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EXTENSIONS OF THE CATEGORY $S - Act$

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Abstract. We define a new category $SS-Act$ whose objects are S-acts and whose morphisms are defined so that each set $Hom_{SS-Act}(A, B)$ is an S-act. It is proved that this category has a reflective subcategory $FS - Act$ that is naturally isomorphic to the category $S - Act$. The set $Hom_{FS-Act}(A, B)$ coincides with the set of all fixed points of the S-act $Hom_{SS-Act}(A, B)$. In the case when S is a group, it is proved that the category $SS - Act$ is a Grothendieck topos and the construction of limits and colimits is considered.

Keywords: S-act, limits and colimits of functors, adjoint functor, Cartesian Closed Category.

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1. Introduction

In [4, 5], we consider the category of Chu spaces over the category $S - Act$, whose objects are S-acts and whose morphisms are homomorphisms of S-acts. In this case, S is a commutative monoid, the set of homomorphisms $Hom_{S-Act}(A, B)$ is an S-act, and the tensor product is used as the product required for the definition of the Chu spaces. If S is a noncommutative monoid then the tensor product of S-acts is not an S-act but it is possible to consider the Cartesian product as a product. The category $S - Act$ is Cartesian closed, but, unlike in the commutative case, the set $Hom_{S-Act}(A, B)$ is not endowed by a natural structure of an S-act.

In this paper, we define the category $SS-Act$ whose objects are S-acts and whose morphisms are defined so that each set $Hom_{SS-Act}(A, B)$ is an S-act. It is proved that this category has a reflective subcategory $FS-Act$ that is naturally isomorphic to the category $S - Act$. In this case, $Hom_{FS-Act}(A, B)$ coincides with the set of fixed points of the S-act $Hom_{SS-Act}(A, B)$, so that every homomorphism of S-acts is characterized as a fixed point. It is also proved that the category $SS - Act$ is Cartesian closed. This property is useful in the study of the category of Chu spaces over this category. In addition, we consider the completeness property for the category $SS - Act$. The case when S is a group is considered separately.

2. Basic Notions

Let K be a category. We use the following notations: $Ob(K)$ is the class of all objects of K, $Hom_K(a, b)$ is the set of all morphisms from an object a to an object b, $1_a \in Hom_K(a, a)$ is the identity morphism of a. The composition of morphisms $f \in Hom_K(a, b)$ and $g \in Hom_K(b, c)$ is denoted by $g \circ f$. In the category $SS - Act$ defined below, morphisms are mappings of sets but the composition of morphisms differs from the superposition of mappings and the identity morphisms differ from the identity maps. Therefore, special notations will be introduced for compositions and the identity morphisms.

Let us recall some definitions from act theory (see [1]). Let S be a monoid with identity 1. A set A is an *S-act* if there is a map $S \times A \to A, (s, a) \mapsto sa$ such that

$$1a = a \text{ and } s(ta) = (st)a$$

for all $a \in A$ and $s, t \in S$. A *subact* of an S-act A is a subset B of A closed under the action of S. An element a of S-act A is called a *zero* (*fixed point*) if $|Sa| = 1$, i.e. $sa = a$ for any $s \in S$. A *homomorphism* from an S-act A to an S-act B is a map $\theta : A \longrightarrow B$ such that $\theta(sa) = s\theta(a)$ for all $a \in A, s \in S$. The class of all S-acts together with homomorphisms as morphisms forms a category $S - Act$.

3. Preliminary Results

We consider some operations on mappings of sets. Let A, B, S be sets, $t \in S$, and $u : S \times A \longrightarrow B$, $v : S \times B \longrightarrow C$, $x : A \longrightarrow B$, $y : B \longrightarrow C$ be mappings. The mapping $e_A : S \times A \longrightarrow A$ is defined as follows: $e_A(s, a) = a$. Introduce the notations:

(S1) the mapping $(v \cdot u) : S \times A \longrightarrow C$ is defined as follows:

$$(v \cdot u)(s, a) = v(s, u(s, a));$$

(S2) if S is a monoid then the mapping $tu : S \times A \longrightarrow B$ is defined as follows:

$$(tu)(s,a)=u(st,a);$$

(S3) the mapping $\bar{u} : S \times A \longrightarrow S \times B$ is defined as follows:

$$\bar{u}(s,a)=(s,u(s,a));$$

(S4) the mapping $\tilde{x} : S \times A \longrightarrow B$ is defined as follows:

$$\tilde{x}=x \circ e_A,$$

i.e. $\tilde{x}(s,a) = x(a)$;

(S5) the mapping $u_t : A \longrightarrow B$ is defined as follows:

$$(u_t)(a)=u(t,a),$$

where $a \in A$.

Lemma 1. *Suppose that S is a monoid with identity 1, A, B, and S are sets, $t_0 \in S$, $u : S \times A \longrightarrow B$, $v : S \times B \longrightarrow C$, $w : S \times C \longrightarrow D$, $x : A \longrightarrow B$, $y : B \longrightarrow C$, and $z : C \longrightarrow D$ are mappings, and $e_A : S \times A \longrightarrow A$ is the projection mapping.*

(1) If A, B, C, D are S-acts and u, v, x, y are morphisms of the category $S - Act$ than the mappings $v \cdot u$, t_0u, $\bar{u}$, $\tilde{x}$ are also morphisms of the category $S - Act$.

(2) The mappings (S1)-(S5) possess the following properties:

(1.1) $w \cdot (v \cdot u) = (w \cdot v) \cdot u$; $u \cdot e_A = u = e_B \cdot u$;

(2.1) $t'(tu) = (t't)u$; $1u = u$ for any $t, t' \in S$;

(2.2) $t(v \cdot u) = tv \cdot tu$; $te_A = e_A$ for any $t \in S$;

(3.1) $\overline{v \cdot u} = \bar{v} \circ \bar{u}$; $\bar{e}_A = 1_{S \times A}$;

(3.2) $v \cdot u = v \circ \bar{u}$;

(4.1) $\widetilde{y \circ x} = \tilde{y} \cdot \tilde{x}$; $\tilde{1}_A = e_A$;

(4.2) $\tilde{y} \cdot u = y \circ u = \tilde{y} \circ \bar{u}$;

(5.1) $(v \cdot u)_t = v_t \circ u_t$; $(e_A)_t = 1_A$ for any $t \in S$;

(5.2) $(tu)_s = u_{st}$ for any $s, t \in S$;

(5.3) $\tilde{x}_s = x$ for any $s \in S$;

(5.4) if A and B are S-acts then $tu(s,a) = tu_s(a)$ and $u(ts,ta) = u_{ts}(ta)$ for any $s, t \in S$ and $a \in A$.

Proof. Suppose the fulfillment of the conditions of the lemma.

(1) Let A, B, C, D be S-acts and let u, v, x, y be morphisms of $S - Act$.

The mapping $v \cdot u$ is a morphism of $S - Act$:

$$(v \cdot u)(t(s,a)) = (v \cdot u)(ts,ta) = v(ts, u(ts,ta)) =$$

$$= v(ts, tu(s,a)) = tv(s, u(s,a)) = t((v \cdot u)(s,a))$$

for any $s, t \in S$, $a \in A$.

The mapping t_0u is a morphism of $S - Act$:

$$(t_0u)(ts,ta) = u(tst_0,ta) = t(u(st_0,a)) = t((t_0u)(s,a))$$

for any $s, t \in S$, $a \in A$.

The mapping $\bar{u}$ is a morphism of $S - Act$:

$$\bar{u}(ts,ta) = (ts, u(ts,ta)) = (ts, tu(s,a)) = t(s, u(s,a)) = t(\bar{u}(s,a))$$

for any $s, t \in S$, $a \in A$.

The mapping $\tilde{x}$ is a morphism of $S - Act$:

$$\tilde{x}(ts, ta) = x(ta) = tx(a) = t(\tilde{x}(s, a))$$

for any $s, t \in S$, $a \in A$.

(2) (1.1) The equalities

$$((w \cdot v) \cdot u)(s, a) = (w \cdot v)(s, u(s, a)) = w(s, v(s, u(s, a))) =$$
$$= w(s, (v \cdot u)(s, a)) = (w \cdot (v \cdot u))(s, a),$$

where $s \in S$, $a \in A$, imply $(w \cdot v) \cdot u = w \cdot (v \cdot u)$.Moreover,

$$(u \cdot e_A)(s, a) = u(s, e_A(s, a)) = u(s, a) = e_B(s, u(s, a)) = (e_B \cdot u)(s, a)$$

for any $s \in S$, $a \in A$; i.e. $u \cdot e_A = e_B \cdot u = u$.

(2.1) The equalities

$$(t'(tu))(s, a) = (tu)(st', a) = u(((st')t), a) = u((s(t't)), a) = ((t't)u)(s, a),$$
$$(1u)(s, a) = u(s1, a) = u(s, a),$$

where $s \in S$ and $a \in A$, imply $t'(tu) = (t't)u$ and $1u = u$ for any $t, t' \in S$.

(2.2) The equalities

$$((tv) \cdot (tu))(s, a) = (tv)(s, (tu)(s, a)) = v(st, u(st, a)) = (v \cdot u)(st, a) = (t(v \cdot u))(s, a),$$
$$(te_A)(s, a) = e_A(st, a) = a = e_A(s, a),$$

where $s \in S$, $a \in A$, imply $t(v \cdot u) = tv \cdot tu$ and $te_A = e_A$ for any $t \in S$.

(3.1) The equalities

$$(\bar{v} \circ \bar{u})(s, a) = \bar{v}(\bar{u}(s, a)) = (s, v(s, u(s, a))) = (s, (v \cdot u)(s, a)) = \overline{v \cdot u}(s, a),$$
$$\bar{e}_A(s, a) = (s, e_A(s, a)) = (s, a),$$

where $s \in S$, $a \in A$, imply $\overline{v \cdot u} = \bar{v} \circ \bar{u}$ and $\bar{e}_A = 1_{S \times A}$.

(3.2) The equalities

$$(v \circ \bar{u})(s, a) = v(\bar{u}(s, a)) = v(s, u(s, a)) = (v \cdot u)(s, a),$$

where $s \in S$, $a \in A$, imply $v \cdot u = v \circ \bar{u}$.

(4.1) The equalities

$$(\tilde{y} \cdot \tilde{x})(s, a) = \tilde{y}(s, \tilde{x}(s, a)) = y(x(a)) = (y \circ x)(a) = \widetilde{y \circ x}(s, a),$$
$$\tilde{1}_A(s, a) = 1_A(a) = a = e_A(s, a),$$

where $s \in S$, $a \in A$, imply $\widetilde{y \circ x} = \tilde{y} \cdot \tilde{x}$ and $\tilde{1}_A = e_A$.

(4.2) The equalities

$$(\tilde{y} \cdot u)(s, a) = \tilde{y}(s, u(s, a)) = y(u(s, a)) = (y \circ u)(s, a),$$

where $s \in S$, $a \in A$ imply $\tilde{y} \cdot u = y \circ u$. By (3.2) $\tilde{y} \cdot u = \tilde{y} \circ \bar{u}$.

(5.1) The equalities

$$(v \cdot u)_t(a) = (v \cdot u)(t, a) = v(t, u(t, a)) = v_t(u(t, a)) = v_t(u_t(a)) = (v_t \circ u_t)(a),$$
$$(e_A)_t(a) = (e_A)(t, a) = a = 1_A(a),$$

where $a \in A$ imply $(v \cdot u)_t = v_t \circ u_t$ and $(e_A)_t = 1_A$ for any $t \in S$.

(5.2) The equalities

$$(tu)_s(a) = (tu)(s, a) = u(st, a) = u_{st}(a),$$

where $a \in A$ imply $(tu)_s = u_{st}$ for any $s, t \in S$.

(5.3) The equalities

$$\tilde{x}_s(a) = \tilde{x}(s, a) = x(a),$$

where $a \in A$ imply $\tilde{x}_s = x$.

(5.4) Assume that A and B are S-acts. Then, by the definition of u_s and u_{ts}, we have $tu(s,a) = tu_s(a)$ and $u(ts,ta) = u_{ts}(ta)$ for any $s,t \in S$. □

Lemma 2. *Let A and B be S-acts, $u : S \times A \longrightarrow B$ and $v : S \times B \longrightarrow C$ be mappings, and $w = v \cdot u$.*

(1) If w and v are morphisms of $S - Act$ and the mappings v_s are injective for all $s \in S$ then u is a morphism of $S - Act$.

(2) If w and u are morphisms of $S - Act$ and the mappings u_s are surjective for all $s \in S$ then v is a morphism of $S - Act$.

(3) If the mappings u_s are bijective for all $s \in S$ and the mapping $v : S \times B \longrightarrow A$ is such that $v(s,b) = u_s^{-1}(b)$ for all $s \in S$ and $b \in B$ then

(a) $v \cdot u = e_A$, $u \cdot v = e_B$;

(b) under the condition that u is a morphism of $S - Act$, v is a morphism of $S - Act$.

Proof. Note that, by (5.4) of Lemma 1, u is a morphism of $S - Act$ if and only if $tu_s(a) = u_{ts}(ta)$ for all $s,t \in S$ and $a \in A$.

(1) Let $s,t \in S$ and $a \in A$. By (5.1) of Lemma 1 and the remark above, we have

$$v_{ts}(u_{ts}(ta)) = w_{ts}(ta) = tw_s(a),$$
$$v_{ts}(tu_s(a)) = tv_s(u_s(a)) = tw_s(a).$$

Hence $v_{ts}(u_{ts}(ta)) = v_{ts}(tu_s(a))$. Since the mapping v_{ts} is injective, $u_{ts}(ta) = tu_s(a)$. By the remark above, u is a morphism of $S - Act$.

(2) Let $s,t \in S$ and $b \in B$. Since u_s is surjective, $u_s(a) = b$ for some $a \in A$. By (5.1) of Lemma 1, and the remark above we have $tv_s(b) = tv_s(u_s(a)) = tw_s(a)$ and $v_{ts}(tb) = v_{ts}(tu_s(a)) = v_{ts}(u_{ts}(ta)) = w_{ts}(ta)$. Since w is a morphism of $S-Act$, $tw_s(a) = w_{ts}(ta)$, i.e., $tv_s(b) = v_{ts}(tb)$. Hence $tv(s,b) = v(ts,tb)$. By the remark above, v is a morphism of $S - Act$.

(3) Suppose that u_s is bijective for all $s \in S$, the mapping $v : S \times B \longrightarrow A$ is such that $v(s,b) = u_s^{-1}(b)$ for all $s \in S$ and $b \in B$.

(a) It is clear that $v_s = (u^{-1})_s$. By (5.1) of Lemma 1,

$$(v \cdot u)(s,a) = (v \cdot u)_s(a) = (v_s \circ u_s)(a) = a = e_A(a).$$

The equality $u \cdot v = e_B$ is obtained likewise.

(b) Let u and $v \cdot u = e_A$ be morphisms of the category $S - Act$. Since u_s are surjective for all $s \in S$, by (2), v is a morphism of $S - Act$. □

By (2.1) of Lemma 1, the mapping $(t,u) \mapsto tu$ $(t \in S)$ defines a left action on each set $Hom_{S-Act}(S \times A, B)$. So we can consider the set $Hom_{S-Act}(S \times A, B)$ as an S-act under this action. Let us consider some properties of the *zeros* of S-act $Hom_{S-Act}(S \times A, B)$; in other words, of the *fixed points* of $Hom_{S-Act}(S \times A, B)$ under this action, that is, such morphisms $u : S \times A \longrightarrow B$ that $tu = u$ for any $t \in S$.

Lemma 3. *(1) If $u : S \times A \longrightarrow B$, $v : S \times B \longrightarrow C$ are fixed points then $v \cdot u$ is a fixed point too; the morphism $e_A : S \times A \longrightarrow A$ is a fixed point.*

(2) The mapping $x \mapsto \tilde{x} \equiv x \circ e_A$ is a bijection of the set $Hom_{S-Act}(A,B)$ onto the set of all fixed points of $Hom_{S-Act}(S \times A, B)$.

(3) Let $s_0 \in S$ and $u \in Hom_{S-Act}(S \times A, B)$. Consider the following conditions:

(i) u is a fixed point;

(ii) $u=\tilde{x}$ *for some* $x\in Hom_{S-Act}(A,B)$*;*
(iii) $u_s=u_t$ *for all* $s,t\in S$*;*
(iv) $\tilde{u}_{s_0}=u$ *and* $u_{s_0}\in Hom_{S-Act}(A,B)$*;*
(v) $u_s\in Hom_{S-Act}(A,B)$ *for all* $s\in S$*;*
(v′) $u_{s_0}\in Hom_{S-Act}(A,B)$*.*
(a) Conditions (i), (ii), (iii), (iv) *are equivalent and* $(iv)\Rightarrow(v)\Rightarrow(v')$*.*
(b) Moreover, if S *is a group then conditions* (i), (ii), (iii), (iv), (v), (v') *are equivalent.*

Proof. (1) immediately follows from (2.2) of Lemma 1.

2) First we prove (3a). Let $s_0\in S$ and $u\in Hom_{S-Act}(A,B)$.

$(i)\Rightarrow(iii)$. By (5.2) of Lemma 1, $u_t=(tu)_1$, and so $u_t=u_1$ for any $t\in S$.

$(iii)\Rightarrow(iv)$. Since $u_s=u_t$ for all $s,t\in S$, we have

$$\tilde{u}_{s_0}(s,a)=u_{s_0}(a)=u_s(a)=u(s,a)$$

for all $s\in S$.

$(iv)\Rightarrow(iii)$. Since $u_{s_0}=u$, we have

$$u_s(a)=u(s,a)=\tilde{u}_{s_0}(s,a)=u_{s_0}(a)=u(s_0,a)=u_{s_0}(a)$$

for all $s,t\in S$.

$(iv)\Rightarrow(ii)$. By hypothesis, $\tilde{u}_{s_0}=u$. By $(iv)\Rightarrow(v)$, we have $x=u_{s_0}\in Hom_{S-Act}(A,B)$. Thus, $\tilde{x}=u$.

$(ii)\Rightarrow(i)$. Since $u=\tilde{x}$, we have

$$(tu)(s,a)=u(st,a)=\tilde{x}(st,a)=x(a)=\tilde{x}(s,a)=u(s,a)$$

for any $s,t\in S$, i.e., u is a fixed point.

$(iii)\Rightarrow(v)$. Since

$$u_1(ta)=u_t(ta)=u(t,ta)=tu(1,a)=tu_1(a)$$

for any $t\in S$, we have $u_1\in Hom_{S-Act}(A,B)$. Hence, $u_s\in Hom_{S-Act}(A,B)$ for any $s\in S$.

$(v)\Rightarrow(v')$ is obvious.

(3b) Now let S be a group. It suffices to prove that $(v)\Rightarrow(iv)$. Let $(s,a)\in S\times A$. Then

$$\tilde{u}_{s_0}(s,a)=u_{s_0}(a)=tu_{s_0}(t^{-1}a)=tu(s_0,t^{-1}a)=u(ts_0,a)$$

for any $t\in S$. If $t=s_0^{-1}s$ then $\tilde{u}_{s_0}(s,a)=u(s,a)$.

(2) Since (i), (ii), and (iv) are equivalent, $x\mapsto\tilde{x}$ is a mapping from

$$Hom_{S-Act}(A,B)$$

onto the set of all fixed points of $Hom_{S-Act}(S\times A,B)$. If $\tilde{x}_1=\tilde{x}_2$ then $x_1(a)=\tilde{x}_1(s,a)=\tilde{x}_2(s,a)=x_2(a)$ for any $a\in A$, i.e. $x_1=x_2$. Therefore, the mapping $x\mapsto\tilde{x}$ is a bijection. □

4. The Categories $SS-Act$ and $FS-Act$

Everywhere below, S is a monoid.

Definition 1. *(the definition of the category $SS-Act$) Define the category $SS-Act$ as follows:*

- $Ob(SS-Act)=Ob(S-Act)$*;*

- the set $Hom_{SS-Act}(A,B)$ of morphisms from an object A to an object B of $SS-Act$ coincides with the set $Hom_{S-Act}(S\times A,B)$ of morphisms from the object $S\times A$ to the object B of $S-Act$;

- the composition of morphisms $u\in Hom_{SS-Act}(A,B)$ and

$$v\in Hom_{SS-Act}(B,C)$$

is defined as follows:

$$(u\cdot v)(s,a)=v(s,u(s,a)),$$

where $s\in S$, $a\in A$;

- for each object A of the category $SS-Act$, the identity morphism $e_A\in Hom_{SS-Act}(A,A)$ is defined as follows: $e_A(s,a)=a$.

By (1.1) of Lemma 1, the category $SS-Act$ is thus well defined.

Definition 2. *(the functors $U_S: S-Act\longrightarrow SS-Act$ and $V_S: SS-Act\longrightarrow S-Act$) For objects, the functors are defined as follows:*

$$U_S(A)=A,\ \ V_S(B)=S\times B,$$

where $A\in Ob(S-Act)$ and $B\in Ob(SS-Act)$. For arrows, the functors are defined as follows:

$$U_S(x)=\tilde{x},\ \ V_S(u)=\bar{u},$$

where $x\in Hom_{S-Act}(A_1,A_2)$, $u\in Hom_{SS-Act}(B_1,B_2)$, i.e., $U_S(x)(s,a)=x(a)$ for any $s\in S$, $a\in A_1$ and $V_S(u)(s,b)=(s,u(s,b))$ for any $s\in S$, $b\in B_1$.

By (3.1) and (4.1) of Lemma 1, the functors U_S and V_S are well-defined.

Definition 3. *(the definition of the category $FS-Act$) The category $FS-Act$ is the subcategory of the category $SS-Act$ such that*

- $Ob(FS-Act)=Ob(S-Act)$

- the set $Hom_{FS-Act}(A,B)$ of morphisms from an object A to an object B of $FS-Act$ coincides with the set of all zeros (fixed points) of the S-act

$$Hom_{SS-Act}(A,B),$$

that is,

$$u\in Hom_{FS-Act}(A,B)\Longleftrightarrow u\in Hom_{S-Act}(S\times A,B)\ \ \text{and}\ \ tu=u\ for\ all\ t\in S.$$

By Lemma 3(1), the composition of fixed points is a fix point and all identity morphisms are fixed points in $SS-Act$. Thus, the category $FS-Act$ is well defined.

Theorem 1. *The category $S-Act$ is isomorphic to a reflective subcategory of $SS-Act$.*

More exactly, let $In=In^{FS-Act}_{SS-Act}: FS-Act\longrightarrow SS-Act$ be the inclusion functor. Then

(1) $U_S=In\circ U': S-Act\longrightarrow SS-Act$, where $U': S-Act\longrightarrow FS-Act$ is an isomorphism of categories;

(2)(a) the functor V_S is a left adjoint for U_S;

(b) the functor $V'=U'\circ V_S: SS-Act\longrightarrow FS-Act$ is a left adjoint for $In: FS-Act\longrightarrow SS-Act$.

Thus, the category $FS-Act$ is isomorphic to the category $S-Act$, and this category is a reflective subcategory of $SS-Act$.

Proof. (1) Define a functor $U' : S - Act \longrightarrow FS - Act$ as follows: $U'(A) = A$ for any $A \in Ob(S - Act)$, $U'(x) = \tilde{x}$ for any $x \in Hom_{S-Act}(A_1, A_2)$. By the definition of the functor U_S, we have $U_S(x) = \tilde{x}$, then $U'(x) = U_S(x)$. Since U_S is a functor, U' is also a functor. By Lemma 3(2), U' is bijective. Therefore, U' is an isomorphism of categories.

(2)(a) For any $A \in Ob(SS - Act)$ and $B \in Ob(S - Act)$, we will construct a bijective mapping $\varphi(A, B) : Hom_{S-Act}(V_S(A), B) \longrightarrow Hom_{SS-Act}(A, U_S(B))$ such that the diagram

$$\begin{array}{ccc}
Hom_{S-Act}(V_S(A), B) & \xrightarrow{\varphi(A,B)} & Hom_{SS-Act}(A, U_S(B)) \\
\Big\downarrow {\scriptstyle Hom_{S-Act}(V_S(f),g)} & & \Big\downarrow {\scriptstyle Hom_{SS-Act}(f,U_S(g))} \\
Hom_{S-Act}(V_S(A'), B') & \xrightarrow[\varphi(A',B')]{} & Hom_{SS-Act}(A', U_S(B'))
\end{array}$$

commutes for any $f \in Hom_{SS-Act}(A', A)$ and $g \in Hom_{S-Act}(B, B')$, i.e.

$$U_S(g) \cdot \varphi(A, B)(u) \cdot f = \varphi(A', B')(g \circ u \circ V_S(f))$$

for any $u \in Hom_{S-Act}(V_S(A), B)$. Since $V_S(A) = S \times A$ and $U_S(B) = B$, we have $Hom_{S-Act}(V_S(A), B) = Hom_{SS-Act}(A, U_S(B))$. Thus, we can take the identity mapping as $\varphi(A, B)$.

Note, that if $\varphi(A, B)$ is the identity mapping then the commutativity of the diagram takes the form

$$U_S(g) \cdot u \cdot f = g \circ u \circ V_S(f)$$

for any $u \in Hom_{S-Act}(V_S(A), B)$. Let us check this equality. Indeed,

$$(U_S(g) \cdot u \cdot f)(s, a') = \tilde{g}(s, u(s, f(s, a')) = g(u(s, f(s, a'))),$$

$$(g \circ u \circ V_S(f))(s, a') = g(u(\bar{f}(s, a'))) = g(u(s, f(s, a')))$$

for any $s \in S$, $a' \in A'$. Thus, the commutativity of the diagram is proved.

(2)(b) Since V_S is a left adjoint for U_S and U' is a left adjoint for U'^{-1}, by the general properties of the adjunction of functors, $V' = U' \circ V_S$ is a left adjoint for $U_S \circ U'^{-1} = In$. □

Note one more useful property of fixed points.

Lemma 4. *Let A, B, C, D be left S-acts and $w \in Hom_{FS-Act}(B, C)$. Then the mappings*

$$w_* : Hom_{SS-Act}(A, B) \longrightarrow Hom_{SS-Act}(A, C)$$

and

$$w^* : Hom_{SS-Act}(C, D) \longrightarrow Hom_{SS-Act}(B, D),$$

given by the formula

$$w_*(u) = w \cdot u, \; w^*(v) = v \cdot w,$$

where $u \in Hom_{SS-Act}(A, B)$, $v \in Hom_{SS-Act}(C, D)$, are morphisms of $S - Act$.

Proof. Suppose the fulfillment of the hypoptheses of the lemma. Let $t \in S$. Since $w \in Hom_{FS-Act}(B, C)$, by (2.2) of Lemma 1,

$$w_*(tu) = w \cdot (tu) = tw \cdot tu = t(w \cdot u) = tw_*(u);$$

$$w^*(tv) = tv \cdot w = tv \cdot tw = t(v \cdot w) = tw^*(v).$$

□

Let K be a category, $u \in Hom_K(k, l)$. By definition:
u is *a monomorphism* if $u \circ g = u \circ h$ implies $g = h$;
u is *an epimorphism* if $g \circ u = h \circ u$ implies $g = h$;
u is *an isomorphism* if there exists $u^{-1} \in Hom_K(l, k)$ such that $u^{-1} \circ u = 1_k$ and $u \circ u^{-1} = 1_l$, i.e. u is left and right invertible.

It follows from the definition of $SS - Act$ that the projection mapping $e_A : S \times A \longrightarrow A$ is an isomorphism, although, as a set mapping, it is neither injective nor surjective.

In this section, we give characterizations of monomorphisms, epimorphisms and isomorphisms in the category $SS - Act$. More complete results for the case when S is a group can be found in Section 9.

Theorem 2. *Let* $u \in Hom_{SS-Act}(A, B)$.
(1) Consider the following conditions:
(1m) u *is a monomorphism in the category* $SS - Act$*;*
(2m) u_s *is injective for any* $s \in S$*;*
(3m) u_1 *is injective.*
Then (2m) $\Rightarrow$ *(1m)* $\Rightarrow$ *(3m). Moreover, if* S *is a group, then all this conditions are equivalent.*
(2) The following conditions are equivalent:
(1e) u *is an epimorphism in the category* $SS - Act$*;*
(2e) u_s *is surjective for any* $s \in S$*;*
(3e) $\bar{u}$ *is surjective;*
(3'e) $\bar{u}$ *is an epimorphism in the category* $S - Act$*.*
(3) The following conditions are equivalent:
(1is) u *is an isomorphism in the category* $SS - Act$*;*
(2is) u_s *is bijective for any* $s \in S$*.*

Proof. Let us prove (1).

(2m)⇒(1m). Let $v, w \in Hom_{SS-Act}(C, A)$ and $u \cdot v = u \cdot w$. By (5.1) of Lemma 1, we have $(v \cdot u)_s = v_s \circ u_s$ for all $s \in S$. Since u_s is injective, the equality $u_s \circ v_s = u_s \circ w_s$ implies $v_s = w_s$ for all $s \in S$. Hence $v = w$. Thus, u is a monomorphism.

(1m)⇒(3m). Let $\{pt\}$ be $S-act$, $x \in A$, $f_x \in Hom_{SS-act}(\{pt\}, A)$ be a morphism such that $f_x(s, pt) = sx$. Then $(u \cdot f_x)(s, pt) = u(s, sx) = su(1, x) = su_1(x)$. If $u_1(a) = u_1(b)$ then $u \cdot f_a = u \cdot f_b$. Since u is a monomorphism, $f_a = f_b$. Hence $a = b$ and u_1 is an injective.

Let S be a group.

(3m)⇒(2m). Assume $u_s(a) = u_s(b)$. Then $u(s, a) = u(s, b)$. So $su(1, s^{-1}a) = su(1, s^{-1}b)$. Multiplying both sides of the equality from the left by s^{-1}, we obtain $u(1, s^{-1}a) = u(1, s^{-1}b)$. By hypothesis, u_1 is injective. Hence $s^{-1}a = s^{-1}b$. Thus, $a = b$ and u_s is injective.

Let us prove (2).

(3'e)$\Leftrightarrow$ (3e). It is known that any epimorphism in $S{-}Act$ is a surjective morphism. The opposite is obvious.

(1e)$\Leftrightarrow$ (3'e). By (3.2) of Lemma 1, $v \cdot u = v \circ \bar{u}$. Hence,

$$v \cdot u = w \cdot u \Leftrightarrow v \circ \bar{u} = w \circ \bar{u}.$$

If $\bar{u}$ is an epimorphism in $S - Act$ and $v \cdot u = w \cdot u$, we have $v = w$, i.e., u is an epimorphism in $SS - Act$. On the contrary, if u is an epimorphism in $SS - Act$ and $v \circ \bar{u} = w \circ \bar{u}$ then $v = w$, i.e., $\bar{u}$ is an epimorphism in $S - Act$.

(2e)$\Rightarrow$ (1e). Let $v \cdot u = w \cdot u$. By (5.1) of Lemma 1, we have $(v \cdot u)_s = v_s \circ u_s$ for any $s \in S$. Hence, the equality $v \cdot u = w \cdot u$ implies $v_s \circ u_s = w_s \circ u_s$ for any s. Since u_s is surjective, $v_s = w_s$ and $v = w$. Thus, u is an epimorphism in $SS - Act$.

(3e)$\Rightarrow$(2e). Let $b \in B$ and $s \in S$. Then $(s, b) \in S \times B$. Since $\bar{u}$ is surjective, there exists $(t, a) \in S \times A$ such that $\bar{u}(t, a) = (s, b)$. Hence, $(t, u(t, a)) = (s, b)$. Therefore, $t = s$ and $u_s(a) = u(s, a) = b$. Thus, u_s is surjective.

Let us prove (3).

(1is)$\Rightarrow$(2is). Let u be an isomorphism in $SS - Act$ and $s \in S$. Then there exists $v \in Hom_{SS-act}(B, A)$ such that $v \cdot u = e_A$ and $u \cdot v = e_B$. By (5.1) of Lemma 1, $v_s \circ u_s = 1_A$ and $u_s \circ v_s = 1_B$. Thus, u_s is bijective.

(2is)$\Rightarrow$(1is). Let u_s be bijective for any $s \in S$. For any u_s, there exists an inverse mapping $v_s : B \longrightarrow A$. Define a mapping $v : S \times B \longrightarrow A$ by $v(s, b) = v_s(b)$. By Lemma 2(3), v is a morphism of $S - Act$ and $v \cdot u = e_A$, $u \cdot v = e_B$. Thus, u is an isomorphism in $SS - Act$. □

Example *of a monoid S such that there exists a morphism h of $SS - Act$ with the following properties:*

(1) h is not a monomorphism, and the mapping h_1 is injective;

(2) h is not an epimorphism, and the mapping h_1 is surjective.

Let $S = \{1, a\}$ be a monoid, where 1 is the unit of the monoid and $a^2 = a$. Consider the morphism $h \in Hom_{SS-Act}(S, S)$ is defined as follows: $h(1, 1) = 1$ and $h(u, v) = a$ for all pairs (u, v) except for $(1, 1)$. It follows from the definition of the morphism h that the mapping h_1 is bijective. Define morphisms $f_1, f_2 \in Hom_{SS-Act}(S, S)$ by the equalities $f_1(1, u) = f_2(1, u) = f_2(a, 1) = 1$, $f_1(a, u) = f_2(a, a) = a$ for any $u \in S$. It is clear that f_1, f_2 are different morphisms.

Show that $h \cdot f_1 = h \cdot f_2$. Indeed, $(h \cdot f_i)(1, u) = h(1, f_i(1, u)) = h(1, 1) = 1$ and $(h \cdot f_i)(a, u) = h(a, f_i(a, u)) = a$ for any $i \in \{1, 2\}$, $u \in S$, i.e. $h \cdot f_1 = h \cdot f_2$. Hence, h is not a monomorphism.

Show that $f_1 \cdot h = f_2 \cdot h$. Indeed, $(f_i \cdot h)(1, u) = f_i(1, h(1, u)) = 1$ and $(f_i \cdot h)(a, u)) = f_i(a, h(a, u)) = f_i(a, a) = a$ for any $i \in \{1, 2\}$, $u \in S$, i.e. $f_1 \cdot h = f_2 \cdot h$. Hence, h is not an epimorphism.

5. Products and Coproducts in the Category $SS - Act$

There are different equivalent definitions of limits and colimits, as well as of their special cases, products, coproducts, equalizers, co-equalizers, pullback squares. We will follow [2].

Let K, L be categories, $c \in Ob(L)$, Δc be a constant functor which has the value c at each object $k \in Ob(K)$ and the value l_c at each arrow of K.

Let $F : K \to L$ be a functor.

A cone to the base F from the vertex c is a natural transformation of functors $\varphi : \Delta c \to F$, that is, a family of morphisms $\varphi = \{\varphi(k) \in Hom_L(c, F(k)) \mid k \in Ob(K)\}$

such that $\varphi(k_2) = F(f) \circ \varphi(k_1)$ for any morphism $f \in Hom_K(k_1, k_2)$. A cone is called a *limiting cone* or a *universal cone to* F if for any cone $\varphi' : \Delta l \to F$ to the base F from the vertex l there is unique morphism $x \in Hom_L(l, c)$ such that $\varphi'(k) = \varphi(k) \circ x$. The vertex c of the universal cone to F is called the *limit object (the inverse limit or the projective limit) of the functor* F. In this case, we write $c = limF$.

A cone from the base F *to the vertex* c is a natural transformation of functors $\psi : F \to \Delta c$, that is, a family of morphisms $\psi = \{\psi(k) \in Hom_L(F(k), c) \mid k \in Ob(K)\}$ such that $\psi(k_2) \circ F(f) = \varphi(k_1)$ for any morphism $f \in Hom_K(k_1, k_2)$. A cone is called a *colimiting cone* or a *universal cone from* F if for any cone $\psi' : F \to \Delta l$ from the base F to the vertex l there is a unique morphism $x \in Hom_L(c, l)$ such that $\psi'(k) = x \circ \psi(k)$. The vertex c of a universal cone from F is called the *colimit object (the direct limit or the inductive limit) of the functor* F. In this case, we write $c = colimF$.

Let K be a discrete category. Then a functor F is a family $\{c_k = F(k) \mid k \in Ob(K)\}$ of objects of the category L. In this case, a universal cone $p = \{p(k) \in Hom_L(c, c_k) \mid k \in Ob(K)\}$ to F is called a *product cone*, the object c is called *the product of a family* $\{c_k \mid k \in K\}$, and the morphism $p_k = p(k) \in Hom_L(c, c_k)$ is called a *projection*, or a *k-th factor projection of the product (k-th projection).* A universal cone $i = \{i(k) \in Hom_L(c_k, c) \mid k \in Ob(K)\}$ from F is called a *coproduct cone*, the object c is called the *coproduct of the family* $\{c_k \mid k \in K\}$, and the morphism $i_k = i(k) \in Hom_L(c, c_k)$ is called the *injection* or the k-th embedding injection of the coproduct (the k-th injection). The product of objects c_k is denoted by $c = \prod_{k\in K} c_k$. The coproduct of objects c_k is denoted by $c = \coprod_{k\in K} c_k$. Note that, for a family $\{f_k \in Hom_L(b_k, c_k) \mid k \in K\}$ of morphisms and $b = \prod_{k\in K} b_k$, there is a unique morphism $f \in Hom_L(b, c)$ such that $p^c_k \circ f = f_k \circ p^b_k$. This morphism f is denoted by $\prod_{k\in K} f_k$.

Let $f, g \in Hom_L(a, b)$. An *equalizer of the pair* f, g is a morphism $e \in Hom_L(d, a)$ such that

(i) $f \circ e = g \circ e$;

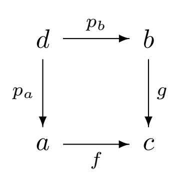
(ii) if $h \in Hom_L(c, a)$ is such that $f \circ h = g \circ h$ then $h = e \circ h'$ for a unique morphism $h' \in Hom_L(c, d)$.

A *coequalizer of the pair* f, g is a morphism $u \in Hom_L(b, e)$ such that

(i) $u \circ f = u \circ g$;

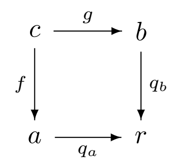
(ii) if $h \in Hom_L(b, c)$ is such that $h \circ f = h \circ g$ then $h = h' \circ u$ for a unique morphism $h' \in Hom_L(e, c)$.

A *pullback square* or a *cartesian square* is a diagram of the form



if it commutes and for any pair (p'_a, p'_b) with $p'_a \in Hom_L(d', a)$, $p'_b \in Hom_L(d', b)$, and $f \circ p'_a = g \circ p'_b$, there is unique morphism $h \in Hom_L(d', d)$ such that $p_a \circ h = p'_a$ and $p_b \circ h = p'_b$. Denote by $a \times_c b$ the vertex d of the cartesian square. The vertex $a \times_c b$ is called a *pullback*, a *fibered product*, or a *product, over the object* c. The morphisms $p_a \in Hom_L(a \times_c b, a)$ and $p_b \in Hom_L(a \times_c b, b)$ are called the *projections of the pullback.*

A *cocartesian square*, or a *pushout of* $< f, g >$ is a diagram of the form



that commutes and is such that for any pair (q'_a, q'_b) with $q'_a \in Hom_L(a, r')$, $q'_b \in Hom_L(b, r')$, and $q'_a \circ f = q'_b \circ g$, there is unique morphism $h \in Hom_L(r, r')$ such that $h \circ q_a = q'_a$ and $h \circ q_b = q'_b$. The vertex r of the pushout is called a *fibered sum*, or a *coproduct, over the object* c. The notation $r = a \sqcup_c b$ is often used.

Since there is a pair of adjoint functors $U_S : S - Act \to SS - Act$, $V_S : SS - Act \to S - Act$, we can make some conclusions about the limits and colimits in the category $SS - Act$. Recall that, given $f \in Hom_{S-Act}(A, B)$ and $u \in Hom_{SS-Act}(A, B)$, we have $U_S(f) = \tilde{f} \in Hom_{SS-Act}(A, B)$ and $V_S(u) = \bar{u} \in Hom_{S-Act}(S \times A, S \times B)$, where $\tilde{f}(s, a) = f(a)$ and $\bar{u}(s, a) = (s, u(s, a))$.

Theorem 3. *(1) Let $F : K \to S - Act$ be a functor and let $U_S \circ F : K \to SS - Act$ be the superposition of functors, i.e., the functor such that $(U_S \circ F)(k) = F(k)$ and $(U_S \circ F)(f) = \widetilde{F(f)}$ for any object k and any morphism f of the category K.*

(a) If $p_F = \{p_k \in Hom_{S-Act}(C, F(k)) \mid k \in Ob(K)\}$ is a limiting cone to F then $U_S(p_F) = \{\widetilde{p_k} \in Hom_{SS-Act}(C, F(k)) \mid k \in Ob(K)\}$ is a limiting cone to $U_S \circ F$, that is, $lim\, U_S \circ F = limF = C$.

In particular,

(b) if K is a set and $p = \{p_k : C \to C_k \mid k \in K\}$ is a product cone in $S - Act$ then $U_S(p) = \{\widetilde{p_k} \in Hom_{SS-Act}(C, C_k) \mid k \in K\}$ is a product cone in $SS - Act$. Thus, the product of a family of S-acts in $S - Act$ coincides with the product of this family in $SS - Act$.

(2) Let $G : K \to SS - Act$ be a functor and let $V_S \circ G : K \to S - Act$ be the superposition of functors, i.e., the functor such that $(V_S \circ G)(k) = S \times G(k)$, $(V_S \circ G)(f) = \overline{G(f)}$.

(a) If $i_G = \{i_k \in Hom_{SS-Act}(G(k), C) \mid k \in Ob(K)\}$ is a cone from G such that $V_S(i_G) = \{\overline{i_k} \in Hom_{S-Act}(S \times G(k), S \times C) \mid k \in Ob(K)\}$ is not a colimiting cone from $V_S \circ G$ then i_G is not a colimiting cone from F. Thus, if $S \times C$ is not a colimit of functor $V_S \circ G$ then C is not a colimit of G.

In particular,

(b) if $u, v \in Hom_{SS-Act}(A, B)$, $w \in Hom_{SS-Act}(B, C)$, $w \cdot u = w \cdot v$, and $\bar{w} \in Hom_{S-Act}(S \times B, S \times C)$ is not a coequalizer of the pair $\bar{u}$, $\bar{u} : S \times A \to S \times B$ in $S - Act$ then w is not a coequalizer of the pair u, v.

Proof. It is known (see [2]) that if a functor has a left adjoint then it preserves limits, that is, this functor maps every limiting cone to a limiting cone. If a functor has a right adjoint then it preserves colimits, that is, this functor maps every colimiting cone to a colimiting cone. By Theorem 1, the functor V_S is a left adjoint for U_S. Since a product is a limit, we have (1). Since the functor U_S is a right adjoint for V_S and a coequalizer is a colimit, we have (2). □

This theorem allows us to conclude both the presence of certain types of limits and the absence of colimits in the category $SS - Act$.

Theorem 4 (the product completeness and coproduct completeness of the category $SS-Act$)**.** *Any family $\{C_k \mid k \in K\}$ of S-acts admists a product and a coproduct in the category $SS-Act$. More precisely,*

(1) if $\prod_{k\in K} C_k$ is a Cartesian product of S-acts, $p_m \in Hom_{S-Act}(\prod_{k\in K} C_k, C_m)$ is the projection onto the m-th factor of the product then

$$\{p_m \in Hom_{S-Act}(\prod_{k\in K} C_k, C_m) \mid m \in K\}$$

is the product cone in $S-Act$ and

$$\{\tilde{p}_m \in Hom_{SS-Act}(\prod_{k\in K} C_k, C_m) \mid m \in K\}$$

is the product cone of the family $\{C_k \mid k \in K\}$ in $SS-Act$, i.e., $\prod_{k\in K} C_k$ is the product of the family $\{C_k \mid k \in K\}$ in $SS-Act$;

(2) if $\coprod_{k\in K} C_k$ is a disjoint union of S-acts and $i_m \in Hom_{S-Act}(C_m, \coprod_{k\in K} C_k)$ is the embedding then

$$\{i_m \in Hom_{S-Act}(C_m, \coprod_{k\in K} C_k) \mid m \in K\}$$

is the coproduct cone in $S-Act$ and

$$\{\tilde{i}_m \in Hom_{SS-Act}(C_m \to \coprod_{k\in K} C_k) \mid m \in K\}$$

is the coproduct cone of the family $\{C_k \mid k \in K\}$ in $SS-Act$, i.e., $\coprod_{k\in K} C_k$ is the coproduct of the family $\{C_k \mid k \in K\}$ in $SS-Act$.

Proof. (1) It is well known and easily verified directly that

$$\{p_m \in Hom_{SS-Act}(\prod_{k\in K} C_k, C_m) \mid m \in K\}$$

is the product cone in the category $S-Act$. Therefore, by item (1)(b) of Theorem 3, we get (1).

(2) It is well known and easily verified directly that

$$\{i_m : C_m \to \coprod_{k\in K} C_k \mid m \in K\}$$

is the coproduct cone in the category $S-Act$.

Let us prove that $\coprod_{k\in K} C_k$ is the coproduct of the family $\{C_k \mid k \in K\}$ in $SS-Act$. Define $m_i \in Hom_{SS-Act}(C_i, \coprod\limits_{i\in I} C_i)$ as follows: $m_i(s, c_i)$=c_i for all $c_i \in C_i$ and $i \in I$.

We will prove that $(\coprod\limits_{i\in I} C_i, (m_i)_{i\in I})$ is the coproduct of the S-acts C_i, $i \in I$, in $SS-Act$. Let D be an S-act and let $u_i \in Hom_{SS-Act}(C_i, D)$, $i \in I$. Consider the mapping $h : S \times \coprod\limits_{i\in I} C_i \longrightarrow D$: $h(s, c_i) = u_i(s, c_i)$ for any $s \in S$, $c_i \in C_i$, $i \in I$. The equalities

$$h(ts, tc_i) = u_i(ts, tc_i) = tu_i(s, c_i) = th(s, c_i)$$

imply $h \in Hom_{SS-Act}(\coprod\limits_{i\in I} C_i, D)$. Since $h \cdot m_i(s, c_i) = h(s, m_i(s, c_i)) = h(s, c_i) = u_i(s, c_i)$ for any $s \in S$, $c_i \in A_i$, we have $h \cdot m_i = u_i$ for all $i \in I$. Let us show that a morphism h such that $h \cdot m_i = u_i$ is unique. Suppose that $h' \in$

$Hom_{SS-Act}(\coprod_{i\in I} C_i, D)$ and $h'\cdot m_i = u_i$ for any $i\in I$. Prove that $h=h'$, i.e. $h'(s,c_i)=h(s,c_i)$ for all $s\in S$, $c_i\in C_i$ and $i\in I$. Indeed, for $a_i\in C_i\subseteq \coprod_{i\in I} C_i$ we have

$$h(s,c_i)=h(s,m_i(s,c_i))=u_i(s,c_i)=h'(s,m_i(s,c_i))=h'_i(s,c_i),$$

i.e. $h=h'$. Thus, h is unique and $\coprod_{k\in K} C_k$ is the coproduct of the family $\{C_k \mid k\in K\}$ in $SS-Act$. □

6. $SS-Act$ is Cartesian Closed

Let K be a category. Consider a functor $H_K : K^o\times K \longrightarrow Set$. For objects, the functor is defined as follows: $H_K(k,l)$ is the set of all morphisms from an object k into an object l in K. For arrows the functor is defined as follows: for any $f\in Hom_{K^o}(k,k')$ and $g\in Hom_K(l,l')$,

$$H_K(f,g)(u)=g\circ u\circ f,$$

where $u\in H_K(k,l)$.

Let n, k be any pair of objects of K. Fix a product (p_n,p_k), where $p_n=p_n^{n,k}\in Hom_K(n\times k,n)$, $p_k=p_k^{n,k}\in Hom_K(n\times k,k)$. Then for any pair of morphisms $h\in Hom_K(n',n)$ and $f\in Hom_K(k',k)$ there exists a unique morphism $h\times f\in Hom_K(n'\times k', n\times k)$ such that

$$p_n\circ(h\times f)=h\circ p_{n'},\quad p_k\circ(h\times f)=f\circ p_{k'}.$$

Thus the correspondence $(n,k)\mapsto n\times k$, $(h,f)\mapsto h\times f$ defines a functor $K^o\times K\longrightarrow K$. This definition is correct because $(h\circ h')\times(f\circ f')=(h\times f)\circ(h'\times f')$ for any $h'\in Hom_K(n'',n')$, $f'\in Hom_K(k'',k')$.

Therefore, we can define the functor $H_KT : K^o\times K^o\times K\longrightarrow Set$. For objects, the functor is defined as follows:

$$H_KT(n,k,l)=H_K(n\times k,l)=Hom_K(n\times k,l),$$

where $n,k\in Ob(K^o)$, $l\in Ob(K)$. For arrows, the functor is defined as follows:

$$H_KT(h,f,g)(u)=g\circ u\circ(h\times f),$$

where $h\in Hom_K(n',n)$, $f\in Hom_K(k',k)$, $g\in Hom_K(l,l')$, $u\in Hom_K(n\times k,l)$.

Let $k,l\in Ob(K)$. Suppose that there are an object $\mathcal{H}_K(k,l)\in Ob(K)$ and an isomorphism of functors from K^o into Set:

$$\{p_K(n,k,l): Hom_K(n\times k,l)\longrightarrow Hom_K(n,\mathcal{H}_K(k,l))\mid n\in Ob(K)\}. \tag{1}$$

Then the object $\mathcal{H}_K(k,l)\in Ob(K)$ is called *internal hom*. From category-theoretic facts it follows that if for every pair of objects $k,l\in Ob(K)$ there are an object $\mathcal{H}_K(k,l)$ and an isomorphism of functors (1) then the mapping $(k,l)\mapsto\mathcal{H}_K(k,l)$ can be uniquely extended to a functor $\mathcal{H}_K : K^o\times K\longrightarrow K$ such that

$$\mathcal{H}_K(f,g)\circ p_K(n,k,l))(u)\circ h=p_K(n',k',l')(g\circ u\circ(h\times f)) : n'\longrightarrow\mathcal{H}_K(k',l')$$

for any $f \in Hom_K(k', k)$, $g \in Hom_K(l, l')$, $h \in Hom_K(n', n)$ and $u \in Hom_K(n \times k, l)$, i.e., the following diagram commutes:

$$\begin{array}{ccc} Hom_K(n \times k, l) & \xrightarrow{p_K(n,k,l)} & Hom_K(n, \mathcal{H}_K(k,l)) \\ \downarrow{\scriptstyle H_K(h,f,g)} & & \downarrow{\scriptstyle H_K(h,\mathcal{H}_K(f,g))} \\ Hom_K(n' \times k', l') & \xrightarrow[p_K(n',k',l')]{} & Hom_K(n', \mathcal{H}_K(k', l')) \end{array}$$

The functor $\mathcal{H}_K$ is also called the internal *hom*.

A category with finite products and a functor $\mathcal{H}_K$ is called *Cartesian closed*. This category is a monoidal category with respect to the product, and the Cartesian closure of this category is equivalent to its closure as a monoidal category.

It is known (see [3]) that the category $S - Act$ is a topos and hence is Cartesian closed.

Let us write a functor $\mathcal{H}_S = \mathcal{H}_{S-Act} : (S - Act)^o \times (S - Act) \longrightarrow S - Act$. For objects, the functor is defined as follows:

$$\mathcal{H}_S(A, B) = Hom_{S-Act}(S \times A, B),$$

where $A, B \in Ob(S - Act)$. For arrows, the functor is defined as follows:

$$\mathcal{H}_S(f, g)(w) = g \circ w \circ (1_S \times f) : S \times A' \longrightarrow B',$$

where $f \in Hom_{S-Act}(A', A)$, $g \in Hom_{S-Act}(B, B')$, $w \in Hom_{S-Act}(S \times A, B)$.

Define a mapping

$$p_S(U, A, B) : Hom_{S-Act}(U \times A, B) \longrightarrow Hom_{S-Act}(U, Hom_{S-Act}(S \times A, B))$$

as follows

$$(p_S(U, A, B)(u)(d))(t, a) = u(td, a),$$

where $u \in Hom_{S-Act}(U \times A, B)$, $d \in U$, $a \in A$. Since u is a morphism of S-acts then $p_S(U, A, B)(u)(d)$ is a morphism of S-acts. Since

$$(p_S(U, A, B)(u)(td))(s, a) = u(std, a) =$$

$$= (p_S(U, A, B)(u)(d))(st, a) = (tp_S(U, A, B)(u)(d))(s, a),$$

where $s, t \in S$, $d \in U$, $a \in A$, then $p_S(U, A, B)(u)$ is a morphism of S-acts. Let us write the mapping

$$q_S(U, A, B) : Hom_{S-Act}(U, Hom_{S-Act}(S \times A, B)) \longrightarrow Hom_{S-Act}(U \times A, B),$$

the inverse to $p_S(U, A, B)$, as follows

$$q_S(U, A, B)(v)(d, a) = v(d)(1, a),$$

where $v \in Hom_{S-Act}(U, Hom_{S-Act}(S \times A, B))$, $d \in U$, $a \in A$.

Thus, the functor $\mathcal{H}_S$ is internal *hom*.

Theorem 5. *The category $SS - Act$ is Cartesian closed.*

Proof. The product of objects U, S in the category $SS-Act$ is the Cartesian product $U \times A$ of this objects in the category $S - Act$, and the canonical projections are the morphisms

$$p_U = p_U^{U,A} \in Hom_{SS-Act}(U \times A, U), \quad p_A = p_A^{U,A} \in Hom_{SS-Act}(U \times A, A)$$

such that

$$p_U(s,d,a) = d, \quad p_A(s,d,a) = a,$$

where $s \in S$, $d \in U$, $a \in A$. Moreover, for

$$f \in Hom_{SS-Act}(A', A), \quad g \in Hom_{SS-Act}(B, B')$$

the morphism $h \times f \in Hom_{SS-Act}(U' \times A', U \times A)$ such that

$$p_U \cdot (h \times f) = h \cdot p_{U'}, \quad p_A \cdot (h \times f) = f \cdot p_{A'}$$

is given by the equality

$$(h \times f)(s, d', a') = (h(s, d'), f(s, a')),$$

where $s \in S$, $d' \in U'$, $a' \in A'$.

Define a functor

$$\mathcal{H}_{SS} = \mathcal{H}_{SS-Act} : (SS - Act)^o \times (SS - Act) \longrightarrow SS - Act.$$

For objects, the functor is defined by

$$\mathcal{H}_{SS}(A, B) = Hom_{S-Act}(S \times A, B),$$

where $A, B \in Ob(SS - Act)$. For arrows, the functor

$$\mathcal{H}_{SS}(f, g) : S \times Hom_{S-Act}(S \times A, B) \longrightarrow Hom_{S-Act}(S \times A', B')$$

is defined by

$$\mathcal{H}_{SS}(f, g)(s, w) = (sg) \cdot w \cdot (sf),$$

where

$$(s, w) \in S \times Hom_{S-Act}(S \times A, B), \quad f \in Hom_{SS-Act}(A', A), \quad g \in Hom_{SS-Act}(B, B').$$

Since

$$\mathcal{H}_{SS}(f, g)(ts, tw) = (tsg) \cdot tw \cdot (tsf) = t((sg) \cdot w \cdot (sf)) = t\mathcal{H}_{SS}(f, g)(s, w),$$

where $s, t \in S$, it follows that $\mathcal{H}_{SS}(f, g)$ is a morphism of S-acts.

Let us show that $\mathcal{H}_{SS}$ is a functor. It suffices to prove two equalities

$$\mathcal{H}_{SS}(e_A, e_B) = e_{Hom_{SS-act}(A,B)},$$

$$\mathcal{H}_{SS}(f \cdot f', g' \cdot g) = \mathcal{H}_{SS}(f', g') \circ \mathcal{H}_{SS}(f, g),$$

where $f' \in Hom_{SS-Act}(A'', A')$, $g' \in Hom_{SS-Act}(B', B'')$. By (1.1) of Lemma 1, the equalities

$$\mathcal{H}_{SS}(e_A, e_B)(s, w) = (se_A) \cdot w \cdot (se_B) = (e_A) \cdot w \cdot (e_B) = w = e_{Hom_{SS-act}(A,B)}(s, w),$$

where $s \in S$, imply $\mathcal{H}_{SS}(e_A, e_B) = e_{Hom_{SS-act}(A,B)}$. Check the second equality:

$$\mathcal{H}_{SS}(f \cdot f', g' \cdot g)(s, w) = s(g' \cdot g) \cdot w \cdot s(f \cdot f') = sg' \cdot sg \cdot w \cdot sf \cdot sf' =$$
$$= sg' \cdot (sg \cdot w \cdot sf) \cdot sf' = sg' \cdot \mathcal{H}_{SS}(f, g)(s, w) \cdot sf' =$$
$$= \mathcal{H}_{SS}(f', g')(\mathcal{H}_{SS}(f, g)(s, w)) = (\mathcal{H}_{SS}(f', g') \circ \mathcal{H}_{SS}(f, g))(s, w),$$

where $s \in S$, imply $\mathcal{H}_{SS}(f \cdot f', g' \cdot g) = \mathcal{H}_{SS}(f', g') \circ \mathcal{H}_{SS}(f, g)$. Thus, $\mathcal{H}_{SS}$ is a functor.

Let us prove that this functor is an internal functor. Define $p_{SS-Act}(U, A, B) : Hom_{SS-Act}(U \times A, B) \longrightarrow Hom_{SS-Act}(U, \mathcal{H}_{SS}(A, B))$ as

$$p_{SS-Act}(U, A, B) = p_{SS}(U, A, B) = p_S(S \times U, A, B).$$

By the definition of p_{SS-Act}, if $u \in Hom_{SS-Act}(U \times A, B)$ then

$$p_{SS}(U, A, B)(u)(s, d)(t, a) = p_S(S \times U, A, B)(u)(s, d)(t, a) = u(ts, td, a),$$

where $s,t \in S$, $a \in A$, $d \in U$. Since p_S is bijective and hence so is $p_S S$, it suffices to prove the commutativity of the diagram

$$\begin{array}{ccc} Hom_{SS-Act}(U \times A, B) & \xrightarrow{p_{SS}(U,A,B)} & Hom_{SS-Act}(U, \mathcal{H}_{SS}(A,B)) \\ \Big\downarrow {\scriptstyle H_{SS}(h,f,g)} & & \Big\downarrow {\scriptstyle H_{SS}(h,\mathcal{H}_{SS}(f,g))} \\ Hom_{SS-Act}(U' \times A', B') & \xrightarrow{p_{SS}(U',A',B')} & Hom_{SS-Act}(U', \mathcal{H}_{SS}(A',B')) \end{array}$$

i.e.

$$\mathcal{H}_{SS}(f,g) \cdot p_{SS}(U,A,B)(u) \cdot h = p_{SS}(U',A',B')(g \cdot u \cdot (h \times f))$$

for any $u \in Hom_{SS-Act}(U \times A, B)$. That is, we need to show that

$$(sg) \cdot (p_{SS}(U,A,B)(u))(h(s,d')) \cdot (sf) = (p_{SS}(U',A',B')(g \cdot u \cdot (h \times f)))(s,d')$$

for all $(s,d') \in S \times U'$, $u \in Hom_{SS-Act}(U \times A, B)$. Both sides of this equality are mappings $S \times A' \to B'$. Apply the left-hand side of the equality to an arbitrary element $(t,a') \in S \times A'$:

$$((sg) \cdot (p_{SS}(U,A,B)(u)(h(s,d')) \cdot (sf))(t,a') =$$

$$= (sg)(t, (p_S(S \times U, A, B)(u)(s, h(s,d'))(t, (sf)(t,a'))) = g(ts, u(ts, th(s,d'), f(ts,a')));$$

then apply the right-hand side of the equality to $(t,a') \in S \times A'$:

$$(p_{SS}(U',A',B')(g \cdot u \cdot (h \times f)))(s,d'))(t,a') = (g \cdot u \cdot (h \times f)(ts, td', a') =$$

$$= g(ts, u(ts, (h \times f)(ts, td', a') = g(ts, u(ts, h(ts,td'), f(ts,a'))).$$

Therefore, the diagram commutes. Thus, $\mathcal{H}_{SS}$ is an internal functor. □

7. Incompleteness of the Category $SS - Act$

In this section, we prove that the category $SS-Act$ is incomplete. By Theorem 4, $SS - Act$ is product complete and coproduct complete. But there exist morphisms in this category for which there are no equalizers and coequalizers.

Proposition 1. *The category $SS - Act$ is not complete and not cocomplete for some monoid S.*

Proof. Let $S = \{1, a\}$ be the monoid in which 1 is the unit of the monoid and $a^2 = a$.

Prove that the category $SS - Act$ is not complete for this monoid. Consider the morphisms $f_1, f_2 \in Hom_{SS-Act}(S,S)$ defined by the equalities $f_1(1,u) = f_1(a,1) = 1$, $f_1(a,a) = a$, $f_2(1,u) = f_2(a,a) = a$, $f_2(a,1) = 1$ for any $u \in S$. Demonstrate that there is no equalizer for f_1 and f_2. Suppose that $(E,e) = Eq(f_1,f_2)$. Then $e \in Hom_{SS-Act}(E,S)$ and $f_1 \cdot e = f_2 \cdot e$. Therefore,

$$1 = f_1(1, e(1,v)) = (f_1 \cdot e)(1,v) = (f_2 \cdot e)(1,v) = f_2(1, e(1,v)) = a$$

for any $u \in E$. A contradiction.

We now prove that $SS-Act$ is not cocomplete. Consider the morphisms $f_1, f_2 \in Hom_{SS-Act}(S,S)$ defined by the equalities $f_1(1,u) = f_2(1,u) = f_2(a,1) = 1$, $f_1(a,u) = f_2(a,a) = a$ for any $u \in S$. Show that there is no coequalizer for f_1 and f_2. Suppose that $(Q,h) = Coeq(f_1,f_2)$. Then $h \in Hom_{SS-Act}(S,Q)$ and $h \cdot f_1 = h \cdot f_2$.

Since h is a coequalizer, h is an epimorphism. So $Q = h_1(S) = \{h(1,1), h(1,a)\} = Q$. Let $h(1,1) = q_1$ and $h(1,a) = q_2$. Then $aq_1 = aq_2 = q_i$ for some $i \in \{1,2\}$. We may assume that $q_2 = aq_2$ (if $q_1 = aq_1$ then the argument is similar). So $q_2 = aq_2 = ah(1,a) = h(a,a)$. Since $h(a,1) = h(a, f_2(a,1)) = h(a, f_1(a,1)) = h(a,a)$, we have $h(a,1) = h(a,a) = q_2$.

Consider an S-act $D = \{d_1, d_2, d\}$, where $ad_1 = ad_2 = ad = d$, and the morphism $g \in Hom_{SS-Act}(S, D)$ such that $g(1,1) = d_1$, $g(1,a) = d_2$, and $g(a,1) = g(a,a) = d$. Note that $g \cdot f_1 = g \cdot f_2$. Indeed,

$$(g \cdot f_1)(a,1) = g(a, f_1(a,1)) = g(a,a) = g(a,1) = g(a, f_2(a,1)) = (g \cdot f_2)(a,1),$$

$$(g \cdot f_1)(a,a) = g(a, f_1(a,a)) = g(a, f_2(a,a)) = (g \cdot f_2)(a,a)$$

and

$$(g \cdot f_1)(1,u) = g(1, f_1(1,u)) = g(1, f_2(1,u)) = (g \cdot f_2)(1,u)$$

for any $u \in S$. Since (Q,h) is a coequalizer for f_1 and f_2, there exists $g' \in Hom_{SS-Act}(Q, D)$ such that $g = g' \cdot h$. Therefore,

$$g'(1,q_1) = g'(1, h(1,1)) = (g' \cdot h)(1,1) = g(1,1) = d_1,$$

$$g'(1,q_2) = g'(1, h(1,a)) = (g' \cdot h)(1,a) = g(1,a) = d_2,$$

$$g'(a,q_2) = g'(a, h(a,1)) = (g' \cdot h)(a,1) = g(a,1) = d.$$

We may assume that $g'(a,q_1) \neq d_1$ (if $g'(a,q_1) = d_1$ then d_1 is replaced by d_2 in the argument below). Consider the mapping $g'' : S \times Q \longrightarrow D$ defined as follows: $g''(a,q_1) = d_1$ and $g''(u,v) = g'(u,v)$ for any $(u,v) \neq (a,q_1)$. Show that g is well defined. Indeed, $g''(au, aq) = g''(a,q_2) = g'(a,q_2) = d = ag''(u,q)$ for any $u \in S$ and $q \in Q$. So $g'' \in Hom_{SS-Act}(Q, D)$. Show that $g = g'' \cdot h$. Since $(1, h(1,u)) \neq (a, q_1)$ for $u \in S$, we have $g''(1, h(1,u)) = g'(1, h(1,u)) = g(1,u)$; since $h(a,u) = q_2 \neq q_1$ for $u \in S$m we have $g''(a, h(a,u)) = g'(a, h(a,u)) = g(a,u)$. Thus, by definition of a coequalizer, $g' = g''$. A contradiction. □

8. The Category $SetS - Act$

In this section, we consider the category $SetS - Act$. As explained below, on the one hand, this category is equivalent to the category of sets; on the other hand, it contains information about the categories $S - Act$ and $SS - Act$. This information can be obtained by using the functor $Q_r : SS - Act \longrightarrow SetS - Act$. In particular, Q_r defines an isomorphism from the category $FS - Act$ to the category $S - Act$ for all $r \in S$.

Define *the category* $SetS - Act$ as follows:

- $Ob(SetS - Act) = Ob(S - Act)$;
- $Hom_{SetS-Act}(A, B)$ is the set of all mapping from the set A into the set B;
- the composition of morphisms is defined as the superposition of mappings;
- the identity morphism is the identity mapping.

It is clear that the category $SetS - Act$ contains $S - Act$ as a subcategory and is closely related to $SS - Act$. The main properties of $SetS - Act$, which are not directly related to the properties of the monoid S, follow from the equivalence of this category and the category Set. We define explicitly the equivalence and the adjoint mapping.

Consider the following *functor* $Ac : Set \longrightarrow SetS - Act$. For objects, $Ac(A)$ is the S-act A with the trivial action of the monoid S, i.e., $as = a$ for any $a \in A$, where $A \in Ob(Set)$; for arrows, $Ac(x)=x$, where $x \in Hom_{Set}(A_1, A_2)$. Clearly, $Ac(x) \in Hom_{SetS-Act}(Ac(A_1), Ac(A_2))$.

In the following lemma, $I : SetS - Act \longrightarrow Set$ stads for the forgetful functor.

Lemma 5. *(equivalence of categories $SetS - Act$ and Set). The functors I and Ac are equivalences of the categories $SetS - Act$ and Set. The functor I is a left and right adjoint for Ac, and $I \circ Ac = 1_{Set} : Set \to Set$ is the identity functor.*

Proof. It is clear that I and Ac are full and faithful functors. The equality $I \circ Ac = 1_{Set}$ is obvious too. Since every set is endowed with the structure of an S-act, any object A of Set is isomorphic to $I(A)$. Hence, I is an equivalence. Similarly, Ac is an equivalence.

Prove that Ac is left adjoint to I. Let $A, A' \in Ob(Set)$, $B, B' \in Ob(SetS - Act)$, $f \in Hom_{Set}(A', A)$, $g \in Hom_{SetS-Act}(B, B')$. Consider the diagram

$$\begin{array}{ccc} Hom_{SetS-Act}(Ac(A), B) & \xrightarrow{\alpha(A,B)} & Hom_{Set}(A, I(B)) \\ \Big\downarrow {\scriptstyle Hom_{SetS-Act}(Ac(f),g)} & & \Big\downarrow {\scriptstyle Hom_{Set}(f,I(g))} \\ Hom_{SetS-Act}(Ac(A'), B') & \xrightarrow{\alpha(A',B')} & Hom_{Set}(A', I(B')) \end{array}$$

where $\alpha(A, B)$ is the identity mapping. Since

$$g \circ u \circ Ac(f) = I(g) \circ u \circ f$$

for any $u : Ac(A) \longrightarrow B$, the diagram commutes. Thus, Ac is left adjoint to I.

Similarly, the following diagram commutes:

$$\begin{array}{ccc} Hom_{Set}(I(B'), A') & \xrightarrow{\beta(B',A'))} & Hom_{SetS-Act}(B', Ac(A')) \\ \Big\downarrow {\scriptstyle Hom_{S-Act}(I(g),f)} & & \Big\downarrow {\scriptstyle Hom_{SS-Act}(g,Ac(f))} \\ Hom_{Set}(I(B), A) & \xrightarrow{\beta(B,A)} & Hom_{SetS-Act}(B, Ac(A)) \end{array}$$

where $\beta(B, A)$ is the identity. Therefore, Ac is right adjoint to I. □

The following propositions are consequences of Lemma 5.

Proposition 2. *(1) $SetS - Act$ is a Grothendieck topos and so an elementary topos.*

(2) Any monomorphism in $SetS - Act$ is an equalizer, and any epimorphism in $SetS - Act$ is a coequalizer.

(3) Let $f \in Hom_{SetS-Act}(A, B)$. Then

f is a monomorphism in $SetS - Act$ if and only if f is left invertible if and only if f is an injective mapping;

f is an epimorphism in $SetS - Act$ if and only if f is right invertible if and only if f is a surjective mapping;

f is an isomorphism in $SetS - Act$ if and only if f is a bijective mapping.

Proof. (1) The category Set coincides with the category of sheaves of sets on a category with one object and one morphism. Consequently, any category equivalent to Set is a Grothendieck topos. But any Grothendieck topos is an elementary topos (see [3]).

(2) In Set, any monomorphism is an equalizer and any epimorphism is a coequalizer. The category $SetS-Act$ is equivalent to Set. Hence, (2) is proved.

(3) Every injective mapping has a left inverse, by the assumption the axiom of choice, every surjective mapping has a right inverse. Hence, (3) is proved. □

Proposition 3. *The category $SetS-Act$ is complete and cocomplete. Limits and colimits for an arbitrary functor $F : K \longrightarrow SetS-Act$ can be found as follows:*

(1) Let $\{\varphi(k) : E \longrightarrow I(F(k)) \mid k \in Ob(K)\}$ be a limiting cone of the functor $I \circ F : K \longrightarrow Set$ (it always exists). We fix an arbitrary action of the monoid S on E (for example, the trivial one) and denote the obtained S-act by D. Then $\{\varphi(k) : D \longrightarrow F(k) \mid k \in Ob(K)\}$ is a limiting cone of the functor F and $limF = D$.

(2) Let $\{\psi(k) : I(F(k)) \longrightarrow E \mid k \in Ob(K)\}$ a colimiting cone of the functor $I \circ F : K \longrightarrow Set$ (it always exists). We fix an arbitrary action of the monoid S on E (for example, a trivial one) and denote the obtained S-act by D. Then $\{\psi(k) : F(k) \longrightarrow D \mid k \in Ob(K)\}$ is a colimiting cone of the functor F and $colimF = D$.

Proof. (1) Apply the functor I to the cone $\{\varphi(k) : D \longrightarrow F(k) \mid k \in Ob(K)\}$. Since $I(D) = E$, the resulting cone coincides with the limiting cone of the functor $I \circ F$. Since I is an equivalence of categories, $\{\varphi(k) : D \longrightarrow F(k) \mid k \in Ob(K)\}$ is the limiting cone of F.

(2) Apply the functor I to the cone $\{\psi(k) : F(k) \longrightarrow D \mid k \in Ob(K)\}$. Since $I(D) = E$, the resulting cone coincides with the colimiting cone of the functor $I \circ F$. Since I is an equivalence of categories, $\{\psi(k) : F(k) \longrightarrow D \mid k \in Ob(K)\}$ is the colimiting cone of F. □

Definition 4. *(the functor $Q_r : Hom_{SS-Act}(A, B) \longrightarrow Hom_{Set-Act}(A, B)$, where $r \in S$)*

For objects, put $Q_r(A) = A$, where $A \in Ob(SS-Act)$; for arrows, put $Q_r(u) = u_r$, where $u \in Hom_{SS-Act}(A, B)$, i.e., $(u_r)(a) = u(r, a)$ for all $a \in A$.

By (5.1) of Lemma 1, we have $(v \cdot u)_r = v_r \circ u_r$ and $(e_A)_r = 1_A$. Therefore, the functors $Q_r : SS-Act \longrightarrow SetS-Act$, $r \in S$, are well defined.

The following proposition gives one more characterization of monomorphisms, epimorphisms, and isomorphisms in $SS-Act$ with the use of the functors Q_r.

Proposition 4. *Let $u \in Hom_{SS-Act}(A, B)$.*

(1) Consider the following conditions:

(1m) u is a monomorphism in $SS-act$;

(2m) $Q_s(u) \in Hom_{SetS-Act}(A, B)$ is a monomorphism in $SetS-Act$ for any $s \in S$;

(3m) $Q_1(u) \in Hom_{SetS-Act}(A, B)$ is a monomorphism in $SetS-Act$.

Then (2m)⇒(1m)⇒(3m). Moreover, if S is a group, then all this conditions are equivalent.

(2) The following conditions are equivalent:

(1e) u is an epimorphism in $SS-Act$;

(2e) $Q_s(u) \in Hom_{SetS-Act}(A, B)$ is an epimorphism in $SetS-Act$ for any $s \in S$.

(3) The following conditions are equivalent:

(1is) u *is an isomorphism in* $SS - Act$*;*

(2is) $Q_s(u) \in Hom_{SetS-Act}(A,B)$ *is an isomorphism in* $SetS - Act$ *for any* $s \in S$*.*

Note that any functor Q_r defines an isomorphism from the category $FS - Act$ onto the category $S - Act$.

Theorem 6. *Let* $r \in S$*. The mappings* $A \mapsto Q_r(A)$ *and* $u \mapsto Q_r(u)$ *define an isomorphism from the subcategory* $FS - Act$ *of* $SS - Act$ *onto the subcategory* $S - Act$ *of* $SetS - Act$*.*

Proof. Since Q_r is a functor, it suffices to check that $Q_r(Hom_{FS-Act}(A,B)) = Hom_{S-Act}(A,B)$ for any A,B and that the mapping $Q_r \mid Hom_{FS-Act}(A,B) : Hom_{FS-Act}(A,B) \longrightarrow Hom_{SetS-Act}(A,B)$ is injective. Let $u \in Hom_{FS-Act}(A,B)$. By Lemma 3, $u = \tilde{f}$ for some $f \in Hom_{S-Act}(A,B)$. Hence $Q_r(u)(a) = \tilde{f}(r,a) = f(a)$ for any $a \in A$, i.e. $Q_r(u) = f$. Thus, $Q_r(Hom_{FS-Act}(A,B)) \subset Hom_{S-Act}(A,B)$. If $f \in Hom_{S-Act}(A,B)$ then $\tilde{f} \in Hom_{FS-Act}(A,B)$ and $Q_r(\tilde{f}) = f$, i.e., we have $Q_r(Hom_{FS-Act}(A,B)) \supset Hom_{S-Act}(A,B)$.

It remains to prove that $Q_r \mid Hom_{FS-Act}(A,B)$ is injective. Suppose that $u,v \in Hom_{FS-Act}(A,B)$ and $Q_r(u) = Q_r(v)$. Then $u = \tilde{f}$ and $v = \tilde{g}$ for some $f,g \in Hom_{S-Act}(A,B)$. Therefore, $Q_r(u) = f$ and $Q_r(v) = g$. Hence, $f = g$ and $u = \tilde{f} = \tilde{g} = v$. Thus, the injectivity is proved. □

9. The Category $SS - Act$ When S is a Group

In this section, we assume that S is a group. Here we prove that $SS - Act$ is a Grothendieck topos, characterize epimorphisms, monomorphisms, and isomorphisms, describe methods for constructing limits and colimits.

Lemma 6. *Let* $r \in S$*.*

(1) Define a mapping $(t,f) \mapsto tf$ *as follows:*

$$(tf)(a) = rtr^{-1}f(rt^{-1}r^{-1}a)$$

where $t \in S$*,* $a \in A$*. Then this mapping defines an action of* S *on* $Hom_{SetS-Act}(A,B)$*.*

(2) If $g \in Hom_{SetS-Act}(B,C)$ *then* $t(g \circ f) = tg \circ tf$*.*

(3) The mapping $f : A \longrightarrow B$ *is a fixed point under this action if and only if* $f \in Hom_{S-Act}(A,B)$*.*

Proof. (1) The equalities

$$s(tf)(a) = rsr^{-1}(tf(rs^{-1}r^{-1}a)) = rsr^{-1}(tf(b)) = rsr^{-1}rtr^{-1}f(rt^{-1}r^{-1}b)) =$$
$$= rstr^{-1}f(rt^{-1}r^{-1}b)) = rstr^{-1}f(rt^{-1}r^{-1}rs^{-1}r^{-1}a) =$$
$$= rstr^{-1}f(rt^{-1}s^{-1}r^{-1}a) = r(st)r^{-1}f(r(st)^{-1}r^{-1}a) = (st)f(a),$$

where $s,t \in S$ and $a \in A$, imply $s(tf) = (st)f$ for any $s,t \in S$.

$$1f(a) = rr^{-1}f(rr^{-1}a) = f(a),$$

i.e., $1f = f$.

(2) Let $g \in Hom_{SetS-Act}(B,C)$. The equalities

$$(tg \circ tf)(a) = (tg)(tf(a)) = rtr^{-1}g(rt^{-1}r^{-1}(tf(a)) =$$
$$= rtr^{-1}g(rt^{-1}r^{-1}(rtr^{-1}f(rt^{-1}r^{-1}a)) = rtr^{-1}g(f(rt^{-1}r^{-1}a)) = t(g \circ f)(a),$$

where $t \in S$, $a \in A$, imply $(tg \circ tf) = t(g \circ f)$ for any $t \in S$.

(3) Let $a\in A$, $s,t,r\in S$. If $t=rsr^{-1}$ then $(tf)(sa)=sf(a)$. If f is a fixed point then $f(sa)=(tf)(sa)=sf(a)$. Thus, $f\in Hom_{S-Act}(A,B)$.

Conversely, if $f\in Hom_{S-Act}(A,B)$ then

$$(tf)(a)=rtr^{-1}f(rt^{-1}r^{-1}a)=f(rtr^{-1}rt^{-1}r^{-1}a)=f(a),$$

hence $tf=f$ for any $t\in S$. □

Thus, by Lemma 5, for any $r\in S$ the set $Hom_{SetS-Act}(A,B)$ is S-act and $(tf)(a)=rtr^{-1}f(rt^{-1}r^{-1}a)$, where $t\in S$, $a\in A$.

Theorem 7. *(on the isomorphism of the categories $SS-Act$ and $SetS-Act$). Let $r\in S$.*

(1) The functor $Q_r: SS-Act\longrightarrow SetS-Act$ is an isomorphism of categories. The inverse functor $P_r: SetS-Act\longrightarrow SS-Act$ is defined by $P_r(A)=A$, where $A\in Ob(SetS-Act)$, and $P_r(f)(s,a)=sr^{-1}f(rs^{-1}a$, where $s\in S$, $a\in A$, $f\in Hom_{SetS-Act}(A,B)$.

(2)(a) The mapping $u\mapsto Q_r(u)$ is an isomorphism of the S-acts $Hom_{SS-Act}(A,B)$ and $Hom_{SetS-Act}(A,B)$.

(b) for $u\in Hom_{SS-Act}(A,B)$, we have

$$u\in Hom_{FS-Act}(A,B)\Longleftrightarrow Q_r(u)\in Hom_{S-Act}(A,B).$$

Proof. (1) Prove that $P_r(f)\in Hom_{S-Act}(A,B)$ for any $f\in Hom_{SetS-Act}(A,B)$. Indeed,

$$P_r(f)(t(s,a))=P_r(f)(ts,ta)=tsr^{-1}f(r(ts)^{-1}ta)=$$
$$=t(sr^{-1}f(rs^{-1}a))=t(P_r(f)(s,a))$$

for any $s\in S$, $a\in A$.

Show that $P_r: SetS-Act\longrightarrow SS-Act$ is a functor. Indeed, for any S-acts A, B, C, $f\in Hom_{SetS-Act}(A,B)$, $g\in Hom_{SetS-Act}(B,C)$, and the identity morphism $1_A\in Hom_{SetS-Act}(A,A)$, we have

$$(P_r(g)\cdot P_r(f))(s,a)=P_r(g)(s,P_r(f)(s,a))=P_r(g)(s,sr^{-1}f(rs^{-1}a))=$$
$$=sr^{-1}g(rs^{-1}sr^{-1}f(rs^{-1}a))=sr^{-1}(g\circ f)(rs^{-1}a)=P_r(g\circ f)(s,a),$$
$$P_r(1_{I(A)})(s,a)=sr^{-1}1_{I(A)}(rs^{-1}a)=a=e_A(s,a)$$

for any $s\in S$, $a\in A$, i.e., $P_r(g\circ f)=P_r(g)\cdot P_r(f)$ and $P_r(1_{I(A)})=e_A$.

It remains to prove that the functors Q_r and H_r are mutually inverses. The functors Q_r and H_r coincide for objects. Let A, B be left S-acts, $u:\in Hom_{SS-Act}(A,B)$ and $f\in Hom_{SetS-Act}(A,B)$. Then

$$(Q_r\circ P_r)(f))(a)=Q_r(P_r(f))(a)=P_r(f)(r,a)=rr^{-1}f(rr^{-1}a)=f(a),$$
$$P_r(Q_r(u))(s,a)=sr^{-1}Q_r(u)(rs^{-1}a)=sr^{-1}u(r,rs^{-1}a)=u(s,a)$$

for any $s\in S$, $a\in A$, i.e., $Q_r(H_r(f))=f$ and $P_r(Q_r(u))=u$.

(2) (a) By (1), the mapping $Q_r: Hom_{SS-Act}(A,B)\longrightarrow Hom_{SetS-Act}(A,B)$ is bijective. Let us prove that Q_r is a morphism of $S-Act$. Indeed, for any $u\in Hom_{SS-Act}(A,B)$, $t\in S$ and $a\in A$, we have

$$(tQ_r(u))(a)=(tu_r)(a)=u_{rt}(a)=u(rt,a)=(tu)(r,a)=(tu)_r(a)=Q_r(ut)(a),$$

i.e., $Q_r(tu)=tQ_r(u)$.

(b) If $u\in Hom_{FS-Act}(A,B)$ then, by Theorem 9, $Q_r(u)\in Hom_{S-Act}(A,B)$.

If $Q_r(u) \in Hom_{S-Act}(A, B)$ then $Q_r(u)$ is a fixed point of the S-act $Hom_{SetS-Act}(A, B)$. Since, by (2)(a), Q_r is isomorphism of S-acts, it follows that u is a fixed point of the S-act $Hom_{SS-Act}(A, B)$, i.e., $u \in Hom_{FS-Act}(A, B)$. □

The following results are consequences of Theorem 7. Some of these results are concerned with arbitrary topoi and some are consequences of the specific properties of the categories $SetS - Act$.

Theorem 8. *Let S be a group. The category $SS - Act$ is a Grothendieck topos, in particular, it is an elementary topos.*

Proof. By Proposition 2, category $SetS - Act$ is a Grothendieck topos. By Theorem 7, $SS - Act$ is isomorphic to $SetS - Act$. Therefore, $SS - Act$ is a Grothendieck topos. □

Theorem 9. *(on monomorphisms, epimorphisms, and isomorphism in $SS - act$).*
Let $r \in S$ and $u \in Hom_{SS-Act}(A, B)$.
(1) The following are equivalent:
(1m) u is a monomorphism in $SS - act$;
(1'm) u is left invertible, i.e. there is $v \in Hom_{SS-Act}(B, A)$ such that $v \cdot u = e_A$;
(3'm) u_r is injective for any $r \in S$.
(2) The following are equivalent:
(1e) u is an epimorphism in $SS - Act$;
(1'e) u is right invertible, i.e., there is $w \in Hom_{SS-Act}(B, A)$ such that $u \cdot w = e_B$;
(3'e) $u_r : A \to B$ is surjective for any $s \in S$.
(3) The following are equivalent:
(1is) u is an isomorphism in $SS - Act$;
(2'is) u_r is bijective for any $r \in S$.
(4) Any monomorphism in $SS - Act$ is an equalizer, and any epimorphism in $SS - Act$ is a co-equalizer.

Proof. Suppose the fulfillment of the hypotheses of the theorem.

(1) $(1m) \Rightarrow (3'm)$ By Proposition 4, the mapping $u_r = Q_r(u)$ is a monomorphism in $SetS - Act$. By Proposition 2(3), it is injective.

$(3'm) \Rightarrow (1'm)$ Since $u_r : A \to B$ is an injective mapping, there exists $h : B \to A$, such that $h \circ u_r = 1_A$. Applying the functor P_r, we have $P_r(h \circ u_r) = P_r(h) \cdot P_r(u_r) = e_A$. Let $v = P_r(h)$. Then $P_r(u_r) = P_r(Q_r(u)) = u$ implies $v \cdot u = e_A$.

$(1'm) \Rightarrow (1m)$ A left invertible morphism is a monomorphism in any category.

(2) $(1e) \Rightarrow (3'e)$ By Proposition 4, the mapping $u_r = Q_r(u)$ is an epimorphism in $SetS - Act$. By Proposition 2(3), it is surjective.

$(3'e) \Rightarrow (1'e)$ Since $u_r : A \to B$ is a surjective mapping, by the axiom of choice, there exists a mapping $h : B \to A$ such that $u_r \circ h = 1_B$. Applying the functor P_r, we infer $P_r(u_r \circ h) = P_r(u_r) \cdot P_r(h) = e_B$. Let $w = P_r(h)$. Since $P_r(u_r) = P_r(Q_r(u)) = u$, we have $u \cdot w = e_B$.

$(1'e) \Rightarrow (1e)$. A right invertible morphism is an epimorphism in any category.

(3) $(1is) \Rightarrow (2'is)$ was proved in Theorem 4.

$(2'is) \Rightarrow (1is)$). Since the mapping $u_r : A \to B$ is injective and surjective, by (1) and (2), u is left and right invertible. Thus, u is an isomorphism.

(4) By Proposition 2, any monomorphism is an equalizer and any epimorphism is a coequalizer in $SetS - Act$. Since the categories $SS - Act$ and $SetS - Act$ are isomorphic, these properties hold in $SS - Act$. □

Theorem 10. *(on limits and colimits in the category $SS - act$). Let S be a group. Then the category $SS - Act$ is complete and cocomplete.*

More exactly, let $F : K \to SS - Act$ be a functor, $r \in S$.

(1) Suppose D is an $S - act$ and $\{\varphi(k) \in Hom_{SetS-Act}(D, Q_r(F(k))) \mid k \in Ob(K)\}$ is a limiting cone in the category $SetS - Act$. Let $\Phi(k) = P_r(\varphi(k))$ be the mapping from $S \times D$ into $F(k)$ such that $\Phi(k)(s, d) = sr^{-1}\varphi(k)(rs^{-1}d)$ for all $s \in S$, $d \in D$; then

$$\{\Phi(k) \in Hom_{SS-Act}(D, F(k)) \mid k \in Ob(K)\}$$

is a limiting cone for the functor F.

(2) Let E be an $S - act$ and let $\{\psi(k) : Q_r(F(k)) \longrightarrow E \mid k \in Ob(K)\}$ be a colimiting cone in the category $SetS - Act$.

Let $\Psi(k) = P_r(\psi(k)) : S \times F(k) \longrightarrow E$ be such that $\Psi(k)(s, a) = sr^{-1}\psi(k)(rs^{-1}a)$ for all $s \in S$, $a \in F(k)$. Then

$$\{\Psi(k) \in Hom_{SS-Act}(F(k), E) \mid k \in Ob(K)\}$$

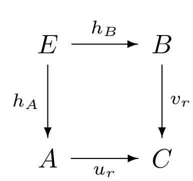
is a colimiting cone for F.

Proof. By Proposition 3, for any functor with codomain in the category $SetS - Act$, in particular, with codomain in $Q_r \circ F$, there exist a limiting cone and a colimiting cone. Since P_r is an isomorphism of categories, we may apply it to the limiting or colimiting cone of the functor $Q_r \circ F$ and obtain the corresponding cone of F. □

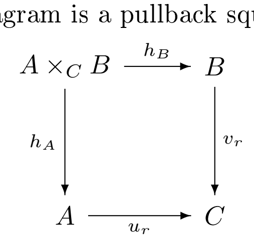
We will now give the constructions of the fibered products, fibered sums, equalizers, and coequalizers.

Let S be a group, $r \in S$.

Fibered products. Let $u \in Hom_{SS-Act}(A, C)$, $v \in Hom_{SS-Act}(B, C)$, $u_r : A \to C$ and $v_r : B \to C$ be the mappings such that $u_r(a) = u(r, a)$ and $v_r(b) = v(r, b)$ for any $a \in A$, $b \in B$. Introduce the notations: $E = u_r \times_C v_r = \{(a, b) \in A \times B \mid u(r, a) = v(r, b)\}$, $h_A : E \to A$ and $h_B : E \to B$ are the projections, i.e., $h_A(a, b) = a$, $h_B(a, b) = b$ for all $a \in A$, $b \in B$. Fix an arbitrary action of the group S on the set E and denote the resulting S-act by $A \times_C B$. The following diagram is a pullback square in Set:



By Lemma 5, the following diagram is a pullback square in $SetS - Act$:



Let $p_A = P_r(h_A)$, $p_B = P_r(h_B)$, i.e.

$$p_A(s,a,b) = sr^{-1}h_A(rs^{-1}(a,b)), p_B(s,a,b) = sr^{-1}h_B(rs^{-1}(a,b))$$

for all $s \in S$, $a \in A$, $b \in B$. Then, by Theorem 7, the following diagram is a pullback square in the category $SS - Act$:

$$\begin{array}{ccc} A \times_C B & \xrightarrow{p_B} & B \\ \downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[u]{} & C \end{array}$$

In particular, if $r = 1$, E is an S-act with the trivial action of S, and $p_A(s,a,b) = sh_A(a,b) = sa$, $p_B(s,a,b) = sh_B(a,b) = sb$ for all $s \in S$, $a \in A$, $b \in B$, then the square is a pullback.

Fibered sums. Let $u \in Hom_{SS-Act}(C,A)$, $v \in Hom_{SS-Act}(C,B)$. The mappings $u_r : C \to A$, $v_r : C \to B$ are defined by $u_r(c) = u(r,c)$, $v_r(c) = v(r,c)$. Suppose that the following diagram is a cocartesian square (pushout) in Set:

$$\begin{array}{ccc} C & \xrightarrow{v_r} & B \\ \downarrow{\scriptstyle u_r} & & \downarrow{\scriptstyle h_B} \\ A & \xrightarrow[h_A]{} & Q \end{array}$$

Fix an arbitrary action of the group S on Q and denote the resulting S-act by $A \sqcup_C B$. By Lemma 5, the following diagram is a pushout in $SetS - Act$:

$$\begin{array}{ccc} C & \xrightarrow{v_r} & B \\ \downarrow{\scriptstyle u_r} & & \downarrow{\scriptstyle h_B} \\ A & \xrightarrow[h_A]{} & A \sqcup_C B \end{array}$$

Let $q_A = P_r(h_A)$, $q_B = P_r(h_B)$, i.e.,

$$q_A(s,a) = sr^{-1}h_A(rs^{-1}a), q_B(s,b) = sr^{-1}h_B(rs^{-1}b)$$

for all $s \in S$, $a \in A$, $b \in B$. By Theorem 7, the following diagram is a pushout in $SS - Act$:

$$\begin{array}{ccc} C & \xrightarrow{v} & B \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & A \sqcup_C B \end{array}$$

In particular, if $r = 1$, $A \sqcup_C B$ is an S-act with the trivial action of S, and $q_A(s,a) = h_A(s^{-1}a)$, $q_B(s,b) = h_B(s^{-1}b)$ for all $s \in S$, $a \in A$, $b \in B$, then $A \sqcup_C B$ is fibered sum.

Equalizers. Let $u, v \in Hom_{SS-Act}(A,B)$, let $u_r, v_r \in Hom_{Set}(A,B)$, where $u_r(a) = u(r,a)$, $v_r(a) = v(r,a)$ for any $a \in A$, and let $i \in Hom_{Set}(T,A)$ be an equalizer of the pair u_r, v_r, and so $T = \{a \in A \mid u(r,a) = v(r,a)\}$, $i(a) = a$ for all $a \in A$. Fix an arbitrary action of the group S on T and denote the resulting S-act by E. By Lemma 5, $i \in Hom_{SetS-Act}(E, A)$ is an equalizer of the pair u_r, v_r in $SetS - Act$. Apply the functor P_r. By Theorem 7, the morphism $e = P_r(i) \in Hom_{SS-Act}(E, A)$ is an equalizer of the pair u, v in the category $SS - Act$, where $e(s,a) = sr^{-1}i(rs^{-1}a)$ for all $s \in S$, $a \in A$.

In particular, if $r = 1$, E is an S-act with trivial action of S, and $e(s,a) = sa$ for all $s \in S$, $a \in A$, then $e \in Hom_{SS-Act}(E, A)$ is an equalizer of the pair u, v.

Coequalizers. Let $u, v \in Hom_{SS-Act}(A, B)$, let $u_r, v_r \in Hom_{Set}(A, B)$, where $u_r(a) = u(r,a)$, $v_r(a) = v(r,a)$ for any $a \in A$, and let a mapping $j \in Hom_{Set}(B, T)$ be a coequalizer of the pair u_r, v_r in Set. Fix an arbitrary action of S on T and denote the resulting S-act by E. By Lemma 5, $j \in Hom_{SetS-Act}(B, E)$ is a coequalizer of the pair u_r, v_r in $SetS - Act$. Apply the functor P_r. By Theorem 7, the morphism $e = P_r(j) \in Hom_{SS-Act}(B, E)$ is a coequalizer of the pair u, v in $SS - Act$, where $e(s,b) = sr^{-1}j(rs^{-1}b)$ for all $s \in S$, $b \in B$.

In particular, if $r = 1$, E is an S-act with the trivial action of S, and $e(s,b) = j(s^{-1}b)$ for all $s \in S$, $b \in B$, then $e \in Hom_{SS-Act}(B, E)$ is a coequalizer of the pair u, v.

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