S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 18, №2, стр. 1390–1396 (2021) УДК 515.162
DOI 10.33048/semi.2021.18.105 MSC 57M15

SPATIAL GRAPHS AND THEIR ISOTOPY CLASSIFICATION

V.M. NEZHINSKIJ

ABSTRACT. The author's results related to the isotopic classification of orientable spatial framed graphs and contained in his recent paper are generalized to not necessarily orientable spatial framed graphs.

Keywords: spatial graph, skeleton, tangle, isotopy.

1. Introduction

By a spatial framed graph we mean a (smooth) connected compact two-dimensional submanifold with boundary of the space $\mathbb{R}^3$, whose handle decomposition satisfy the following properties. (1). The number of handles is finite. (2). Indices of handles are equal to either zero or one. (3). Handles of the same index do not intersect with each other. (4). Handles of different indices either do not intersect, or intersect along their boundaries only. (5). Each index one handle intersects the union of all index zero handles along two pairwise disjoint arcs. The fundamental problem of the spatial framed graph theory is the classification problem of spatial framed graphs up to (smooth) isotopy of $\mathbb{R}^3$ that preserves handle decompositions.

In the recent paper by the author [4], for *orientable* spatial framed graphs a theory was constructed that is needed to solve the problem formulated above. The main goal of this paper is to construct a theory of *not necessarily orientable* spatial framed graphs, developing and generalizing it.

The content of this article is as follows. First, as in the orientable case [4], we equip the spatial graphs with a skeleton and a marked point, but this time we equip the skeleton with an orientation. (The skeleton is always orientable, regardless of whether the spatial framed graph is orientable or not.) Further, when necessary, we correct the definitions of invariants from [4], so that the resulting set of invariants (with respect to isotopies that maps handle decompositions to handle

---

NEZHINSKIJ, V.M., SPATIAL GRAPHS AND THEIR ISOTOPY CLASSIFICATION.

Received September, 13, 2021, published November, 26, 2021.

decompositions, oriented skeletons onto oriented skeletons and marked points to marked points) turned out to be complete. Finally, we list all the values of the invariants (both old, for orientable spatial framed graphs, and new ones, for non-orientable spatial framed graphs) that they can take. All these results are new, that is, as far as the author knows, they are absent in the scientific literature.

This paper can be regarded as a continuation of [4]. (See also [5].) However, the acquaintance with [4] is not necessary, since we present all the statements needed.

2. Object of study

2.1. **Spatial framed graph.** Consider a smooth compact two-dimensional submanifold with boundary of the space $\mathbb{R}^3$. We will say that this submanifold *is equipped with the structure of a framed graph,* if its handle decomposition is given, which has the following properties:
(i) the number of handles is finite;
(ii) all handles are only handles of indices zero (that is, "disks") or one (that is, "bands");
(iii) the handles of index zero are pairwise disjoint, the handles of index one do not intersect pairwise, handles of different indices intersect only along their boundaries, and each handle of index one intersects with the union of all handles of index zero along two pairwise disjoint arcs.

Note, although this paper will not need it, that, as it is easy to see, *any smooth compact two-dimensional submanifold with boundary can be equipped with a framed graph structure* and that *this structure is not unique.*

A submanifold equipped with the structure of a framed graph will be called *a spatial framed graph*, the submanifold itself is *the support* of a spatial framed graph, the handles of index zero are *framed vertices*, the handles of index one are *framed edges.*

2.2. **Skeleton.** A spatial framed graph whose support is connected and simply connected will also be called *spatial framed tree.* Note that, as a simple direct check shows, *the support of any spatial framed tree is diffeomorphic to the standard two-dimensional disk.*

For any spatial framed graph, its *subgraph* is a graph consisting of some of its (whole) handles. *A skeleton* of a spatial framed graph we will call (any) its subgraph, which is a tree and contains all the framed vertices of the graph. Let me formulate two properties of the skeletons that are needed for what follows. First, *any skeleton is diffeomorphic to the standard two-dimensional disk.* This is a direct consequence of the remark in the previous paragraph. Second, *any skeleton is orientable.* This is a consequence of the previous property.

The skeleton existence problem is solved by the following (obvious) theorem. *A spatial framed graph has a skeleton if and only if its support is connected.* Note that, generally speaking, there is no uniqueness. For example, any spatial complete graph with four vertices, somehow framed, has two skeletons that do not admit a diffeomorphic handle-preserving mapping of one skeleton onto the other.

In this paper, we restrict ourselves to considering only spatial framed graphs whose supports are connected, so that *all the graphs considered in this paper have skeletons.*

2.3. **Marked point.** Let a spatial framed graph and its skeleton be given. For the *marked point* we take some point that belongs to the boundary of the skeleton and is not contained in any framed edge of the graph. The existence of such a point follows from the definition of a spatial framed graph, see 2.1.

2.4. **Object of study.** We call a spatial framed graph *totally framed* if a skeleton, an orientation of the skeleton, and a marked point (chosen as in 2.3) are fixed.

Spatial totally framed graphs are the object of our study.

3. Problem statement

Two spatial totally framed graphs are called *isotopic* if there is a smooth isotopy of the space $\mathbb{R}^3$, which maps, preserving handle decompositions, the first spatial graph to the second, the oriented skeleton of the first graph to the oriented skeleton of the second, the marked point of the first graph to the marked point of the second.

Our problem is to classify spatial totally framed graphs up to isotopy.

4. Fundamental invariants

Throughout what follows we will denote by D^2 the closed disk centered at the point (0,0) of radius one, located in the plane $\mathbb{R}^2$, by S^1 its boundary circle, by D^3 a closed ball located in the space $\mathbb{R}^3$ centered at the point (0,0,0) of radius one and by S^2 its boundary sphere, we will identify the plane $\mathbb{R}^2$ with the subspace $\mathbb{R}^2 \times 0$ of the space $\mathbb{R}^3$, and assume that the disk D^2 is the equator of the ball D^3 and that the circle S^1 is the equator of the sphere S^2.

4.1. **Preparatory material.** Consider a spatial framed graph. Take some framed edge of this spatial framed graph. It is clear that the boundary of this framed edge is the union of four pairwise disjoint arcs: two arcs are the arcs along which the framed edge intersects the union of all framed vertices; the other two are complements to the union of these arcs. Consider in this framed edge some simple smooth curve intersecting its edge only at its ends, intersecting transversally, different ends of the curve belong to the interiors of different arcs; we call this curve an *axis* of the framed edge if its ends belong to the interiors of the first two arcs, and we call it a *meridian* of the framed edge if its ends belong to the interiors of the other two arcs.

For any natural number n and any integer i put

$$v_{n,i} = (\cos\frac{2\pi i}{n},\ \sin\frac{2\pi i}{n}, 0).$$

It is clear that: all points $v_{n,i}$ lie on the circle S^1; points $v_{n,0}, v_{n,1}, \dots, v_{n,n-1}$ are pairwise distinct; for any natural numbers n and m, the equality $v_{n,0} = v_{m,0}$ holds.

Consider a spatial totally framed graph. In each framed edge contained in the skeleton of this spatial totally framed graph, we choose some meridian, and in each framed edge not contained in the skeleton, we choose some axis; we call a spatial totally framed graph *special* if:

(i) the support of its skeleton is the (standard oriented) disk D^2,

(ii) the point $v_{2q+1,0}(= v_{1,0})$ is its marked point,

(iii) points $v_{2q+1,1}, v_{2q+1,2}, \dots, v_{2q+1,2q}$ are all edge points of the above meridians and axes (where q is the number of framed edges of the graph),

(iv) all the above meridians are (metric) chords of the disk D^2,

(v) the interiors of all axes of framed edges mentioned above are contained in the space $\mathbb{R}^3 \setminus D^3$.

Lemma 1. *Any spatial totally framed graph is isotopic to a special spatial totally framed graph.*

This follows from the fact that, as noted in 2.2, any skeleton of a spatial framed graph is diffeomorphic to the closed standard disc D^2, and that, as is well known (see, for example, [2]), for any oriented two-dimensional submanifold with boundary of the space $\mathbb{R}^3$ diffeomorphic to the closed disk D^2, there exists a smooth isotopy of the space $\mathbb{R}^3$, taking with preserving the orientation of this submanifold to the closed disc $D^2(\subset \mathbb{R}^3)$.

4.2. **Combinatorial invariant.** Let's start with combinatorial preparation.

Let k and l be non-negative integers such that $k \geq l$. Consider a sequence of the first k numbers of the natural series and choose from it some l-term subsequence $(i_1, i_2, \ldots, i_l)$. Next, put $m = k - l$ and denote by $(j_1, j_2, \ldots, j_m)$ a subsequence of the sequence of the first k numbers of the natural series, complementary to the subsequence $(i_1, i_2, \ldots, i_l)$. Finally, suppose that the numbers k and l (and therefore the number m) are even and that each of this subsequences is split into pairwise disjoint pairs. We will say that these subsequences and their partitions into pairs define *a scheme of the type* (k, l), if the pairing of the second (m-term) subsequences have the following additional property: for any two pairs of numbers of this partition, both numbers of one of the pairs are greater than the smallest and less than the largest number of the other pair.

Let us now define the required invariant.

According to Lemma 1, it suffices to define it for *special* spatial totally framed graphs. Take some special spatial totally framed graph. We denote by p and q the numbers of its framed vertices and edges, respectively. From the sequence of points $(v_{2q+1,1}, v_{2q+1,2}, \ldots, v_{2q+1,2q})$ of the circle S^1 choose two subsequences. First subsequence $(v_{2q+1,i_1}, v_{2q+1,i_2}, \ldots, v_{2q+1,i_{2q-2p+2}})$ consists of all points that are the ends of the axes of framed edges that are not contained in the skeleton of the spatial totally framed graph. Let's split this subsequence into pairs: two points make up a pair, if they are the ends of the same axis. Second subsequence $(v_{2q+1,j_1}, v_{2q+1,j_2}, \ldots, v_{2q+1,j_{2p-2}})$ – additional to the first. We will also split it into pairs: two points make up a pair if they are the ends of the same meridian.

Our *combinatorial* invariant of the special spatial totally framed graph is a scheme of the type $(2q, 2q-2p+2)$, consisting of subsequences $(i_1, i_2, \ldots, i_{2q-2p+2})$ and $(j_1, j_2, \ldots, j_{2p-2})$ of the sequence $(1, 2, \ldots, 2q)$, each of the subsequences, in turn, is split into pairs of numbers, which are the second indices of pairs of points $v_{2q,r}$, which are the ends of one axis, or one meridian, respectively.

Lemma 2. *If special spatial framed graphs are isotopic, then their combinatorial invariants are equal.*

This follows from [4, Lemma 5].

4.3. **Numerical invariants.** Throughout the subsection, we will assume that (some) spatial totally framed graph is fixed.

Let's start with the two we need for further observations.

Take a framed edge of the above spatial totally framed graph that is not contained in its fixed skeleton, and choose some axis of this edge. Our first observation is the following.

Proposition 1. *This axis has standard orientation.*

It is well known that the orientation of the framed skeleton (as of any oriented manifold) determines the orientation of its boundary. Let's equip the boundary of the framed skeleton with this orientation.

Note, that by the definition, an axis of the handle intersects the boundary of the framed skeleton along its end points. Clearly that to set an orientation of an axis, it is sufficient to choose one of its end points as a starting one. We choose as a starting point of the axis the first of its end points after the marked point $v_{1,0}$ when passing the boundary of the framed skeleton in the direction consistent with the defined above orientation of the boundary of the framed sceleton.

Now consider the set of *all* framed edges of this graph, not contained in its framed skeleton. Our second observation is the following.

Proposition 2. *There is a standard ordering of this set.*

Indeed, the order of framed edges is given by the order of the starting points of (some) their axes when passing the boundary of the framed skeleton from the marked point $v_{1,0}$ in the direction consistent with its (above defined) orientation.

Let us now turn to the definition of auxiliary invariants.

In the original spatial totally framed graph, take some framed edge not contained in its framed skeleton.

Consider the topological space, which is the union of this framed edge with the framed skeleton of the graph. Obviously, this space is a smooth compact connected two-dimensional submanifold with boundary of the space $\mathbb{R}^3$; let's call it *elementary.* It is clear that if an elementary submanifold is orientable, then it is diffeomorphic to a cylinder, and that if not, then it is diffeomorphic to the Mobius strip.

Let us first consider the case when the elementary submanifold orientable. In this case, the orientation of the framed skeleton determine the orientation of the entire elementary submanifold; we equip an elementary submanifold with this orientation. Let's choose some axis of the framed edge and equip it with standard orientation. (For the standard orientation of the axis, see the beginning of this subsection, Proposition 1.) We construct a closed curve as a union of this axis and some curve contained in the skeleton. We will equip this curve with the orientation that continues the orientation of the axis. An *auxiliary invariant corresponding to the selected framed edge,* (in the orientable case) we define as the linking number of the oriented closed curve constructed above and the curve obtained by shifting it from the submanifold along the normal to the elementary submanifold in the positive direction. Obviously, this number is a well-defined invariant.

Let us now consider the case when the elementary submanifold is nonorientable. Let's choose some axis of the framed edge, equip it with the standard orientation and twist the framed edge half a turn along the axis in the positive direction. We got new spatial totally framed graph with new selected framed edge, this time elementary submanifiold is orientable. An *auxiliary invariant corresponding to the selected framed edge,* (in the nonorientable case) we define as the difference between the auxiliary invariant of the new spatial totally framed graph corresponding to the new framed edge, and the number $1/2$.

A simple straight verification shows that this number is a well-defined invariant too.

We define *numerical* invariant of the spatial totally framed graph as the sequence of auxiliary invariants, corresponding to all framed edges not contained in the framed skeleton, ordered according to the standard ordering of these edges. (For the standard ordering of these edges see the beginning of this subsection, Proposition 2.)

Lemma 3. *If spatial totally framed graphs are isotopic, then their numerical invariants are equal.*

This follows from well defindness of the auxiliary invariants and Proposition 2.

4.4. **Topological invariant.** Consider a smooth compact one-dimensional submanifold with boundary of the ball D^3, whose interior is contained in the interior of the ball D^3, the boundary is contained in the sphere S^2, the intersection with the sphere S^2 is orthogonal. Following the tradition dating back to Conway [1], we call such a submanifold a *tangle* if the boundary of each of its components is nonempty, its boundary is in the equator S^1 of the sphere S^2 and it does not contain the point $v_{1,0}$. (Note that there are papers in which the already mentioned submanifolds without any restrictions are named as tangles; see, for example, [3]. Our paper uses the term tangle in the classical sense only.) In what follows, a tangle, the number of connected components of which is equal to l, will also be called an *l-component tangle.*

Two tangles are called *isotopic* if there is an isotopy of the ball D^3 mapping the first tangle on the second one, the interior of the ball D^3 into itself, the boundary sphere S^2 of the ball D^3 into itself, the equator S^1 into itself and fixed at the point $v_{1,0}$. (It is clear that isotopic tangles have the same number of components.) Note that, as is well known and easy to verify, *isotopy of tangles is an equivalence relation.*

Let us now define our topological invariant.

According to Lemma 1, it sufficient to define the invariant for *special* spatial totally framed graphs. Let us take some special spatial totally framed graph and in each of its framed edges not contained in its framed skeleton we will choose an axis. Remove from the space $\mathbb{R}^3$ the interior of the ball D^3, from the framed graph we leave only the axes of the framed handles selected above and attach an infinitely distant point to the space $\mathbb{R}^3 \setminus IntD^3$; it is not hard to see that we have built the tangle. We define a *topological* invariant of a spatial graph as the isotopic class of this tangle.

Lemma 4. *If special spatial totally framed graphs are isotopic, then their topological invariants are equal.*

This follows from [4, Lemma 5].

5. Main Theorem

Theorem 1. *A. Two spatial totally framed graphs are isotopic if and only if their combinatorial, numerical and topological invariants are equal.*

B. Let p and q be non-negative integers, $q \geq p - 1$. For any scheme of the type $(2q, 2q - 2p + 2)$, any $(q - p + 1)$-term sequence of numbers, each of which is either integer or half-integer, and any isotopic class $(p-1)$-component tangle there *exists a totally framed spatial graph, whose combinatorial, numerical and topological invariants are equal to this scheme, sequence and isotopic class, respectively.*

The fact that if spatial totally framed graphs are isotopic than their combinatorial, numerical and topological invariants are equal follows from lemmas 2-4. The rest can be cheked by simple direct verification.

References

[1] J.H. Conway, *An enumaration of knots and links, and some of their algebraic properties*, Computational problems in abstract algebra, Proc. Conf., Pergamon, Oxford, 1967, 329–358.

[2] M.W. Hirsch, *Differential topology*, Graduate texts in Mathematics, **33**, Springer-Verlag, New York etc., 1976. Zbl 0356.57001

[3] W.B.R. Lickorish, *Prime knots and tangles*, Trans. Am. Math. Soc., **267** (1981), 321–332. Zbl 0472.57004

[4] V.M. Nezhinskij, *Isotopy invariants of spatial graphs*, Sib. Èlektron. Mat. Izv., **17** (2020), 769–776. Zbl 1446.57015

[5] V.M. Nezhinskij, *Spatial graphs, tangles and plane trees*, St. Petersburg Math. J., **31**:6 (2020), 1055–1063. Zbl 1453.05023

Vladimir Mikhajlovich Nezhinskij
St.-Petersburg State University,
7-9, Universitetskaja nab.,
St.-Petersburg, 199034, Russia
Herzen State Pedagogical University of Russia,
48, nab. r. Mojki,
St.-Petersburg, 191186, Russia
Email address: nezhin@pdmi.ras.ru