contains inner points. Therefore (52) implies that all functions $v_{i_1\dots i_{m-1}}$ belong to $C^1(D_+)$ and we can introduce the tensor field $v = (v_{i_1\dots i_{m-1}}) \in C^1(D_+; S^{m-1}\mathbb{R}^n)$. The same formula (52) implies the validity of the boundary condition

$$v|_\gamma = 0. \tag{53}$$

After the formula (47), some arguments are presented which show that (44)–(45) imply (48). Inverting these arguments, we now show that (52)–(53) imply

$$dv = f \quad \text{in the domain } D_+. \tag{54}$$

It remain to use again the ellipticity of the operator d: since the right-hand side of the equation (54) belongs to $C^k(D_+; S^m\mathbb{R}^n)$, then $v \in C^{k+1}(D_+; S^{m-1}\mathbb{R}^n)$. Now formulas (53)–(54) mean that f is a γ-potential tensor field. □

Proof of Lemma 1. The restriction of the function f to any affine plane of dimension $\leq n-2$ is a polynomial of degree m (which does not need to be homogeneous) since such a plane can be included into an $(n-1)$-dimensional vector subspace. The terminology remark should be added: a polynomial of degree $k < m$ is also a polynomial of degree m.

Let us prove that the restriction of f to every affine hyperplane P is a polynomial of degree $2m$. Choose affine coordinates $(x_1,\dots,x_{n-2},t)$ on the hyperplane P. Instead of $(x_1,\dots,x_{n-2},t)$, we write (y,t) where $y = (x_1,\dots,x_{n-2})$. The function $f(y,t)$ is a polynomial of degree m in t:

$$f(y,t) = \alpha_0(y) + \alpha_1(y)t + \dots + \alpha_m(y)t^m. \tag{55}$$

Let us demonstrate that all coefficients of the polynomial are polynomials of degree m in y. Setting $t = 0, 1, \dots, m$ in (55), we obtain the system

$$\begin{aligned} \alpha_0(y) &= f(y,0), \\ \alpha_0(y) + \alpha_1(y) + \dots + \alpha_m(y) &= f(y,1), \\ \alpha_0(y) + 2\alpha_1(y) + \dots + 2^m\alpha_m(y) &= f(y,2), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_0(y) + m\alpha_1(y) + \dots + m^m\alpha_m(y) &= f(y,m). \end{aligned}$$

We express $\alpha_0(y)$ from the first equation and substitute the expression to other equations. In this way we arrive to the system

$$\begin{aligned} \alpha_1(y) + \dots + \alpha_m(y) &= f(y,1) - f(y,0), \\ 2\alpha_1(y) + \dots + 2^m\alpha_m(y) &= f(y,2) - f(y,0), \\ \dots\dots\dots\dots\dots\dots \\ m\alpha_1(y) + \dots + m^m\alpha_m(y) &= f(y,m) - f(y,0). \end{aligned} \tag{56}$$

Right-hand sides of these equations are polynomials of degree m in y since the restriction of f to the $(n-2)$-dimensional affine plane $\{(y, t_0) \in P\}$ is a polynomial of degree m for any t_0. The Vandermonde determinant of the system (56) is not equal to zero. Therefore (56) implies that all coefficients of the polynomial (55) are polynomials of degree m in y. We see now from (55) that $f|_P$ is a polynomial of degree $2m$.

Now, we prove that f is a polynomial of degree $3m$. We actually repeat our previous arguments. Namely, choose affine coordinates $(x_1, \dots, x_{n-1}, t) = (y, t)$ in V. Since the restriction of f to any one-dimensional affine subspace is a polynomial of degree m, the representation (55) is valid. Repeating our arguments, we prove that all $\alpha_k(y)$ are polynomials of degree $2m$. Therefore the function f itself is a polynomial of degree $3m$.

Since the restriction of f to every one-dimensional vector subspace is a homogeneous polynomial of degree m, the function f itself is a homogeneous polynomial of degree m. □

5. Further plans

In the case of $m = 0$, Theorem 1 states that the restriction $f|_{D_+}$ of a function $f \in C(D)$ is uniquely recovered from the data $I_\Gamma f$. Is the recovering stable in any sense? Most probably, not. Indeed, this problem is closely related to the so called *exterior problem for the Radon transform* [10, §VI.3]. As well known, the latter problem is strongly ill-posed. Most probably, the same is true for our problem.

In the case of $m = 1$, Theorem 1 states that, for a vector field $f \in C^1(D; \mathbb{C}^2)$, the restriction $f|_{D_+}$ can be recovered from the data $I_\Gamma f$ uniquely up to a summand that is a γ-potential vector field. What information on the field $f|_{D_+}$ can be extracted from the data $I_\Gamma f$? To answer the question, the following considerations are useful.

Given a closed bounded strictly convex domain $D \subset \mathbb{R}^2$ of the class C^1, let D_+, γ, Γ be defined as in Theorem 1; set also $\gamma_0 = \overline{\partial D_+ \setminus \gamma}$. Then $\partial D_+ = \gamma \cup \gamma_0$. Let $\nu \in \mathbb{R}^2$ be the unit vector orthogonal to the segment γ_0. Let us show that every vector field $f \in C^1(D_+; \mathbb{C}^2)$ can be uniquely represented in the form

$$f = {}^s f + dv, \tag{57}$$

where the vector field ${}^s\!f \in C^1(D_+;\mathbb{C}^2)$ satisfies the equation

$$\operatorname{div} {}^s\!f = 0 \quad \text{in the domain } D_+ \tag{58}$$

and boundary condition

$$\langle {}^s\!f, \nu\rangle|_{\gamma_0} = 0, \tag{59}$$

and the function $v \in C^2(D_+)$ satisfies the boundary condition

$$v|_\gamma = 0. \tag{60}$$

Indeed, applying the operator div to the equality (57), we obtain

$$\Delta v = \operatorname{div} f \quad \text{in the domain } D_+. \tag{61}$$

On the other hand, restricting (57) to γ_0 and taking the dot product with ν, we obtain in virtue of (59)

$$\langle dv, \nu\rangle|_{\gamma_0} = \frac{\partial v}{\partial \nu}\Big|_{\gamma_0} = 0. \tag{62}$$

Thus, the function v solves the mixed boundary value problem for the Poisson equation (61) with the Dirichlet boundary condition (60) on the part γ of the boundary ∂D_+ and Neumann boundary condition (62) on the rest of the boundary. As well known, the solution to the problem exists, is unique, and well depends on the right-hand side div f of the equation (61).

We call the summands ${}^s\!f$ and dv of the decomposition (57) γ-*solenoidal* and γ-*potential* parts of the vector field f respectively. The summands are orthogonal to each other with respect to the scalar product

$$(f,g)_{L^2(D_+)} = \int\limits_{D_+} \langle f(x), g(x)\rangle\, dx \quad \big(f,g \in C(D_+;\mathbb{C}^2)\big).$$

Therefore Theorem 1 for $m = 1$ can be rephrased as follows: The ray transform $I_\Gamma f$ uniquely determines the γ-solenoidal part of a vector field f while its γ-potential part can be arbitrary.

In our forthcoming work we are going to generalize latter arguments to higher rank symmetric tensor fields and the multi-dimensional ray transform with incomplete data

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