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# ON THE EXISTENCE OF GLOBAL SOLUTION OF THE SYSTEM OF EQUATIONS OF ONE-DIMENSIONAL MOTION OF A VISCOUS LIQUID IN A DEFORMABLE VISCOUS POROUS MEDIUM

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Abstract. The initial-boundary value problem for the system of one-dimensional motion of a viscous liquid in a deformable viscous porous medium is considered. Local theorem of existence and uniqueness of the problem is proved in the case of compressible liquid. In the case of incompressible liquid the theorem of global solvability in time is proved in Holder classes.

**Keywords:** Darcy's law, poroelasticity, filtration, global solvability, porosity.

## 1. Introduction

An interest in the mathematical modeling of multiphase flows arises when considering the problems of describing such technogenic systems as modeling processes in oil wells and near-surface formations. In many practical problems, the porosity of the medium is variable, and the porous medium is deformable. At present, rigorous mathematical results in the field of models of filtration in deformable porous media are presented only in a few works and in the case of single-phase filtration.

Mathematical models of fluid filtration in a porous medium apply to a broad range of practical problems. The examples include, but are not limited to filtration near river dams, irrigation and drainage of agricultural fields, dynamics of hydraulic

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fracturing during oil and gas mining, methane extraction from coal and shale deposits, flow of magma in the earth's crust, physiological fluids motion in tissues, tumour growth processes etc. The problems of heat and mass transfer in multiphase media and the dynamics of multicomponent media are widely represented in natural processes and human activities. This motivates the mathematical modeling of the processes of interpenetrating motion of continuous media. The main attention in the paper is paid to investigation of the problem of heat and mass transfer in porous and poroelastic media. The mathematical model considered in this paper takes into account poroelasticity and deformation of the medium, including motion of solid particles of the medium and variable porosity. The resulting mathematical models are usually non-classical and require new approaches, both to investigate their correctness and to numerically simulate them. Mathematical analysis of multiphase and multicomponent systems provides an opportunity to predict the nature of complex flows in various situations that do not have a pre-experimental background. In addition, such analysis serves as a basis for the construction of numerical algorithms, which plays a key role in the development of specific technological processes. The parameters of these models strongly depend on the properties of fluids as well as the porous solid medium. Thus, a vast number of models are currently available (see [1], [2] and references therein). Majority of these models, however, make a simplifying assumption of a static solid porous skeleton, and treat the porosity as a given function. In this work, we try to relax this assumption to account the mobility and poroelastic properties of a solid component. Models with a given porosity function of a solid component are based on the Muskat-Leverett filtering theory. S.N. Antontsev and V.N. Monakhov [3] developed the theory for a special case of two-phase motion of immiscible incompressible liquids in a non-deformable porous medium. A large number of papers are devoted to numerical studies (see, for example, [4]).

The construction of mathematical models of fluid filtration processes in porous media is complicated by the fact that flow is often considered in a mobile inhomogeneous medium, which is characterised by the presence of variable porosity. A special feature of the model in this paper is the consideration of mobility of solid skeleton and its poroelastic properties. This model is a generalization of the classical Muskat-Leverett model in which porosity is treated as a given function. The consideration of the compressibility of the porous medium is fundamental.

Terzaghi [5] was the first to develop models of poroelastic media that would take into account the mobility of the skeleton and its poroelastic properties. He introduced the principle of effective stress, defined as the difference between the total stress and the pressure of the liquid phase. This position reflects the fact that the liquid carries a part of the load. The relation between the deformation of the solid matrix skeleton and the fluid flow is of key importance here. Bio [6] further developed Terzaghi's theory: he introduced a joint deformation model of a fluid-saturated porous medium and established the theory of poroelasticity. Almost simultaneously and independently, Frenkel [7] developed a similar theory. Later, V.N. Nikolayevsky, P.P. Zolotarev, and Kh.A. Rakhmatullin [8], [9], [10] proposed analogous models in their studies.

O.B. Bocharov [11], allowed the porosity to depend on the pressure, but no deformation of the porous skeleton was considered. V.V. Vedernikov and V.N. Nikolaevskii [12] proposed a two-phase filtration model in a deformable porous medium with a solid skeleton motion being described analogous to the Terzaghi principle and the modified Hooke linear law. There are no results to validate the model. This was done later in [13] where particular solutions were derived. O.B. Bocharov et al. [14] derived the properties of the solutions for a degenerate case. All filtration models are very complex both from a theoretical point of view as well as in their application to specific problems. Only a handful of studies on deformable porous media models published to date included result justification. A mainstream research is based on the classical theory of filtration, and model justification is only examined for a limited number of specific cases. Strict mathematical results are only presented in a few papers exploring the existence and uniqueness of such problems solutions. For example, A. M. Abourabia et al. in [15], [16], [17] reduced the initial system of equations to a single equation of higher order by making a number of simplifying assumptions. M. Simpson et al. [17] proved a local solvability of the Cauchy problem in S.L. Sobolev spaces. Y. Geng et al. [15], [16], investigated solutions of the "simple wave"type. Numerical studies of such problems were carried out, for example, in [18].

## 2. Problem Statement

The concepts of volumes of the solid skeleton $V_s$ and pores $V_p$ are introduced for each component of a two-phase medium (the $s$ skeleton and the $f$ liquid phase contained in it). Then the specific pore volume (porosity) can be stated as $\phi = \frac{V_p}{V_t}$, where the total volume is $V_t = V_p + V_s$.

Darcy flow, which describes the fluid velocity relative to the solid velocity, is defined as [19]

$$\vec{q}_D = \phi(\vec{v}_f - \vec{v}_s),$$

where $\vec{v}_f, \vec{v}_s$ are the velocities of fluid and porous skeleton respectively.

Mass conservation laws for liquid and solid phases in absence of phase transitions have the form [20]

$$\frac{\partial(\rho_f \phi)}{\partial t} + \bigtriangledown \cdot (\rho_f \phi \vec{v}_f) = 0,$$

$$\frac{\partial(1-\phi)\rho_s}{\partial t} + \bigtriangledown \cdot ((1-\phi)\rho_s \vec{v}_s) = 0,$$

where $t$ is the time, $\rho_f$ is the density of liquid, $\rho_s$ is the density of solid phase, $\bigtriangledown = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ is the gradient operator, $(x_1, x_2, x_3)$ are the Eulerian coordinates.

Laws of conservation of mass can be written in terms of material derivative: $(\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}_s \cdot \bigtriangledown)$. Then we obtain

$$\frac{d\rho_f \phi}{dt} = - \bigtriangledown \cdot (\rho_f(\vec{q}_D + \phi \vec{v}_s)),$$

$$\nabla \cdot \vec{v}_s = \frac{1}{1-\phi} \frac{d\phi}{dt}.$$

By motion of fluid in the deformable medium it is assumed that [20], [21]:

1. the deviator of stress tensor in the liquid phase is neglected ($S_f = 0$), because the fluid viscosity is much lower than the skeleton shear viscosity.

2. the total stress tensor $\sigma$ is described via stress tensors of solid phase $\sigma_s$ and liquid phase $\sigma_f$ by the rule:

$$\sigma = (1-\phi)\sigma_s + \phi\sigma_f = (1-\phi)(S_s - p_s I) - \phi p_f I,$$

and the total pressure is described as $p_{tot} = (1-\phi)p_s + \phi p_f$, where $\sigma_s, p_s$ are the stress tensor and the pressure of solid phase, $S_s = 2\eta\dot{\varepsilon}_D$ is deviator of stress tensor, $\dot{\varepsilon}_D = \frac{1}{2}\left(\frac{\partial v_s}{\partial \vec{x}} + \left(\frac{\partial v_s}{\partial \vec{x}}\right)^*\right)$ is the deviatoric strain rate tensor, $\eta$ is the skeleton shear viscosity, $\sigma_f, p_f$ are the stress tensor and the pressure of liquid phase.

Terzaghi's principle states that deformation of the solid matrix is determined by an effective stress defined as [5] $\sigma_e = \sigma + p_f I$. Then, for a fully saturated, low-porosity media, the effective dynamic pressure is $p_e = p_{tot} - p_f$ [22].

Notice, that

$$d\phi = \frac{dV_p}{V_t} - V_p\frac{dV_t}{V_t^2} = \frac{dV_p}{V_t} - \phi\frac{dV_t}{V_t}. \tag{1}$$

If density of the solid phase $\rho_s$ is constant, then $dV_s = 0$ and $dV_t = dV_p$. From equation (1) we obtain

$$d\phi = (1-\phi)\frac{dV_t}{V_t}. \tag{2}$$

The assumption that porosity is a function of effective pressure was used in work [23] $\phi = \phi(p_e)$, in particular: $\phi = \phi_0 \exp\{-bp_e\}$. In approach, used in [23], the bulk compressibility of the two-phase medium $\beta_t$ is defined as relativity summary change of volume, responding on changing applied effective dynamic pressure: $\beta_t = -\frac{1}{V_t}(\frac{\partial V_t}{\partial p_e})$.

Equation (2) in this case can be written as

$$d\phi = -(1-\phi)\beta_t dp_e.$$

Volumetric compressibility is also a function of porosity, for example [24]: $\beta_t(\phi) = \phi^b\beta_\phi$, where $\beta_\phi$ is the bulk compressibility, $b$ is a positive constant

$$\beta_\phi = -1/V_p(\partial V_p/\partial p_e).$$

Thus the temporal variation of the porosity owing to mechanical compaction can be written as [25]:

$$\frac{1}{1-\phi}\frac{d\phi}{dt} = -\beta_t(\phi)\frac{dp_e}{dt}.$$

The constitutive creep law can be written as [26], [27]

$$\frac{1}{1-\phi}\frac{d\phi}{dt} = -\frac{p_e}{\xi},$$

where $\xi$ corresponds to a bulk viscosity. This formulation is analogous to a creep-controlled viscous compaction law used in studies dealing with magma transport in the Earth's mantle [28].

The bulk viscosity as rule depends on $\phi$, for example: $\xi(\phi) = \frac{\eta}{\phi^m}$, where $m$ is a positive constant [29].

Thus the rheological law combining mechanical and viscous compaction can be expressed as [24], [26]

$$\frac{1}{1-\phi}\frac{d\phi}{dt} = -\beta_t(\phi)\frac{dp_e}{dt} - \frac{p_e}{\xi(\phi)}.$$

The conservation of momentum for the fluid can be stated as Darcy's law [19], [30]

$$\vec{q}_D = -K \bigtriangledown \left(\frac{P_{ex}}{\rho_f g}\right),$$

where $K$ is the hydraulic conductivity, $K = (k'\rho_f g)/\mu$, $k'$, $\mu$ are the permeability and the fluid dynamic viscosity, $g$ is the density of the mass forces ($\vec{g} = (0, 0, -g)$), $P_{ex}$ is the excess fluid pressure, defined as the difference between the fluid pressure and the hydrostatic pressure: $P_{ex} = p_f - p_h$. In this way, we have

$$\vec{q}_D = -\frac{k'}{\mu}(\triangledown p_f + \rho_f \vec{g}).$$

In some cases coefficients $k', \beta_t, \xi$ could be determined in experiment another way. In particular they can be determined as: $\beta_t = \phi^b \beta_\phi, \xi = \eta/\phi^m, k' = k\phi^n$, where $k$ is the permeability, $b = 1/2, m \in [0, 2], n = 3$ [24].

The laws of momentum conservation for each phase can be written as [31]

$$\nabla \cdot (\phi\sigma_f) - \rho_f \phi \vec{g} + M = 0,$$
$$\nabla \cdot ((1-\phi)\sigma_s) - \rho_s (1-\phi)\vec{g} - M = 0,$$

where $M$ is the interphase pulse exchange.

Adding these equations, we obtain the momentum conservation equation of the system "solid matrix - pore fluid"[20], [21], [25], namely the equation of incompressible deformation of the solid skeleton matrix, taking into account the effect of the pore fluid pressure:

$$\nabla \cdot \sigma + \rho_{tot}\vec{g} = 0,$$

where $\rho_{tot} = (1-\phi)\rho_s + \phi\rho_f$ is the density of the two-phase medium.

In expanded form, the previous equation can be written as follows [31]

$$\rho_{tot}\vec{g} + div\left((1-\phi)\eta\left(\frac{\partial \vec{v}_s}{\partial \vec{x}} + (\frac{\partial \vec{v}_s}{\partial \vec{x}})^*\right)\right) - \nabla p_{tot} = 0.$$

In some applications, the force balance equation is written as [20], [31]

$$-\nabla p_{tot} + \rho_{tot}\vec{g} = 0.$$

The equation of energy conservation is taken in the following form [31], [32]

$$(\rho_f c_f \phi + \rho_s c_s (1-\phi))\frac{\partial \theta}{\partial t} + (\rho_f c_f \phi \vec{v}_f + \rho_s c_s (1-\phi)\vec{v}_s)\nabla\theta = div(\lambda\nabla\theta),$$

where $\theta$ is the temperature of the medium (the same for each phases), $c_s$ and $c_f$ are the heat capacities for at constant volume of phases. The thermal conductivity coefficient of the medium $\lambda(\phi)$ is taken in the form $\lambda(\phi) = \lambda_f\phi + \lambda_s(1-\phi)$, where $\lambda_f, \lambda_s$ are the thermal conductivity of liquid and solid phase (averaged thermal conductivity) [31].

Thus, taking into account the dependencies coefficients of skeleton viscosity and compactness of porosity and temperature, the equations of model in the absence of phase transitions have the form [20], [21], [24], [31]:

$$\frac{\partial (1-\phi)\rho_s}{\partial t} + div((1-\phi)\rho_s \vec{v}_s) = 0, \quad \frac{\partial(\rho_f \phi)}{\partial t} + div(\rho_f \phi \vec{v}_f) = 0, \tag{3}$$

$$\phi(\vec{v}_f - \vec{v}_s) = -\frac{K(\phi)}{\mu(\theta)}(\nabla p_f + \rho_f \vec{g}), \tag{4}$$

$$\nabla \cdot \vec{v}_s = -a_1(\phi)\xi_1(\theta)p_e - a_2(\phi)\xi_2(\theta)(\frac{\partial p_e}{\partial t} + \vec{v}_s \cdot \nabla p_e), \tag{5}$$

$$\nabla \cdot \sigma + \rho_{tot}\vec{g} = 0, \quad \rho_{tot} = \phi\rho_f + (1-\phi)\rho_s, \tag{6}$$

(7) $$(\rho_f c_f \phi + \rho_s c_s(1-\phi))\frac{\partial \theta}{\partial t} + (\rho_f c_f \phi \vec{v}_f + \rho_s c_s(1-\phi)\vec{v}_s)\nabla\theta = div(\lambda\nabla\theta),$$

(8) $$p_{tot} = \phi p_f + (1-\phi)p_s, p_e = (1-\phi)(p_s - p_f).$$

This quasilinear composite type system describes the spatial unsteady non-isothermal motion of a compressible fluid in a viscoelastic medium. Here $K(\phi)$, $a_1(\phi), a_2(\phi)$, $\xi_1(\theta), \xi_2(\theta)$ are the parameters of poroelastic medium.

For the permeability coefficient $K(\phi)$, a well-known dependence of the form is used $K(\phi) = K'\phi^n$, where $K' = const > 0$, $n = 3$, [24]. In what follows, the notations are used $k(\phi) = K(\phi)/\mu(\theta)$, $a_1(\phi) = \phi^m, \xi_1(\theta) = 1/\eta(\theta)$, where $\eta(\theta)$ is the coefficient of dynamic viscosity of the skeleton, which characterizes the relationship between the strain rate tensor and the stress tensor and is determined from the experiment under uniaxial compression [33], [34]. The following dependence is taken as a model one: $\eta(\theta) = \eta_r exp(Q_r(1-\theta/\theta_r)/R\theta)$, $\eta_r, Q_r, \theta_r, R$ are positive constants (analog of the Arrhenius formula for the dependence of the reaction rate on temperature) [24].

Numerical studies of various initial-boundary value problems for the system of equations (2)–(8) were carried out in the works [21], [24], [35]. A numerical analysis of the initial-boundary value problem for the system (3)–(8) is carried out in [36]: difference schemes are constructed and their convergence is established. Questions of justification in these papers were not considered. In some particular cases, the issues of justifying this model are discussed in [37], [38]. The local solvability of the Cauchy problem in Sobolev spaces was established in [17].

The system of equations describing the one-dimensional unsteady motion of a compressible fluid in a viscous porous medium ($a_2(\phi) = 0$) in the domain $(x,t) \in Q_T = \Omega \times (0,T), \Omega = (0,1)$, is as follows [21], [31]:

(9) $$\frac{\partial (1-\phi)\rho_s}{\partial t} + \frac{\partial}{\partial x}((1-\phi)\rho_s v_s) = 0, \qquad \frac{\partial(\rho_f\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho_f \phi v_f) = 0,$$

(10) $$\phi(v_f - v_s) = -\frac{k(\phi)}{\mu(\theta)}(\frac{\partial p_f}{\partial x} - \rho_f g),$$

(11) $$\frac{\partial v_s}{\partial x} = -a_1(\phi)\xi_1(\theta)p_e, \quad p_e = p_{tot} - p_f, p_{tot} = \phi p_f + (1-\phi)p_s,$$

(12) $$\rho_{tot} g + \frac{\partial p_{tot}}{\partial x} = 0,$$

(13) $$(\rho_f c_f \phi + \rho_s c_s(1-\phi))\frac{\partial \theta}{\partial t} + (\rho_f c_f \phi v_f + \rho_s c_s(1-\phi)v_s)\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}(\lambda\frac{\partial \theta}{\partial x}).$$

The problem is written in the Eulerian coordinates $x, t$. The real density of the solid particles $\rho_s$ is assumed constant, and Clapeyron dependence is taken as $p_f = R\theta\rho_f, R = const > 0$. At the boundary of the region $\Omega$, the velocities of the phases $v_s, v_f$ are set, and at the initial moment of time the density $\rho_f^0(x)$ and the porosity $\phi^0(x)$ are set.

The following conditions are considered for the system (9)–(13):

(14) $$v_s|_{x=0,1} = v_f|_{x=0,1} = 0, \frac{\partial \theta}{\partial x}\,|_{x=0,x=1} = 0, \rho_f|_{t=0} = \rho^0(x), \phi|_{t=0} = \phi^0(x), \theta\,|_{t=0} = \theta^0(x).$$

Following [3], [39], [40] we rewrite the system (9)–(13). Suppose that $\bar{x} = \bar{x}(\tau, x, t)$ is the solution of the Cauchy problem

$$\frac{\partial \bar{x}}{\partial \tau} = v_s(\bar{x}, \tau), \quad \bar{x} \mid_{\tau=t} = x.$$

We set $\hat{x} = \bar{x}(0, x, t)$ and take $\hat{x}$ and $t$ for the new variables. Then $1 - \phi(\hat{x}, t) = (1 - \phi^0(\hat{x}))\hat{J}(\hat{x}, t)$, where $\hat{J}(\hat{x}, t) = \frac{\partial \hat{x}}{\partial x}(\hat{x}, t)$ is the Jacobian of the transformation. Instead of (9)–(13) we have

$$\frac{\partial(1-\hat{\phi})}{\partial t} + \frac{(1-\hat{\phi})^2}{1-\phi^0}\frac{\partial \hat{v}_s}{\partial \hat{x}} = 0, \quad \frac{\partial}{\partial t}(\hat{\rho}_f\hat{\phi}) + \frac{(1-\hat{\phi})}{1-\phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}\hat{v}_f) = \hat{v}_s\frac{(1-\hat{\phi})}{1-\phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}),$$

$$\hat{\phi}(\hat{v}_s - \hat{v}_f) = \frac{k(\hat{\phi})}{\mu(\hat{\theta})}\frac{(1-\hat{\phi})}{1-\phi^0}\frac{\partial \hat{p}_f}{\partial \hat{x}}, \quad \frac{(1-\hat{\phi})}{1-\phi^0}\frac{\partial \hat{v}_s}{\partial \hat{x}} = -a_1(\hat{\phi})\xi_1(\hat{\theta})\hat{p}_e.$$

Since

$$\hat{v}_s\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}) = \frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}\hat{v}_s) - \hat{\rho}_f\hat{\phi}\frac{\partial \hat{v}_s}{\partial \hat{x}},$$

the continuity equation for the liquid phase can be reduced to the form

$$\frac{1}{(1-\hat{\phi})}\frac{\partial}{\partial t}(\hat{\rho}_f\hat{\phi}) + \frac{1}{1-\phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}(\hat{v}_f - \hat{v}_s)) + \frac{1}{1-\phi^0}\hat{\rho}_f\hat{\phi}\frac{\partial \hat{v}_s}{\partial \hat{x}} = 0.$$

Using the continuity equation for the solid phase, we have

$$\frac{\partial}{\partial t}(\hat{\rho}_f\frac{\hat{\phi}}{1-\hat{\phi}}) + \frac{1}{(1-\phi^0)}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}(\hat{v}_f - \hat{v}_s)) = 0.$$

Finally, passing from $(\hat{x}, t)$ to the mass Lagrangian variables $(y, t)$ by the rule [3] $(1-\phi^0(\hat{x}))d\hat{x} = dy, \quad y(\hat{x}) = \int_0^{\hat{x}}(1-\phi^0(\eta))d\eta \in [0,1]$ and formally replacing $y$ by $x$, system (9)–(13) can be written in the form

(15) $$\frac{\partial(1-\phi)}{\partial t} + (1-\phi)^2\frac{\partial v_s}{\partial x} = 0,$$

(16) $$\frac{\partial}{\partial t}\left(\rho_f\frac{\phi}{1-\phi}\right) + \frac{\partial}{\partial x}(\rho_f\phi(v_f - v_s)) = 0,$$

(17) $$\phi(v_s - v_f) = \frac{k(\phi)}{\mu(\theta)}((1-\phi)\frac{\partial p_f}{\partial x} - \rho_f g),$$

(18) $$(1-\phi)\frac{\partial v_s}{\partial x} = -a_1(\phi)\xi_1(\theta)p_e, \quad p_e = p_{tot} - p_f,$$

(19) $$\rho_{tot}g + (1-\phi)\frac{\partial p_{tot}}{\partial x} = 0,$$

(20)
$$(\rho_f c_f\phi + \rho_s c_s(1-\phi))\frac{\partial \theta}{\partial t} = (1-\phi)\frac{\partial}{\partial x}(\lambda(1-\phi)\frac{\partial \theta}{\partial x}) + \frac{k(\phi)}{\mu(\theta)}c_f\rho_f((1-\phi)\frac{\partial p_f}{\partial x} - \rho_f g)\frac{\partial \theta}{\partial x}.$$

Taking into account equation (19), we obtain the representation for $p_{tot}$ :

$$p_{tot} = P^0(t) - \int_0^x \frac{\rho_{tot}g}{1-\phi}d\xi.$$

We represent the equation (18) as

$$\frac{\partial v_s}{\partial x} = -\frac{a_1(\phi)}{1-\phi}\xi_1(\theta)(p_{tot} - p_f).$$

After integrating this equation by $x$ from 0 to 1 and taking into account the submission for $p_{tot}$ we get

$$P^0 = (\int\limits_0^1 \left(\frac{a_1(\phi)}{1-\phi}\xi_1(\theta)(p_f + \int_0^x \frac{\rho_{tot} g}{1-\phi} d\xi)\right) dx - v_s(1,t) + v_s(0,t))\cdot$$

$$\cdot \left(\int\limits_0^1 \frac{a_1(\phi)}{1-\phi}\xi_1(\theta) dx\right)^{-1}.$$

Converting the original system of equations we get a system for finding porosity, pressure of fluid and temperature (20) (using equation (16), taking into account equation (17) and using equation (15), taking into account equation (18)):

(21) $$\frac{\partial}{\partial t}(\frac{p_f}{R\theta}\frac{\phi}{1-\phi}) = \frac{\partial}{\partial x}(\frac{k(\phi)}{\mu(\theta)}\frac{p_f}{R\theta}((1-\phi)\frac{\partial p_f}{\partial x} - \frac{g}{R\theta}p_f)),$$

(22) $$\frac{\partial G}{\partial t} = \xi_1(\theta)(p_f - p_{tot}),$$

(23)
$$(\rho_f c_f \phi + \rho_s c_s (1-\phi))\frac{\partial \theta}{\partial t} = (1-\phi)\frac{\partial}{\partial x}(\lambda(1-\phi)\frac{\partial \theta}{\partial x}) + \frac{k(\phi)}{\mu(\theta)}c_f \rho_f((1-\phi)\frac{\partial p_f}{\partial x} - \rho_f g)\frac{\partial \theta}{\partial x},$$

where

$$p_{tot} = \int\limits_0^1 \left(\frac{a_1(\phi)}{1-\phi}\xi_1(\theta)(p_f(\rho_f) + \int_0^x \frac{\rho_{tot} g}{1-\phi} d\xi)\right) dx \left(\int\limits_0^1 \frac{a_1(\phi)}{1-\phi}\xi_1(\theta) dx\right)^{-1} -$$

$$- \int_0^x \frac{\rho_{tot} g}{1-\phi} d\xi,$$

and the function $G$ is defined as follows:

$$\frac{\partial G}{\partial \phi} = \frac{1}{(1-\phi)a_1(\phi)}.$$

## 3. Compressible Fluid

For system (21)–(23) conditions (14) could be written as

(24)
$$((1-\phi)\tfrac{\partial p_f(\rho_f)}{\partial x} - \rho_f g) \mid_{x=0,x=1} = 0, \quad \tfrac{\partial \theta}{\partial x} \mid_{x=0,x=1} = 0,$$
$$p_f \mid_{t=0} = p^0(x), \quad \phi \mid_{t=0} = \phi^0(x), \quad \theta \mid_{t=0} = \theta^0(x),$$

here

$$p^0(x) = R\theta^0 \rho^0.$$

In the notation of function spaces we follow [41]: $C^{k+\alpha, m+\beta}(Q_T)$ – Hölder's space, where $k, m$ are natural numbers, $(\alpha, \beta) \in (0, 1]$, with the norm $||f||_{C^{k+\alpha,m+\beta}(Q_T)}$.

**Definition 1.** *The solution of problem (21)–(24) is the set of functions $\phi, \rho_f, \theta \in C^{2+\alpha,1+\beta}(Q_T)$, such that $0 < \phi < 1, \rho_f > 0, \theta > 0$. These functions satisfy the equations (21)–(23) and the initial and boundary conditions (24) and are regarded as continuous functions in $Q_T$.*

**Theorem 1.** *Suppose that the data of problem (21)–(24) satisfy the following conditions:*
*1. the functions $k(\phi), \xi_1(\theta), \lambda(\phi), \mu(\theta), a_1(\phi)$ and their derivatives up to the second order are continuous for $\phi \in (0,1), p_f > 0, \theta > 0$, and satisfy the conditions*

$$k_0^{-1}\phi^{q_1}(1-\phi)^{q_2} \le k(\phi) \le k_0\phi^{q_3}(1-\phi)^{q_4}, k_0^{-1}\phi^{q_5}(1-\phi)^{q_6} \le \lambda(\phi) \le k_0\phi^{q_7}(1-\phi)^{q_8},$$

$$0 < \xi_1^0 \le \xi_1(\theta) \le \xi_1^1 < \infty, \quad 0 < \mu_0 \le \mu(\theta) \le \mu_1 < \infty,$$

$$a_1(\phi) = a_0(\phi)\phi^{\alpha_1}(1-\phi)^{\alpha_2-1}, \quad 0 < R_1 \le a_0(\phi) \le R_2,$$

*where $k_0, \xi_1^0, \xi_1^1, \mu_0, \mu_1\alpha_i, R_i, i = 1,2$ are positive constants, $q_1, ..., q_8$ are fixed real parameters,*
*2. the initial functions $\phi^0, p^0, \theta^0$ and function $g$ satisfy the following smoothness conditions: $\phi^0 \in C^{2+\alpha}(\overline{\Omega}), p^0 \in C^{2+\alpha}(\overline{\Omega}), \theta^0 \in C^{2+\alpha}(\overline{\Omega}), g \in C^{1+\alpha,1+\alpha/2}(\bar{Q}_T)$, and the matching conditions*

$$((1-\phi^0)\frac{dp^0}{dx} - \rho(p^0,\theta^0)g(x,0)) \mid_{x=0,x=1} = 0,$$

*as well as satisfy the inequalities*

$$0 < m_0 \le \phi^0(x) \le M_0 < 1, 0 < m_1 \le p^0(x) \le M_1 < \infty, 0 < m_2 \le \theta^0(x) \le M_2 < \infty,$$

$$0 < g(x,t) \le g_0 < \infty, x \in \overline{\Omega},$$

*where $m_0, M_0, m_1, M_1, g_0$ are given positive constants.*
*Then problem (21)–(24) has a local solution, i.e., there exists a value of $t_0 \in (0,T)$ such that*

$$(\phi(x,t), p_f(x,t), \theta(x,t)) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0}).$$

*Moreover, $0 < \phi(x,t) < 1$, $p_f(x,t) > 0$, $\theta(x,t) > 0$ в $\overline{Q}_{t_0}$.*

*Proof.* The solvability of problem (21)–(24) is established by using the Tikhonov-Schauder Fixed-Point Theorem [3].

Since the function $\psi = G(\phi)$ is strictly monotone, at $\phi \in (0,1)$, then the inverse function exists: $\phi = G^{-1}(\psi)$. Assuming that $p(x,t) = p_f(x,t) - p^0(x), \quad \omega(x,t) = G(\phi) - G(\phi^0), \quad \theta_1 = \theta - \theta^0$. We represent the equations (21)–(23) in the form

$$\frac{\partial}{\partial t}(\frac{p+p^0}{R(\theta_1+\theta^0)}\frac{\phi(\omega)}{1-\phi(\omega)}) = \tag{25}$$
$$= \frac{\partial}{\partial x}\left(\frac{k(\phi(\omega))}{\mu(\theta_1+\theta^0)}\frac{p+p^0}{R(\theta_1+\theta^0)}((1-\phi(\omega))\frac{\partial p+p^0}{\partial x} - \frac{g(p+p^0)}{R(\theta_1+\theta^0)}\right),$$

$$\frac{\partial \omega}{\partial t} = \xi_1(\theta_1+\theta^0)((p+p^0)$$
$$-\int_0^1\left(\frac{a_1(\phi(\omega))}{1-\phi(\omega)}\xi_1(\theta_1+\theta^0)((p+p^0) + \int_0^x \frac{\rho_{tot}(\omega)g}{1-\phi}d\xi)\right)dx\cdot \tag{26}$$
$$\cdot\left(\int_0^1 \frac{a_1(\phi(\omega))}{1-\phi(\omega)}\xi_1(\theta_1+\theta^0)dx\right)^{-1} + \int_0^x(\frac{p+p^0}{R(\theta_1+\theta^0)}\frac{\phi(\omega)}{1-\phi(\omega)} + \rho_s)gd\xi),$$

$$(\frac{c_f(p+p^0)}{R(\theta_1+\theta^0)}\frac{\phi}{1-\phi}+c_s\rho_s)\frac{\partial\theta_1}{\partial t}=\frac{\partial}{\partial x}(\lambda(1-\phi)\frac{\partial(\theta_1+\theta^0)}{\partial x})+ \tag{27}$$
$$+\frac{k(\phi)}{\mu(\theta_1+\theta^0)}\frac{c_f(p+p^0)}{R(\theta_1+\theta^0)}((1-\phi)\frac{\partial(p+p^0)}{\partial x}-\frac{g(p+p^0)}{R(\theta_1+\theta^0)})\frac{\partial(\theta_1+\theta^0)}{\partial x}.$$

For the Banach space, we choose the space $C^{2+\beta,1+\beta/2}(\overline{Q}_{t_0})$, where $\beta$ is any number from the interval $(0,\alpha)$, $\alpha\in[0,1)$. Let

$$V=\{\bar{p}(x,t),\bar{\omega}(x,t),\bar{\theta}\in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0})|$$
$$\bar{p}\mid_{t=0}=\bar{\omega}\mid_{t=0}=\bar{\theta}\mid_{t=0}=\frac{\partial\bar{\theta}}{\partial x}\mid_{x=0,x=1}=((1-\phi(\bar{\omega}))\frac{\partial(\bar{\rho}+\rho^0)}{\partial x}-(\bar{\rho}+\rho^0)g)\mid_{x=0,x=1}=0,$$
$$m_1^*-p^0(x)\le\bar{p}(x,t)\le M_1^*-p^0(x)<\infty,$$
$$m_1^*=\frac{m_1}{2}(1+\frac{g_0}{Rm_2(1-M_0)})^{-1},$$
$$M_1^*=2M_1(1+\frac{g_0}{Rm_2(1-M_0)})^{-1},$$
$$m_2-\theta^0\le\bar{\theta}\le M_2-\theta^0,$$
$$G(\frac{m_0}{2})-G(\phi^0)\le\bar{\omega}(x,t)\le G(\frac{M_0+1}{2})-G(\phi^0)<\infty,\quad (x,t)\in Q_{t_0},$$
$$(|\bar{\omega}|_{1+\alpha,(1+\alpha)/2,Q_{t_0}},|\bar{p}|_{1+\alpha,(1+\alpha)/2,Q_{t_0}},|\bar{\theta}|_{1+\alpha,(1+\alpha)/2,Q_{t_0}})\le K_1,$$
$$(|\bar{\omega}|_{2+\alpha,(2+\alpha)/2,Q_{t_0}},|\bar{p}|_{2+\alpha,(2+\alpha)/2,Q_{t_0}},|\bar{\theta}|_{2+\alpha,(2+\alpha)/2,Q_{t_0}})\le K_1+K_2\},$$

where $K_1$ is an arbitrary positive constant, while the positive constant $K_2$ will be given later. We note that on the set $V$ following inequalities hold: $0<m_0/2\le\phi(\bar{\omega})\le(M_0+1)/2<1$.

Let us construct an operator $\Lambda$ mapping $V$ in $V$. Suppose that $\bar{\omega},\bar{p},\bar{\theta}\in V$. Using (26), we define the function $\omega$ by the equality

$$\omega=\int\limits_0^t\xi_1(\bar{\theta}+\theta^0)((\bar{p}(x,\tau)+p^0(x))-\int\limits_0^1(\frac{a_1(\phi(\bar{\omega}))}{1-\phi(\bar{\omega})}\xi_1(\bar{\theta}+\theta^0)((\bar{p}+p^0)+$$
$$+\int_0^x(\frac{\bar{p}+p^0}{R(\bar{\theta}+\theta^0)}\frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})}+\rho_s)g)d\xi))dx(\int\limits_0^1\frac{a_1(\phi(\omega))}{1-\phi(\omega)}\xi_1(\theta_1+\theta^0)dx)^{-1}+ \tag{28}$$
$$+\int\limits_0^x g(\rho_s+\frac{\bar{p}+p^0(\xi)}{R(\bar{\theta}+\theta^0)}\frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})})d\xi)d\tau.$$

From the representation (28) it follows that smoothness $\omega$ is determined by the smoothness of functions $\bar{p},\bar{\omega},\bar{\theta},\bar{\theta}^0,p^0,P^0$ and $g$. Therefore there exists a value $t_1=t_1(m_0,M_0,m_1,M_1,m_2,M_2)$, such that for all $t_0\le t_1$ the following inequality holds

$$0<\frac{m_0}{2}\le\phi(x,t)\le\frac{M_0+1}{2},\quad (x,t)\in Q_{t_0}. \tag{29}$$

In particular, we have an estimate

$$|\omega|_{2+\alpha,1+\alpha/2,Q_{t_0}}\le C_0(m_0,M_0,m_1,M_1,K_1,m_2,M_2,T,|g|_{1+\alpha,\Omega},|p^0|_{2+\alpha,\Omega},$$
$$|\theta^0|_{2+\alpha,\Omega},|\phi^0|_{2+\alpha,\Omega}|P^0|_{\alpha/2,[0,T]})(1+t_0(|\bar{\rho}_{xx}|_{\alpha,\alpha/2,\Omega}+|\bar{\theta}_{xx}|_{\alpha,\alpha/2,\Omega})).$$

Taking into account (29) we also have the estimate for function $\omega(x,t)$:

$$G(\frac{m_0}{2}) \leq \omega(x,t) + G(\phi^0) \leq G(\frac{M_0+1}{2}).$$

Using (25), $\bar{p}, \bar{\theta}$ and $\bar{\omega}$, we find the function $p(x,t)$ as a solution of the problem (here and elsewhere, we assume that the initial and boundary conditions are matched):
(30)

$$\frac{\partial}{\partial t}(\frac{p+p^0}{R(\bar{\theta}+\theta^0)}\frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})}) = \frac{\partial}{\partial x}(\frac{k(\phi(\bar{\omega}))}{\mu(\bar{\theta}+\theta^0)}\frac{\bar{p}+p^0}{R(\bar{\theta}+\theta^0)}((1-\phi(\bar{\omega}))\frac{\partial p+p^0}{\partial x} - \frac{g(p+p^0)}{R(\bar{\theta}+\theta^0)}),$$

$$p\mid_{t=0}=0, \quad ((1-\phi(\bar{\omega}))\frac{\partial(p+p^0)}{\partial x} - \frac{g(p+p^0)}{R(\bar{\theta}+\theta^0)})\mid_{x=0,x=1}=0.$$

The equation for $p(x,t)$ is uniformly parabolic. In view of the properties of

$$\bar{\omega}(x,t), \quad \bar{\theta}(x,t)$$

and $p^0(x), \theta^0(x)$ the problem (30) has a classical solution [41]. In addition, we have the following estimate:

$$|\frac{1}{a(\bar{\omega})}\frac{\partial a(\bar{\omega})}{\partial t}| \leq C_1(m_0, M_0, m_1, M_1, m_2, M_2, \max_{0\leq t\leq T}|p^0(t)|, \max_{0\leq t\leq T}|\theta^0(t)|).$$

Under the additional condition smallness for the value of the time interval the following statement holds.

**Lemma 1.** *There exists such a $t_2$, that when $t_0 \leq \min(t_1, t_2)$, the classical solution of problem (25) satisfies the following inequality in $Q_{t_0}$:*

$$0 < \hat{m}_1 \leq \rho(x,t) + \rho^0(x) \leq \hat{M}_1 < \infty.$$

*Proof.* Further, setting $U(x,t) = p(x,t) + p^0(x)$, we can express problem (30) in the form

(31)

$$\frac{\partial}{\partial t}(a(\bar{\omega},\bar{\theta})U) = \frac{\partial}{\partial x}\left(K(\bar{\omega})b(\bar{p},\bar{\theta})\frac{\partial U}{\partial x} - k(\bar{\omega})c(\bar{p},\bar{\theta})U\right),$$

$$(\frac{\partial U}{\partial x} - \tilde{d}U)\mid_{x=0,x=1}=0, \quad U\mid_{t=0}=p^0.$$

Here $a(\omega,\theta) = \frac{\phi(\omega)}{1-\phi(\omega)}\frac{1}{R(\theta+\theta^0)}, b(p,\theta) = \frac{p+p^0}{\mu(\theta+\theta^0)R(\theta+\theta^0)}, c(p,\theta) = \frac{g(p+p^0)}{\mu(\theta+\theta^0)R^2(\theta+\theta^0)^2},$ $\tilde{d}(\omega,\theta) = \frac{g}{(1-\phi(\omega))(\theta+\theta^0)R}, K(\omega) = k(\phi(\omega))(1-\phi(\omega)),\ \phi(\omega) = G^{-1}(\omega + G(\phi^0)).$

First, we show that $U(x,t) \geq 0$, $(x,t) \in Q_{t_0}$. In equation (31), let us make the change $U(x,t) = -z(x,t)$. Then

$$z\frac{\partial a}{\partial t} + a\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}(Kb\frac{\partial z}{\partial x} - kcz).$$

Let

$$z^{(0)}(x,t) = \max\{z, 0\}, \quad z^{(0)}(x,t)\mid_{t=0}=\max\{-\rho^0, 0\} = 0,$$

$$\sigma_\varepsilon(x,t) = z^{(0)}(x,t)(|z^{(0)}(x,t)|^2 + \varepsilon)^{-1/2}, \varepsilon > 0.$$

Let us multiply the equation for the function $z$ by $\sigma_\varepsilon$ and then integrate over $\Omega$. We obtain the equality

$$\frac{d}{dt}\int_0^1 a(|z^{(0)}|^2+\varepsilon)^{1/2}dx+\int_0^1 \frac{\partial a}{\partial t}(z\sigma_\varepsilon-(|z^{(0)}|^2+\varepsilon)^{1/2})dx+ \tag{32}$$
$$+\varepsilon\int_0^1(Kb\frac{\partial z}{\partial x}\frac{\partial z^{(0)}}{\partial x}(|z^{(0)}|^2+\varepsilon)^{-3/2}-kcz\frac{\partial z^{(0)}}{\partial x}(|z^{(0)}|^2+\varepsilon)^{-3/2})dx=0.$$

Let

$$A^+(t)=\{x\in\Omega|\, z(x,t)>0\},\quad A^-(t)=\{x\in\Omega|\, z(x,t)\le 0\}.$$

Following [42], we get an estimates for each term:

$$\int_0^1\frac{\partial a}{\partial t}(z\sigma_\varepsilon-(|z^{(0)}|^2+\varepsilon)^{1/2})dx=-\varepsilon\int_{A^+(t)}\frac{\partial a}{\partial t}(|z|^2+\varepsilon)^{-1/2}dx-\varepsilon^{1/2}\int_{A^-(t)}\frac{\partial a}{\partial t}dx,$$

$$\int_0^1 a(|z^{(0)}|^2+\varepsilon)^{1/2}dx=\int_{A^+(t)}a(|z|^2+\varepsilon)^{1/2}dx+\varepsilon^{1/2}\int_{A^-(t)}adx,$$

$$\int_0^1 a(|z^{(0)}|^2+\varepsilon)^{1/2}\mid_{t=0}dx=\varepsilon^{1/2}\int_0^1 a\mid_{t=0}dx,$$

$$\int_{A^+(t)}a(|z|^2+\varepsilon)^{1/2}dx\ge\int_{A^+(t)}a|z|dx=\int_0^1 az^{(0)}dx.$$

Last term can be estimated as following:

$$\int_0^1\varepsilon kcz\frac{\partial z^{(0)}}{\partial x}(|z^{(0)}|^2+\varepsilon)^{-3/2}dx\le\frac{\varepsilon}{2}\int_{A^+(t)}(\frac{\partial z^{(0)}}{\partial x})^2(|z^{(0)}|^2+\varepsilon)^{-3/2}Kbdx+$$
$$+\frac{\varepsilon}{2}\int_{A^+(t)}(\frac{(kcz)^2}{Kb})^2(|z^{(0)}|^2+\varepsilon)^{-3/2}Kbdx$$

We have

$$\varepsilon z^2(|z^{(0)}|^2+\varepsilon)^{-3/2}\le\varepsilon(|z^{(0)}|^2+\varepsilon)^{-3/2}\le\sqrt{\varepsilon}\longrightarrow 0,\quad if\quad \varepsilon\to 0.$$

Integrating relation (32) with respect to time, we obtain

$$\int_{A^+(t)}a(|z|^2+\varepsilon)^{1/2}dx+\varepsilon^{1/2}\int_{A^-(t)}adx+\frac{\varepsilon}{2}\int_0^t\int_{A^+(\tau)}Kb|\frac{\partial z}{\partial x}|^2(z^2+\varepsilon)^{-3/2}dxd\tau=$$
$$=\varepsilon\int_0^t\int_{A^+(\tau)}\frac{\partial a}{\partial\tau}(|z|^2+\varepsilon)^{-1/2}dxd\tau+\varepsilon^{1/2}\int_0^t\int_{A^-(\tau)}\frac{\partial a}{\partial\tau}dxd\tau+\varepsilon^{1/2}\int_0^1 a\mid_{t=0}dx+$$
$$+\frac{\sqrt{\varepsilon}}{2}\int_0^t\int_{A^+(\tau)}\frac{(kc)^2}{Kb}dx.$$

Following [42], we get an estimate passing to the limit as $\varepsilon \to 0$, we find that $z^{(0)} = 0$, i.e. $U \geq 0$.

We can express problem (31) in the form:

$$U_t - \tilde{a}_{11} U_{xx} + \tilde{a}_1 U_x + \tilde{a} U = 0, \quad (U_x - \tilde{d} U)|_{x=0,1} = 0, \quad U\,|_{t=0} = p^0, \tag{33}$$

where

$$\tilde{a}_{11} = \frac{Kb}{a}, \quad \tilde{a}_1 = \frac{d - (Kb)_x}{a}, \quad \tilde{a} = \frac{a_t + (kc)_x}{a}.$$

Following [41], we move from function $U(x,t)$ to a new function $w(x,t)$ related to it by an equality $w(x,t) = e^{-\lambda t}\varphi(x) U(x,t)$, where

$$\varphi = -mx^2 + mx + 1 > 0, m \equiv 2 \max_{Q_t} |\tilde{d}| = \frac{4 g_0}{(1 - M_0) R m_2},$$

and number $\lambda$ will be indicated later.

Because of (33) the function $w$ is the solution of the equation

$$w_t - \tilde{a}_{11} w_{xx} + (\tilde{a}_1 + \frac{2\tilde{a}_{11}\varphi_x}{\varphi}) w_x + (-2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} + \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} - \tilde{a}_1 \frac{\varphi_x}{\varphi} + \tilde{a} + \lambda) w = 0,$$

$$w_x|_{x=0,1} = ((\frac{\varphi_x}{\varphi} + \tilde{d}) w)|_{x=0,1}, \quad w|_{t=0} = \varphi(x) p^0,$$

where $w_x|_{x=0} \geq 0, \quad w_x|_{x=1} \leq 0$, because $\varphi|_{x=0,1} = 1, \quad \varphi_x|_{x=0} = m > 0, \quad \varphi_x|_{x=1} = -m \equiv -2\max \tilde{d} < 0$. Choosing

$$\lambda > \max_{Q_t}[2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} - \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} + \tilde{a}_1\frac{\varphi_x}{\varphi} - \tilde{a}],$$

the function $w$ reaches a positive maximum at $t = 0$, such

$$U(x,t) e^{-\lambda t}\varphi(x) \leq \max_{Q_t}(U(x,t)e^{-\lambda t}\varphi(x)) = \max_{Q_t} w(x,t) \leq$$

$$\leq \max_x w|_{t=0} = \max_x (U(x,t)e^{-\lambda t}\varphi(x))|_{t=0}.$$

Therefore, we obtain an upper bound for $U$ :

$$U \leq e^{\lambda t} M_1 (1 + \frac{g_0}{(1 - M_0) R m_2}).$$

Then there is such a value $\tilde{t}_2 = \ln 2^{1/\lambda}$, that for all $t \leq \tilde{t}_2$ we have the estimate for $\rho$ from above from Lemma 1.

To obtain a lower estimate we represent equation (31) in the form $(z(x,t) = 1/U(x,t))$

$$z_t - \tilde{a}_{11} z_{xx} + \frac{2\tilde{a}_{11}}{z}(z_x^2) + \tilde{a}_1 z_x - \tilde{a} z = 0.$$

We move from function $z(x,t)$ to a new function $w_1(x,t)$ related to it by an equality $w_1(x,t) = e^{-\lambda_1 t}\varphi(x) z(x,t)$, where $\varphi$ defined as before for the upper estimate and number $\lambda_1$ will be indicated later.

The function $w_1$ is the solution of the problem

$$w_{1t} - \tilde{a}_{11} w_{1xx} + (\tilde{a}_1 + \frac{2\tilde{a}_{11}\varphi_x}{\varphi} - 4\tilde{a}_{11}\frac{\varphi_x}{\varphi^2}) w_{1x} + (-2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} + \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} - \tilde{a}_1\frac{\varphi_x}{\varphi} +$$

$$+ 2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^3} - \tilde{a} + \lambda_1) w_1 + 2\tilde{a}_{11}\frac{w_{1x}^2}{\varphi^2 w} = 0,$$

$$w_{1x}|_{x=0,1} = ((\frac{\varphi_x}{\varphi} - \tilde{d}) w_1)|_{x=0,1}, \quad w_1|_{t=0} = \varphi(x)\frac{1}{p^0},$$

where $w_{1x}|_{x=0} \geq 0, \quad w_{1x}|_{x=1} \leq 0$, because

$$\varphi|_{x=0,1} = 1, \quad \varphi_x|_{x=0} = m > 0, \quad \varphi_x|_{x=1} = -m \equiv -2\max \tilde{d} < 0.$$

Choosing

$$\lambda_1 > \max_{Q_t}[2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} - \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} + \tilde{a}_1\frac{\varphi_x}{\varphi} - 2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^3} + \tilde{a}],$$

the function $w_1$ reaches a positive maximum at $t = 0$, such

$$\max_{Q_t}(\frac{1}{U(x,t)}e^{-\lambda_1 t}\varphi(x)) = \max_{Q_t} w_1(x,t) \leq \max_x w_1|_{t=0} = \max_x(\frac{1}{U(x,t)}e^{-\lambda_1 t}\varphi(x))|_{t=0}.$$

Accordingly, we obtain

$$\frac{1}{U(x,t)} \leq \frac{1}{m_1}e^{\lambda_1 t}(1 + \frac{g_0}{R(1-M_0)m_2}).$$

Therefore

$$U(x,t) \geq m_1 e^{-\lambda_1 t}(1 + \frac{g_0}{R(1-M_0)m_2})^{-1}.$$

Then there exists such a value $\tilde{t}_3 = \ln 2^{1/\lambda_1}$, that for all $t \leq \tilde{t}_3$ we have the estimate for $\rho$ from Lemma 1. □

Choosing $t \leq t_2 = \min\{\tilde{t}_2, \tilde{t}_3\}$, we proof Lemma1.

Using (27), $\bar{p}, \bar{\theta}$ and $\bar{\omega}$, we find the function $\theta_1(x,t)$ as a solution of the problem (here and elsewhere, we assume that the initial and boundary conditions are matched):

$$(34) \quad \begin{gathered} (\frac{c_f(\bar{p}+p^0)}{R(\bar{\theta}+\theta^0)}\frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})} + c_s\rho_s)\frac{\partial\theta_1}{\partial t} = \frac{\partial}{\partial x}(\lambda(1-\phi(\bar{\omega}))\frac{\partial(\theta_1+\theta^0)}{\partial x})+ \\ +k(\phi(\bar{\omega}))\frac{c_f(\bar{p}+p^0)}{R(\bar{\theta}+\theta^0)}((1-\phi(\bar{\omega}))\frac{\partial(\bar{p}+p^0)}{\partial x} - \frac{g(\bar{p}+p^0)}{R(\bar{\theta}+\theta^0)})\frac{\partial(\theta_1+\theta^0)}{\partial x}, \\ \theta_1|_{t=0} = 0, \frac{\partial\theta_1}{\partial x}|_{x=0,x=1} = 0. \end{gathered}$$

Further, setting $\tilde{U} = \theta_1 + \theta^0$ in (34), we have a classical maximum principle for function $\tilde{U}$ : [41] $m_2 = \min_{x\in[0,1]}\theta^0 \leq \tilde{U} \leq \max_{x\in[0,1]}\theta^0 = M_2$. Therefore $m_2 - \theta^0 \leq \theta_1 \leq M_2 - \theta^0$, and we have the following estimates for the function $\theta_1$ :

$$|\theta_1|_{\alpha,\alpha/2,Q_{t_0}} \leq C_2(m_0, M_0, m_1, M_1, m_2, M_2, K_1),$$

$$|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_3(m_0, M_0, m_1, M_1, m_2, M_2, K_1, |\theta^0|_{2+\alpha,\omega,|\phi^0|_{2+\alpha,\omega},|\rho^0|_{2+\alpha,\omega}})\cdot$$
$$\cdot(1 + |\theta_{1x}|_{\alpha,\alpha/2,Q_{t_0}}).$$

Using compact attachments $C^{2+\alpha,1+\alpha/2}(Q_{t_0})$ in $C^{1+\alpha,\alpha/2}(Q_{t_0})$, we get

$$|\theta_{1x}|_{\alpha,\alpha/2,Q_{t_0}} \leq \varepsilon|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} + C_\varepsilon|\theta_1|_{\alpha,\alpha/2,Q_{t_0}},$$

where $C_\varepsilon = C_\varepsilon(\varepsilon)$ is the positive constant depending on $\varepsilon$. Then

$$|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_3(1 + \varepsilon|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} + C_\varepsilon C_5).$$

By choosing $\varepsilon$ we get

$$|\theta_1|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_4(m_0, M_0, m_1, M_1, m_2, M_2, K_1).$$

In view of Lemma 1 and the properties of $\bar{\omega}$, we have the following estimates [41]:

$$|\rho|_{\alpha,\alpha/2,Q_{t_0}} \leq C_5,$$

$$|\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \le C_6 \left(1 + |\rho^0|_{2+\alpha,\Omega} + |\bar{\rho}_x|_{\alpha,\alpha/2,Q_{t_0}} + |\bar{\omega}_t|_{\alpha,\alpha/2,Q_{t_0}} + |\bar{\omega}_x|_{\alpha,\alpha/2,Q_{t_0}}\right),$$

in which the constants $C_5, C_6$ depend on $K_1$, $m_0$, $m_1, m_2$, $M_0$, $M_1, M_2$. Therefore

$$|\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \le C_7(K_1, m_0, m_1, m_2, M_0, M_1, M_2).$$

Let $C_8 = max\{C_0, C_4, C_7\}$. Choose $K_2$ so that $C_8 \le (K_1 + K_2)/2$. Then, for $t_0 < \min(t_1, t_2, (K_1 + K_2)^{-1})$ we obtain

$$|\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \le K_1 + K_2, \quad |\omega|_{2+\alpha,1+\alpha/2,Q_{t_0}} \le K_1 + K_2.$$

It remains to verify conditions

$$|\rho|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \le K_1, \quad |\omega|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \le K_1, |\theta|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \le K_1.$$

Integrating equation (30), (33) with respect to time, we obtain

$$|\rho|_{0,Q_{t_0}} \le C_6 t_0, \quad |\theta|_{0,Q_{t_0}} \le C_6 t_0.$$

From the equation (28) we obtain $|\omega|_{0,Q_{t_0}} \le C_7 t_0$. Further, using for $\rho, \omega, \theta$ an inequality of the form [3]

$$|u|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \le C|u|^c_{2+\alpha,1+\alpha/2,Q_{t_0}} |u|^{1-c}_{0,Q_{t_0}}, \quad c = (1+\alpha)(2+\alpha)^{-1},$$

we find that there exists a sufficiently small value of $t_0$, depending on $K_1$ and $K_2$, such that the required estimates hold: $|\rho|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \le K_1$, $|\omega|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \le K_1$, $|\theta|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \le K_1$.

Thus, the operator $\Lambda$ maps the set $V$ into itself for sufficiently small values of $t_0$. Using the estimates obtained above, we can easily show the continuity of the operator $\Lambda$ in the norm of the space $C^{2+\beta,1+\beta/2}(\overline{Q}_{t_0})$. By the Tikhonov-Schauder theorem, there exists a fixed point $(\rho, \omega) \in V$ of the operator $\Lambda$.

Uniqueness is established in the standard way [43].

Theorem 1 is proved. □

## 4. The Case of Incompressible Medium

**Definition 2.** *By a solution of problem (9)–(14) we mean the set of functions* $\phi, \phi_t, \theta, v_s, v_f \in C^{2+\alpha,1+\beta}(Q_T), p_f, p_s \in C^{1+\alpha,1+\beta}(Q_T)$, *such that* $0 < \phi < 1$, $0 < \theta < \infty$. *These functions satisfy the equations (9)–(13) and the initial and boundary conditions (14) and are regarded as continuous functions in* $Q_T$.

**Theorem 2.** *Suppose that* $\rho_f = const > 0$ *and the data of problem (9)–(14) satisfies the following conditions: 1) the functions* $k(\phi), a_1(\phi), \lambda(\phi), \xi_1(\theta)$ *and their derivatives up to the second order are continuous for* $\phi \in (0,1)$, $\theta \in (0,\infty)$ *and satisfy the conditions*

$$k_0^{-1}\phi^{q_1}(1-\phi)^{q_2} \le k(\phi) \le k_0\phi^{q_3}(1-\phi)^{q_4},$$

$$k_0^{-1}\phi^{q_5}(1-\phi)^{q_6} \le \lambda(\phi) \le k_0\phi^{q_7}(1-\phi)^{q_8}, \xi_1(\theta) > 0, \theta \in (0,\infty),$$

$$a_1(\phi) = a_0(\phi)\phi^{\alpha_1}(1-\phi)^{\alpha_2-1}, \quad 0 < R_1 \le a_0(\phi) \le R_2 < \infty,$$

*where* $k_0, \alpha_i, R_i, i = 1,2$ *are positive constants,* $q_1, ..., q_8$ *are fixed real numbers,* $\mu(\theta) = const$. *2) the function* $g$*, the initial functions* $\phi^0$ *and* $\theta^0$ *satisfy the following smoothness conditions:*

$$g \in C^{1+\alpha,1+\beta}(\bar{Q}_T), \quad \theta^0, \phi^0 \in C^{2+\alpha}(\bar{\Omega}),$$

*and the inequalities*

$$0 < m_0 \le \phi^0(x) \le M_0 < 1, \quad 0 < m \le \theta^0(x) \le M < \infty, \quad |g(x,t)| \le g_0 < \infty,$$

$$x \in \bar{\Omega}, \quad t \in (0,T),$$

*where* $m_0, M_0, m, M, g_0$ *are given positive constants. Then problem (9)–(14) has a local solution, i.e., there exists a value of* $t_0$ *such that*

$$\phi(x,t), \phi_t(x,t), \theta(x,t) \in C^{2+\alpha,1+\beta}(\bar{Q}_{t_0}), (v_s(x,t), v_f(x,t)) \in C^{2+\alpha,\beta}(\bar{Q}_{t_0}),$$
$$(p_f(x,t), p_s(x,t)) \in C^{1+\alpha,\beta}(\bar{Q}_{t_0}).$$

*Moreover,* $0 < \phi(x,t) < 1$, $0 < \theta(x,t) < \infty$ *in* $\bar{Q}_{t_0}$.

**Theorem 3.** *Let, in addition to the conditions of Theorem 2, the functions*

$$k(\phi), \quad \xi(\phi,\theta)$$

*satisfy the conditions*

$$k(\phi) = \frac{K}{\mu}\phi, \quad a_1(\phi) = \phi^4,$$

*where* $K, \mu$ *are positive constants. Then for all* $t \in [0,T]$, $T < \infty$ *uniqueness solution of problem (9)–(14) exists, and there are numbers* $0 < m_1 < M_1 < 1$, $0 < m_2 < M_2$ *such that* $m_1 \le \phi(x,t) \le M_1$, $m_2 \le \theta(x,t) \le M_2$, $(x,t) \in Q_T$.

## 2. Local solvability

We first prove the local theorem.

*Proof.* When proving Theorems 2 and 3, it is convenient to use the Lagrange variables [39]. Suppose that $\bar{x} = \bar{x}(\tau, x, t)$ is a solution of the Cauchy problem

$$\frac{\partial \bar{x}}{\partial \tau} = v_s(\bar{x}, \tau), \quad \bar{x}\mid_{\tau=t} = x.$$

We set $\hat{x} = \bar{x}(0,x,t)$ and take $\hat{x}$ and $t$ for the new variables. Then $\hat{J}(\hat{x},t) = \frac{\partial \hat{x}}{\partial x}(x,t) = (1-\phi(\hat{x},t))/(1-\phi^0(\hat{x}))$ is the Jacobian of the transformation. Following [43], we rewrite the system (9)–(13):

$$\frac{\partial}{\partial t}\left(\frac{\phi}{1-\phi}\right) = \frac{\partial}{\partial x}\left(k(\phi)(1-\phi)\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G(\phi)}{\partial t}\right) - k(\phi)g(\rho_{tot}+\rho_f)\right), \tag{35}$$

$$\left((1-\phi)\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right) - g(\rho_{tot}+\rho_f)\right)\mid_{x=0,x=1} = 0, \quad \phi\mid_{t=0} = \phi^0(x), \tag{36}$$

$$\left(c_s\rho_s + c_f\rho_f\frac{\phi}{1-\phi}\right)\frac{\partial\theta}{\partial t} + c_f\rho_f\phi(v_f - v_s)\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\left(\lambda(1-\phi)\frac{\partial\theta}{\partial x}\right), \tag{37}$$

$$\frac{\partial\theta}{\partial x}\mid_{x=0,x=1} = 0, \quad \theta\mid_{t=0} = \theta^0(x), \tag{38}$$

$$\frac{\partial G(\phi)}{\partial t} = \xi_1(\theta)p_e, \quad \frac{dG}{d\phi} = \frac{1}{a_1(\phi)(1-\phi)}. \tag{39}$$

In the system (35) - (39), the basic equations are (35) and (37) for the required functions $\phi$ and $\theta$.

We substitute in the coefficients of the equation (35) and the boundary condition (36) instead of $\theta(x,t)$ an arbitrary smooth function $\theta_0(x,t) \in C^{2+\alpha_1,1+\beta_1}(Q_T)$, which satisfies the inequalities $0 < m \le \theta^0(x) \le M < \infty$. We retain the previous notation $\phi$ for solving the arising problem and the latter is called Problem I.

**Lemma 2.** *Let the data of problem I satisfy the conditions of the theorem. Then problem I has a unique local solution, i.e., there exists a value of $t_0$ such that*

$$(\phi, \phi_t) \in C^{2+\alpha,1+\beta}(Q_{t_0}), \phi \in (0,1).$$

*Proof.* Suppose that $z = \frac{1}{\xi_1(\theta_0)}\frac{\partial G}{\partial t}$, we arrive at the following problem for $G, z$ :

$$z = \frac{1}{\xi_1(\theta_0)}\frac{\partial G}{\partial t}, \quad G\,|_{t=0} = G(\phi^0) = G^0(x), \tag{40}$$

$$\frac{z}{d(G,\theta_0)} - \frac{\partial}{\partial x}\left(a(G)\frac{\partial z}{\partial x} - b(G)\right) = 0, \quad \left(a(G)\frac{\partial z}{\partial x} - b(G)\right)|_{x=0,x=1} = 0, \tag{41}$$

where

$$d(G,\theta_0))) = \frac{1-\phi(G)}{a_1(\phi(G))\xi_1(\theta_0)},$$

$$a(G) = k(\phi(G))(1-\phi(G)), \quad b(G) = k(\phi(G))g(\rho_{tot}+\rho_f).$$

Since $0 < m_0 \le \phi^0(x) \le M_0 < 1$ and the function $G(\phi)$ is monotone, then $G(m_0) \le G^0(x) \le G(M_0)$. From (40) when the inequality $\max_{(x,t)} |\xi_1(\theta)z(x,t)| \le c_0$ take place we have that there is a value $t_0$, such that for all $t \le t_0$ the estimates take place

$$\begin{gathered} G_1(m_0) = G(m_0) - c_0t_0 \le G(x,t) \le G(M_0) + c_0t_0 = G_2(M_0), \\ 0 \le G^{-1}(G_1(m_0)) \le \phi(x,t) \le G^{-1}(G_2(M_0)) < 1. \end{gathered} \tag{42}$$

Let $G_0(x,t)$ be a function continuous in $x$ and $t$, satisfying inequalities (42) and having a continuous derivative $\partial G_0/\partial x$ with respect to $x$, $t$. Substituting $G_0(x,t)$ instead of $G(x,t)$ into the coefficients of equation (41) and the boundary conditions, we arrive at a linear problem for $z$, in which $a > 0, b > 0$ and $d > 0$. The solution to this problem is unique. The existence follows, for example, from Hilbert's theorem [44] for ordinary linear equations of the second order. The $t$ variable plays the role of a parameter. Thus, $(z, z_x, z_{xx}) \in C(Q_{t_0})$. After finding $z(x,t)$, we can find a new value $G(x,t)$ from the equation (40). This value will satisfy the condition (42).

To prove the solvability of problem I, we use the method of successive approximations. Let $z^i(x,t)$ and $G^i(x,t)$ be a solution to the problem

$$\frac{\partial G^{i+1}}{\partial t} = \xi_1(\theta_0)z^{i+1}, \quad G^{i+1}(x,0) = G^0(x),$$

$$\frac{z^{i+1}}{d(G^i)} - \frac{\partial}{\partial x}\left(a(G^i)\frac{\partial z^{i+1}}{\partial x} - b(G^i)\right) = 0,$$

$$\left(a(G^i)\frac{\partial z^{i+1}}{\partial x} - b(G^i)\right)|_{x=0,x=1} = 0,$$

where $i = 0, 1, 2, \ldots$. Substituting $G^0(x)$ into the equation for $z$ at the first step, we find $z^1(x,t)$. After that, from the equation for $G$ we find $G^1(x,t)$, etc. There is a unique solution $z^i(x,t)$ and $G^i(x,t)$, satisfying (42) for each $i$. It is checked in a standard way that for a small value of $t_0$ the solutions $z^i(x,t)$, $G^i(x,t)$ and their derivatives up to the second order inclusive are bounded uniformly in $i$.

We put $y^{i+1} = z^{i+1} - z^i, \omega^{i+1} = G^{i+1} - G^i$. We have

$$\frac{\partial \omega^{i+1}}{\partial t} = \xi_1(\theta_0) y^{i+1}, \quad \omega^{i+1}\mid_{t=0} = 0,$$

$$\frac{y^{i+1}}{d(G^i)} + A_1\omega^i - \frac{\partial}{\partial x}\left(a y_x^{i+1} + A_2\omega^i\right) = 0,$$

$$(a y_x^{i+1} + A_2\omega^i)|_{x=0,x=1} = 0,$$

where the coefficients $A_1, A_2$ are easily recoverable and are limited. We have following inequalities

$$\int_0^1 (|y^{i+1}|^2 + |y_x^{i+1}|^2)dx \le c_1 \int_0^1 |\omega^i|^2 dx \le c_1 \max_x |\omega^i|^2,$$

$$\max_x |\omega^{i+1}| \le c_1 \int_0^t \max_x |y^{i+1}| d\tau,$$

where the constant $c_1$ does not depend on $i$. Taking into account the last inequality for the function $v^i(t) = \max_x |y^i(x,t)|^2$ we get $v^{i+1}(t) \le c_2 \int_0^t v^i(\tau)d\tau$ and therefore [41], $v^i(t) \le (c_2 T)^i v^0/i! \to 0$ for $i \to \infty$. After that it is easy to establish that the sequences $z^i, G^i$ are fundamental in $C(Q_{t_0})$ and have limits $z(x,t) \in C(Q_{t_0})$ and $G(x,t) \in C(Q_{t_0})$. The sequences $z_x^i, z_{xx}^i, G_t^i$ are also fundamental. Passing to the limit as $i \to \infty$, we obtain that the limit functions satisfy the problem (40), (41). The uniqueness of the solution is proved similarly to [45]. Increasing the smoothness of the initial data to those specified in the conditions of Theorem 2 allows us to obtain that $\phi(x,t), \phi_t(x,t) \in C^{2+\alpha,1+\beta}(\bar{Q}_{t_0})$.

Lemma 2 is proved. □

Substituting $\theta_0(x,t)$ and the solution to Problem I into the coefficients of equation (37), we arrive at a linear problem for $\theta(x,t)$ of the form

$$Q\frac{\partial \theta}{\partial t} + V\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\lambda(1-\phi)\frac{\partial \theta}{\partial x}\right),$$

$$\frac{\partial \theta}{\partial x}\mid_{x=0,x=1} = 0, \quad \theta\mid_{t=0} = \theta^0(x),$$

where

$$Q = \rho_s c_s + \rho_f c_f \frac{\phi}{1-\phi}, \quad V = c_f \rho_f \phi(v_f - v_s) = \rho_f c_f k(\phi)\left((1-\phi)\frac{\partial z}{\partial x} + g(\rho_{tot} + \rho_f)\right).$$

The unique solvability of this problem in Holder classes follows from [41], and the solution satisfies the estimate

$$0 < \underline{\theta} = \min_x \theta^0(x) \le \theta(x,t) \le \max_x \theta^0(x) = \bar{\theta} < \infty.$$

After these remarks, the local solvability of the problem (35) - (38) can easily be obtained using the Schauder theorem according to the scheme used in [45].

After finding $\phi, \theta$, the remaining functions from the system (9)–(13) can be defined as follows. We find the phase velocities from (9)

$$v_f(x,t) = -\frac{1}{\phi}\int_0^x \frac{\partial \phi}{\partial t} d\xi \in C^{2+\alpha,\beta}(Q_{t_0}),$$

$$v_s(x,t) = -\frac{1}{1-\phi}\int_0^x \frac{\partial(1-\phi)}{\partial t}d\xi \in C^{2+\alpha,\beta}(Q_{t_0}).$$

From (12) we find $p_{tot}(x,t) = p^0(t) - \int\limits_0^x \rho_{tot} g d\xi \in C^{3+\alpha,1+\beta}(Q_{t_0})$.

From (11) we have $p_e(x,t) = -\frac{\partial v_s}{\partial x}\xi(\phi,\theta) \in C^{1+\alpha,\beta}(Q_{t_0})$, then

$$p_f(x,t) = p_{tot} - p_e \in C^{1+\alpha,\beta}(Q_{t_0}), \quad p_s(x,t) = \frac{p_{tot}}{1-\phi} - \frac{\phi}{1-\phi}p_f \in C^{1+\alpha,\beta}(Q_{t_0}).$$

Theorem 2 is proved.

□

## 3. Global Solvability

Now we will prove the Theorem 3.

*Proof.* By Theorem 2, we will assume that on the interval $[0,t_0]$ there exists a solution to the problem (9)–(14), and $0 < \phi(x,t) < 1$, $0 < \theta(x,t) < \infty$, $x \in \Omega$, $t \in [0,t_0]$. After obtaining the necessary a priori estimates that do not depend on the value of $t_0$, the local solution can be continued to the entire segment $[0,T]$.

**Lemma 3.** *Under the conditions of Theorem 3, for all $t \in [0,T]$ the following relations hold:*

$$\int_0^1 s(x,t)dx = \int_0^1 s^0(x)dx, \quad s = \frac{\phi}{1-\phi}, \quad s^0 = s(x,0), \tag{43}$$

$$0 < \underline{\theta} \equiv \min_{x\in[0,1]} \theta^0(x) \le \theta(x,t) \le \max_{x\in[0,1]} \theta^0(x) \equiv \overline{\theta} < \infty, \tag{44}$$

$$\int_0^1 \frac{1}{\xi_1(\theta)}\frac{a_1}{1-\phi}\left(\frac{\partial G}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^1 k(\phi)(1-\phi)\left|\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)\right|^2 dx \le$$
$$\le \frac{1}{2}\int_0^1 \frac{k(\phi)}{1-\phi} g^2(\rho_{tot}+\rho_f)^2 dx \le N. \tag{45}$$

*Hereinafter, $N$ denotes a constant that depends only on the data of the problem (9)–(14) and does not depend on $t_0$.*

*Proof.* Let us integrate the equation (35) over $x$ from 0 to 1 and take into account the boundary condition (36). After integrating over time from 0 to the current value of $t$, we arrive at the equality (43).

The equation (37) is written in a divergent form:

$$\frac{\partial}{\partial t}\left(\theta(c_s\rho_s + c_f\rho_f\frac{\phi}{1-\phi})\right) + \frac{\partial}{\partial x}\left(\theta c_f\rho_f\phi(v_f - v_s) - \lambda(1-\phi)\frac{\partial\theta}{\partial x}\right) =$$
$$= \theta\left[\frac{\partial}{\partial t}\left(c_s\rho_s + c_f\rho_f\frac{\phi}{1-\phi}\right) + \frac{\partial}{\partial x}(c_f\rho_f\phi(v_f - v_s))\right]. \tag{46}$$

The right-hand side of this equation is equal to zero, since the second equation from (9) in Lagrange variables becomes [43]

$$\frac{\partial}{\partial t}\left(\frac{\phi}{1-\phi}\right) + \frac{\partial}{\partial x}\left(\phi(v_f - v_s)\right) = 0.$$

In particular, from (46) we have

$$\int_0^1 \left(c_f\rho_f\frac{\phi}{1-\phi}+c_s\rho_s\right)\theta dx=\int_0^1 \left(c_f\rho_f\frac{\phi^0}{1-\phi^0}+c_s\rho_s\right)\theta^0 dx,$$

and therefore $\theta(x,t)\in L_1[0,1]$ for all $t\in[0,T]$.

Let the smooth function $\kappa(\theta)$ satisfy the condition $\kappa''(\theta)=d^2\kappa/d\theta^2\geq 0$. Multiplying the equation (37) by $\kappa'(\theta)\equiv d\kappa/d\theta$, and following the equality (46) we reduce the resulting equality to the form

$$\frac{\partial}{\partial t}\left(\left(c_s\rho_s+c_f\rho_f\frac{\phi}{1-\phi}\right)\kappa(\theta)\right)+\frac{\partial}{\partial x}(c_f\rho_f\phi(v_f-v_s)\kappa(\theta))=$$

$$=\frac{\partial}{\partial x}\left(\lambda(1-\phi)\frac{\partial\kappa(\theta)}{\partial x}\right)-\kappa''(\theta)\left(\frac{\partial\theta}{\partial x}\right)^2\lambda(1-\phi). \tag{47}$$

In the case $\kappa(\theta)=\theta^p, p>1$, from (47) we deduce

$$\int_0^1 \theta^p(x,t)dx\leq \max_{x\in[0,1]}\left(\frac{c_f\rho_f}{c_s\rho_s}\frac{\phi^0(x)}{1-\phi^0(x)}+1\right)\int_0^1|\theta^0(x)|^p dx.$$

Whence, in the standard way, we get that $\theta(x,t)\leq \max_{x\in[0,1]}\theta^0(x)$ for all $t\in[0,T], x\in[0,1]$. Put $\theta_1=1/\theta$ and the equation (35) can be represented as

$$\left(c_s\rho_s+c_f\rho_f\frac{\phi}{1-\phi}\right)\frac{\partial\theta_1}{\partial t}+c_f\rho_f(v_f-v_s)\frac{\partial\theta_1}{\partial x}=$$

$$=\frac{\partial}{\partial x}\left(\lambda(1-\phi)\frac{\partial\theta_1}{\partial x}\right)-2\lambda(1-\phi)\left(\frac{\partial\theta_1}{\partial x}\right)^2\theta.$$

Multiplying (37) by $\kappa_1'(\theta_1)=d\kappa_1/d\theta_1$, $\kappa_1=\theta_1^p$, and integrating it over $x$, we arrive at the relation of the form (43) for $\theta_1(x,t)$. Therefore $\theta(x,t)\geq\min_{x\in[0,1]}\theta^0(x)$ for all $t\in[0,T], x\in[0,1]$.

Multiplying the equation (35) by $\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}$ and integrating it over $x$ we arrive at the relation

$$\int_0^1\frac{1}{\xi_1(\theta)}\frac{a_1(\phi)}{1-\phi}\left(\frac{\partial G}{\partial t}\right)^2dx+\int_0^1 k(\phi)(1-\phi)\left|\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)\right|^2dx=$$

$$=\int_0^1 k(\phi)g(\rho_{tot}+\rho_f)\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)dx\leq$$

$$\leq\frac{1}{2}\int_0^1 k(\phi)(1-\phi)\left|\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)\right|^2dx+\frac{1}{2}\int_0^1\frac{k(\phi)}{1-\phi}g^2(\rho_{tot}+\rho_f)^2dx.$$

The last term on the right-hand side is bounded uniformly in $t_0$, since $\phi<1$ and, therefore, $\rho_{tot}\leq\max(\rho_f,\rho_s)$. Finally, due to (43) we have

$$\int_0^1\frac{dx}{1-\phi}=1+\int_0^1 s^0(x)dx. \tag{48}$$

Lemma 3 is proved. □

**Lemma 4.** *Under the conditions of Theorem 3, for all $t \in [0,T], x \in [0,1]$ the estimate takes place*

$$0 < m \le \phi(x,t) \le M < 1. \tag{49}$$

*Proof.* Under the conditions of Theorem 3, the function $G$ has the form

$$G = ln|\phi| - ln|\phi - 1| - \frac{1}{\phi} - \frac{1}{2\phi^2} - \frac{1}{3\phi^3}.$$

It is monotonically increasing. Therefore, if we prove the boundedness of $G$, we come to the estimates of (49). To prove the boundedness of the function $G$, we first establish an embedding of the form

$$\frac{1}{\phi} \in L_1(0,1).$$

By virtue of the properties of the function $\xi_1(\theta)$ and (45) we have

$$\int_0^1 \frac{a_1(\phi)}{1-\phi}\left(\frac{\partial G}{\partial t}\right)^2 dx = \int_0^1 \frac{1}{\phi^4(1-\phi)^3}\left(\frac{\partial \phi}{\partial t}\right)^2 dx \le N \max_{x,t} \xi_1(\theta) \equiv N_1.$$

From the last inequality and (48) we have

$$\int_0^1 \frac{1}{\phi^2(1-\phi)^2}\left|\frac{\partial \phi}{\partial t}\right| dx \le \left(\int_0^1 \frac{dx}{1-\phi}\right)^{1/2}\left(\int_0^1 \frac{1}{\phi^4(1-\phi)^3}\left|\frac{\partial \phi}{\partial t}\right|^2 dx\right)^{1/2} \le$$

$$\le (1 + \int_0^1 s^0(x)dx)^{1/2} N_1^{1/2} \equiv N_2.$$

Therefore

$$\left|\frac{d}{dt}\int_0^1 r(\phi)dx\right| \le \int_0^1 \frac{1}{\phi^2(1-\phi)^2}\left|\frac{\partial \phi}{\partial t}\right| dx \le N_2,$$

where

$$r(\phi) = -\frac{1}{\phi} + \frac{1}{1-\phi} + 2ln\frac{\phi}{1-\phi}.$$

By integrating the last inequality over t from 0 to the current value, we obtain

$$-N_2 t + \int_0^1 r(\phi^0)dx \le \int_0^1 r(\phi)dx \le N_2 t + \int_0^1 r(\phi^0)dx.$$

From which it follows

$$-\frac{N_2 t}{3} - \frac{1}{3}\int_0^1 r(\phi^0)dx + \frac{1}{3}\int_0^1 s^0(x)dx + \frac{4}{3} \le \int_0^1 \frac{1}{\phi}dx$$

$$\le -1 + 3\int_0^1 s^0(x)dx + N_2 t - \int_0^1 r(\phi^0)dx,$$

and

$$\left|\int_0^1 \frac{1}{\phi}dx\right| \le N_3,$$

where

$$N_3 = \max\left\{\frac{N_2 T}{3} + \frac{1}{3}\int_0^1 r(\phi^0)dx - \frac{1}{3}\int_0^1 s^0(x)dx - \frac{4}{3},\right.$$

$$\left. -1 + 3\int_0^1 s^0(x)dx + N_2 T - \int_0^1 r(\phi^0)dx\right\}.$$

Finally, returning to (45), we have

$$\int_0^1 \left|\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)\right| dx$$

$$\leq \left(\int_0^1 k(\phi)(1-\phi)\left|\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)\right|^2 dx\right)^{1/2} \left(\int_0^1 \frac{1}{k(\phi)(1-\phi)}dx\right)^{1/2}$$

$$\leq (2N\frac{\mu}{K})^{1/2}\left(\int_0^1 \frac{1}{\phi(1-\phi)}dx\right)^{1/2}$$

$$\leq (2N\frac{\mu}{K})^{1/2}\left(N_3^{1/2}+(1+\int_0^1 s^0(x)dx)^{1/2}\right) \equiv N_4,$$

and there is a point $x_0(t)$ at which $\frac{\partial G}{\partial t}(x_0(t),t)=0$, therefore

$$\min_{x\in(0,1)}\left|\frac{1}{\xi_1(\theta)}\right|\left|\frac{\partial G}{\partial t}\right| \leq \left|\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right| \leq \int_0^1 \left|\frac{\partial}{\partial x}\left(\frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}\right)\right| dx \leq N_4.$$

Taking into account (44) and the conditions of Theorem 3, from the last inequality we have

$$|lns - \frac{1+s}{s} - \frac{1}{2}\frac{(1+s)^2}{s^2} - \frac{1}{3}\frac{(1+s)^3}{s^3}| = |G(x,t)| \leq |G^0(x)| + TN_4 \max_{x,t}\xi_1(\theta) \equiv N_5.$$

The constants m and M are restored from here as follows. If $s>1$, then from the last inequality follows

$$lns \leq N_5 + \frac{1}{s} + 1 + \frac{1}{2}\left(\frac{1}{s^2}+\frac{2}{s}+1\right) + \frac{1}{3}\left(\frac{1}{s^3}+\frac{3}{s}+\frac{3}{s^2}+1\right) \leq N_5 + 7.$$

Then we have

$$1 < s \leq e^{N_5+7}.$$

If $0<s<1$, then $ln\frac{1}{s} \leq N_5$ and we have the estimate $e^{-N_5} \leq s$. Therefore

$$m = \frac{1}{1+e^{N_5}}, \quad M = \frac{1}{1+e^{-N_5-7}}.$$

Thus we arrive at (49). □

Let $z = \frac{1}{\xi_1(\theta)}\frac{\partial G}{\partial t}$. The problem (35), (36) takes the form

$$\frac{a_1(\phi)\xi_1(\theta)z}{(1-\phi)} = \frac{\partial}{\partial x}\left(k(\phi)(1-\phi)\frac{\partial z}{\partial x} - k(\phi)g(\rho_{tot}+\rho_f)\right),$$

$$\left(k(\phi)(1-\phi)\frac{\partial z}{\partial x} - k(\phi)g(\rho_{tot}+\rho_f)\right)|_{x=0,x=1} = 0.$$

By Lemmas 3 and 4, we have

$$\int_0^t\int_0^1 \theta_x^2 dxd\tau + \int_0^1 (z^2+z_x^2)dx \leq N_6,$$

where $N_6$ is a positive constant depending on the initial data, parameters and problem constants, but does not depend on $t_0$.

Using the representation

$$G(\phi) = \int_0^t \xi_1(\theta)zd\tau + G(\phi^0),$$

we get

$$G'(\phi)\phi_x = \int_0^t (z_x\xi_1(\theta) + z\xi_1'\theta_x)d\tau + G_x(\phi^0).$$

Therefore

$$\int_0^1 \phi_x^2 dx \le N_7.$$

The equation for the function $z(x,t)$ takes form

$$a_0(\phi,\theta)z = a_1(\phi)z_{xx} + a_1'(\phi)\phi_x z_x + a_2'(\phi)\phi_x.$$

The coefficients $a_0(\phi,\theta) > 0$, $a_1(\phi) > 0$, $a_2(\phi)$ are limited and easy to calculate.

We have

$$\int_0^1 z_{xx}^2 dx \le C_1\left(\int_0^1 (z^2+\phi_x^2)dx + \int_0^1 |z_{xx}z_x\phi_x|dx\right),$$

where

$$I_1 = \int_0^1 |z_{xx}||z_x\phi_x|dx \le \max|z_x|\left(\int_0^1 z_{xx}^2 dx\right)^{1/2}\left(\int_0^1 \phi_x^2 dx\right)^{1/2} \le$$
$$\le C_1\left(\left(\int_0^1 z_{xx}^2 dx\right)^{1/2}\left(\int_0^1 \phi_x dx\right)^{1/2} + \left(\int_0^1 z_{xx}^2 dx\right)^{3/4}\left(\int_0^1 \phi_x dx\right)^{1/2}\right).$$

The constant $C_1$ does not depend on $t_0$.

Therefore

$$\max_x |z_x| + \int_0^1 z_{xx}^2 dx \le N_8.$$

The equation for the function $\theta(x,t)$ is given by

$$\theta_t + a_3(\phi,z_x)\theta_x = a_4(\phi)\theta_{xx} + a_5(\phi)\phi_x\theta_x,$$

where the coefficients $a_4(\phi) > 0$, $a_3(\phi,z_x)$, $a_5(\phi)$ are limited and easy to calculate.

Since

$$\int_0^1 |\theta_x\theta_{xx}\phi_x|dx \le \max_x|\theta_x|\left(\int_0^1 \theta_{xx}^2 dx\right)^{1/2}\left(\int_0^1 \phi_x^2 dx\right)^{1/2} \le$$
$$\le c\left(\int_0^1 \theta_{xx}^2 dx\right)^{3/4}\left(\int_0^1 \phi_x^2 dx\right)^{1/2}\left(\int_0^1 \theta_x^2 dx\right)^{1/4},$$

then from the equation for $\theta$ we have

$$\int_0^1 \theta_x^2 dx + \int_0^t\int_0^1 (\theta_t^2 + \theta_{xx}^2)dxd\tau \le N_9.$$

To complete the proof of Theorem 3, it is necessary to obtain the Holder continuity in $x,t$ of the functions $\phi_x$ and $z_x$ included in the coefficients of the equations for $z$ and $\theta$. From the embedding $z_{xx} \in L_2[0,1]$ and the representation for $\phi$ we have $\phi_{xx} \in L_2[0,1]$. Then for $w = \theta_x$ we get

$$\int_0^1 (\theta_t^2 + w_x^2)dx + \int_0^t\int_0^1 (w_t^2 + w_{xx}^2)dxd\tau \le N_{10}.$$

After that we deduce that $|\phi_{xt}| \leq N_{11}$. Finally, following [45] for the function $\sigma = z_t$ we get $\sigma_x \in L_2[0,1]$.

Theorem 3 is proved. □

## Conclusion

The local solvability in Holder classes of the initial - boundary value problem of one-dimensional fluid motion in a nonisothermal viscous porous medium is proved. An example of decidability is given at any finite time interval for the physical characteristics of the filtration coefficients and the viscosity of a liquid of a special type.

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