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AN OPEN MAPPING THEOREM FOR THE NAVIER-STOKES TYPE EQUATIONS ASSOCIATED WITH THE DE RHAM COMPLEX OVER $\mathbb{R}^n$

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Abstract. We consider an initial problem for the Navier-Stokes type equations associated with the de Rham complex over $\mathbb{R}^n \times [0,T]$, $n \geq 3$, with a positive time T. We prove that the problem induces an open injective mappings on the scales of specially constructed function spaces of Bochner-Sobolev type. In particular, the corresponding statement on the intersection of these classes gives an open mapping theorem for smooth solutions to the Navier-Stokes equations.

Keywords: Navier-Stokes equations, de Rham complex, open mapping theorem.

1. Introduction

The problem of describing the dynamics of incompressible viscous fluid is of great importance in applications. The principal problem consists in finding a sufficiently regular solution to the equations for which a uniqueness theorem is available, cf. [19]. Essential contributions has been published in the research articles [23, 24], [15], [14], as well as surveys and books [18]), [25, 26], [43], [8], etc.

We consider a family of a little bit more general problems associated with the de Rham complex. More precisely, denote by Λ^q the bundle of exterior forms of degree $0 \leq q \leq n$ over $\mathbb{R}^n$. We write $\Omega^q(\mathbb{R}^n)$ for the space of all differential forms of degree q with C^∞ coefficients on $\mathbb{R}^n$. These space constitute the so-called de Rham complex $\Omega^\cdot(\mathbb{R}^n)$ on $\mathbb{R}^n$ whose differential is given by the exterior derivative d. To

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display d acting on q-forms one uses the designation $du := d_q u$ for $u \in \Omega^q(\mathbb{R}^n)$ (see for instance [6], [41]); it is convenient to set $d_q = 0$ if $q < 0$ or $q \geq n$. As usual, denote by d_q^* the formal adjoint for d_q. Then, as it is known, we have

$$(1.1) \qquad d_{q+1} \circ d_q = 0, \qquad d_q^* d_q + d_{q-1} d_{q-1}^* = -E_{m(q)} \Delta, \quad 0 \leq q \leq n,$$

where E_m is the unit matrix of type $(m \times m)$ and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$ is the usual Laplace in the Euclidean space $\mathbb{R}^n$, $n \geq 2$.

We also consider the induced vector bundle $\Lambda^q(t)$ over $\mathbb{R}^n \times [0, +\infty)$ consisting of the differential forms with coefficients depending on both the variable $x \in \mathbb{R}^n$ and on the real parameter $t \in [0, +\infty)$.

In the sequel we consider the following Cauchy problem. Given any sufficiently regular differential forms $f = \sum_{\#I=q} f_I(x,t) dx_I$ and $u_0 = \sum_{\#I=q} u_{I,0}(x) dx_I$ on $\mathbb{R}^n \times [0,T]$ and $\mathbb{R}^n$, respectively, find a pair (u,p) of sufficiently regular differential forms $u = \sum_{\#I=q} u_I(x,t) dx_I$ and $p = \sum_{\#I=q-1} p_I(x,t) dx_I$ on $\mathbb{R}^n \times [0,T]$ satisfying

$$(1.2) \qquad \left\{ \begin{array}{rcll} \partial_t u - \mu \Delta u + \mathcal{N}_q u + d_{q-1} p & = & f, & (x,t) \in \mathbb{R}^n \times (0,T), \\ d_{q-1}^* u & = & 0, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u & = & u_0, & (x,t) \in \mathbb{R}^n \times \{0\} \end{array} \right.$$

with positive fixed numbers T and μ and a non-linear term $\mathcal{N}_q u$ that is specified by the following assumptions (cf. [27], [31] for more general problems in the context of elliptic differential complexes):

$$(1.3) \qquad \mathcal{N}_q u = M_1^{(q)}(d_q u, u) + d_{q-1} M_2^{(q)}(u,u)$$

with two bilinear differential operators with constant coefficients and of zero order:

$$(1.4) \qquad M_1^{(q)}(v,u) : \Omega^{q+1}(\mathbb{R}^n) \times \Omega^q(\mathbb{R}^n) \to \Omega^q(\mathbb{R}^n),$$

$$(1.5) \qquad M_2^{(q)}(v,u) : \Omega^q(\mathbb{R}^n) \times \Omega^q(\mathbb{R}^n) \to \Omega^{q-1}(\mathbb{R}^n).$$

For $n = 1$, $q = 0$ and $\mathcal{N}_0 u = u' u$ relations (1.2) reduce obviously to the Cauchy problem for Burgers' equation, [3].

Next, identifying the sections of the bundle Λ^0 with functions over $\mathbb{R}^n$ and the sections of the bundle Λ^1 with n-vector fields over $\mathbb{R}^n$, we may write $d_0 = \nabla$, $d_0^* = -\mathrm{div}$ where ∇ and div are the gradient operator and the divergence operator, respectively. In this way, the identification of the sections of the bundle Λ^2 with $n(n-1)/2$-vector functions over $\mathbb{R}^n$ yields $d_1 = \mathrm{rot}$, the rotation operator for $n = 3$. In particular, if we denote by $\star$ the $\star$-Hodge operator and by $\wedge$ the exterior product of differential forms then for $n = 3$, $q = 1$ and

$$(1.6) \qquad \mathcal{N}_1 u = (u \cdot \nabla) u = \star(\star d_1 u \wedge u) + d_0 |u|^2/2,$$

relations (1.2) are usually referred to as but the Navier-Stokes equations for incompressible fluid with given dynamical viscosity μ of the fluid under the consideration, density vector of outer forces f, the initial velocity u_0 and the search-for velocity vector field u and the pressure p of the flow, see for instance [22], [43].

Of course, motivated by both the uniqueness theorem and the physical reasons, one have to assume that the data and the solutions are essentially decreasing at the infinity. We do it using proper scale of Bocnher-Sobolev (Banach) spaces providing reasonable Lebesgue integrability for the coefficients of the froms and their derivatives under the consideration.

For $n = 2$ the most important results on the Navier-Stokes equations are due to J. Leray [23, 24] and O.A. Ladyzhenskaya [16]. After Leray [23, 24], a great attention was paid to weak solutions of the Navier-Stokes equations in cylindrical domains in $\mathbb{R}^3 \times [0, +\infty)$. E. Hopf [14] proved the existence of weak solutions to the Navier-Stokes equations satisfying reasonable estimates. However, in this full generality no uniqueness theorem for a weak solution has been known. On the other hand, under stronger conditions on the solution, it is unique, cf. [18, 19] who proved the existence of a smooth solution for the two-dimensional version of problem (1.2). Namely, beginning from the end of 1960-s, it is known that the uniqueness theorem and improvement of regularity actually follow from the existence of a weak solution in the Bochner class $L^{\mathfrak{s}}([0,T], L^{\mathfrak{r}}(\mathbb{R}^n))$ with Ladyzhenskaya-Prodi-Serrin numbers $\mathfrak{s}$, $\mathfrak{r}$, satisfying $2/\mathfrak{s} + n/\mathfrak{r} = 1$ and $\mathfrak{r} > n$, see [34], [38], [18] and [25, 26] (the limit case $n = r = 3$ was added to the list in [7]). On the other hand, the standard energy estimate provides the existence in $L^s([0,T], L^r(\mathbb{R}^n))$ with $2/s + n/r = n/2$, only. Obviously, the uniqueness/regularity class and the existence class coincide for $n = 2$, while the cases $n \geq 3$ require additional investigation. Thus, the scientific community was convinced that the principal problem is to provide conditions for local interior regularity of the solutions to the Navier-Stokes equations or to estimate the set of their singular points, cf., for instance, [4], [35]. Some attention was paid to the so-called periodic setting, where no questions of boundary regularity arise, see [38], [44]. Beginning from Leray [23, 24], many attempts were made to construct a counter-example to the existence of smooth solutions, see for instance, [45] by T. Tao or [36] by G. Seregin. Some arguments that smooth data may generate solutions to the Navier-Stokes equations with singularities for $n \geq 5$ were indicated in [33].

In the present paper we focus on the so-called stability property discovered by O. A. Ladyzhenskaya for the Navier-Stokes equations in some Bochner type spaces (see [18, Ch. 4, § 4, Theorems 10 and 11]). Namely, if for sufficiently regular data (f, u_0) there is a sufficiently regular solution (u, p) to the Navier-Stokes equations, then there is a neighbourhood of the data in which all elements admit solutions with the same regularity (cf. also [9] for the problem in the class of infinitely smooth vector fields with zero force). We extend this property to the spaces of sufficient smoothness, expressing it as an open mapping theorem for (1.2). As the case $n = 2$ is much more easy to handle proving existence/regularity theorems, we will be concentrated on $n \geq 3$.

Namely, we prove that if $n \geq 3$ then for each $0 \leq q \leq n - 1$ and for each finite positive T equations (1.2) induce open injective mappings of each Banach space of the scale under the consideration (Theorem 3.3). In particular, intersection of these classes with respect to the smoothness indexes $s \in \mathbb{N}$ gives an open mapping theorem for smooth solutions to (1.2) for each $T > 0$ and smooth data (Corollary 3.3). Finally, we use the standard topological arguments immediately implying that a nonempty open connected set in a topological vector space coincides with the space itself if and only if the set is closed.

The remaining case $q = n$ is degenerate in some sense; it can be treated in a different way, see Example 2.1.

We emphasize that we do not need any existence theorem for non-linear equations (1.2) in order to achieve the open mapping theorem in the Bochner-Sobolev type spaces. On the other hand, the open mapping theorem changes somehow the accents for investigations of existence theorems related to smooth solutions to (1.2). For instance, it implies that dealing with the problem in the the Bochner-Sobolev type spaces using apriori estimates, one should prove them for the elements of the pre-image of precompact sets, only (see Corollary 3.2). This echoes the idea of using the properness property to study nonlinear operator equations, see for instance [40].

2. Preliminaries

As usual, we denote by $\mathbb{Z}_+$ the set of all nonnegative integers including zero, and by $\mathbb{R}^n$ the Euclidean space of dimension $n \geq 2$ with coordinates $x = (x^1, \dots, x^n)$.

In the sequel we use systematically the Gronwall type lemma in the integral form for continuous functions.

Lemma 2.1. *Let $0 < \gamma \leq 1$ and $A \geq 0$ be constants and let B, C and Y be nonnegative continuous functions defined on a segment $[a, b]$. If moreover Y satisfies the integral inequality*

$$Y(t) \leq A + \int_a^t (B(s)Y(s) + C(s)(Y(s))^{1-\gamma}) ds$$

for all $t \in [a, b]$, then

$$Y(t) \leq \Big(A^\gamma \exp\Big(\gamma \int_a^t B(s)ds\Big) + \gamma \int_a^t C(s) \exp\Big(\gamma \int_s^t B(t')dt'\Big) ds\Big)^{1/\gamma}$$

for all $t \in [a, b]$.

Proof. See for instance [11] or [28, p. 353] for $\gamma = 1$ and [32] or [28, p. 360] for $0 < \gamma < 1$. □

Also the (discrete) Young inequality will be of frequent use in this paper. To wit, given any $N = 1, 2, \dots$, it follows that

$$\prod_{j=1}^N a_j \leq \sum_{j=1}^N \frac{a_j^{p_j}}{p_j} \tag{2.1}$$

for all positive numbers a_j and all numbers $p_j \geq 1$ satisfying $\displaystyle\sum_{j=1}^N 1/p_j = 1$.

We continue with introducing proper function spaces. For $p \in [1, +\infty)$, we denote by $L^p(\mathbb{R}^n)$ the usual Lebesgue space of functions on $\mathbb{R}^n$ with the standard norm. Of course, for $p = 2$ the norm is generated by the standard inner product and so $L^2(\mathbb{R}^n)$ is a Hilbert space. As usual, the scale $L^p(\mathbb{R}^n)$ continues to include the case $p = \infty$, too.

The integral Hölder inequality is one of the frequently used tools for us, to wit,

$$\| \prod_{j=1}^N a_j \|_{L^q(\mathbb{R}^n)} \leq \prod_{j=1}^N \|a_j\|_{L^{q_j}(\mathbb{R}^n)} \tag{2.2}$$

for all $a_j \in L^{q_j}(\mathbb{R}^n)$, provided that $q \geq 1$, $q_j \geq 1$ and $\displaystyle\sum_{j=1}^N 1/q_j = 1/q$, see for instance [1, Corollary 2.6].

For a domain $\mathcal{X}$ in $\mathbb{R}^n$, we denote by $C_0^\infty(\mathcal{X})$ the set of all C^∞ functions with compact support in $\mathcal{X}$. If $s = 1, 2, \dots$, we write $W^{s,p}(\mathcal{X})$ for the Sobolev space of all functions $u \in L^p(\mathcal{X})$ whose generalised partial derivatives up to order s belong to $L^p(\mathcal{X})$, equipping it with the standard norm. Then $\mathring{W}^{s,p}(\mathcal{X})$ denotes the closure of the subspace $C_0^\infty(\mathcal{X})$ in $W^{s,p}(\mathcal{X})$. The space $W^{s,p}_{\rm loc}(\mathcal{X})$ consists of functions belonging to $W^{s,p}(U)$ for each relatively compact domain $U \subset \mathcal{X}$.

We will fairly often use the fact that $C_0^\infty(\mathbb{R}^n)$ is dense in the normed space $W^{s,p}(\mathbb{R}^n)$ if $p \in [1,+\infty)$. Let also $\mathcal{D}'(\mathbb{R}^n)$ stand for the space of distributions over $\mathbb{R}^n$. The space $W^{s,p}_{\rm loc}(\mathbb{R}^n)$ consists of functions belonging to $W^{s,p}(\mathcal{X})$ for each relatively compact domain $\mathcal{X} \subset \mathbb{R}^n$.

As usual, in the case $p=2$ we simply write $H^s(\mathcal{X})$ instead of $W^{s,2}(\mathcal{X})$ equipping it with the standard inner product. It is convenient to identify $H^0(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

The scale of Sobolev spaces continues to include the case of negative s, too. We will use only the space $H^{-s}(\mathbb{R}^n)$ defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H^{-s}(\mathbb{R}^n)} = \sup_{\substack{v \in C_0^\infty(\mathbb{R}^n) \\ v \neq 0}} \frac{|(u,v)_{L^2(\mathbb{R}^n)}|}{\|v\|_{H^s(\mathbb{R}^n)}}.$$

It may be easily identified with the dual of $H^s(\mathbb{R}^n)$, see for instance [1, Theorem 3.12]. The the pairing $\langle \cdot, \cdot \rangle_s$ on $H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ is given by

$$\langle f, v \rangle_s = \lim_{m \to +\infty} (f_m, v)_{L^2(\mathbb{R}^n)}$$

where $v \in H^s(\mathbb{R}^n)$, $f \in H^{-s}(\mathbb{R}^n)$ and the sequence $\{f_m\} \subset C_0^\infty(\mathbb{R}^n)$ approximates f in $H^{-s}(\mathbb{R}^n)$.

The Lebesgue and the Sobolev spaces give us an important tool for obtaining apriori estimates for solutions the Cauchy problem: this is the family of the Gagliardo-Nirenberg inequalities, see [30]. More precisely, let us set for $1 \leq p \leq \infty$

$$\|D^j v\|_{L^p(\mathbb{R}^n)} = \max_{|\alpha|=j} \|\partial^\alpha v\|_{L^p(\mathbb{R}^n)}.$$

Then for all $v \in L^{q_0}(\mathbb{R}^n) \cap L^{s_0}(\mathbb{R}^n)$ such that $D^{j_0} v \in L^{p_0}(\mathbb{R}^n)$ and $D^{m_0} v \in L^{r_0}(\mathbb{R}^n)$ we have

$$\|D^{j_0} v\|_{L^{p_0}(\mathbb{R}^n)} \leq c \, \|D^{m_0} v\|^a_{L^{r_0}(\mathbb{R}^n)} \|v\|^{1-a}_{L^{q_0}(\mathbb{R}^n)} \tag{2.3}$$

with a positive constants $c = c^{(n)}_{j_0,m_0,s_0}(p_0,q_0,r_0)$ independent on v where

$$\frac{1}{p_0} = \frac{j_0}{n} + a\left(\frac{1}{r_0} - \frac{m_0}{n}\right) + \frac{(1-a)}{q_0} \text{ and } \frac{j_0}{m_0} \leq a \leq 1 \tag{2.4}$$

with the following exceptional cases:

1) if $j_0 = 0$, $m_0 r_0 < n$ and $q_0 = +\infty$ then it is necessary to assume additionally that either u tends to zero at infinity or $u \in L^{\tilde{q}}(\mathbb{R}^n)$ for some finite $\tilde{q} > 0$;

2) If $1 < r_0 < +\infty$ and $m_0 - j_0 - n/r_0$ is a non-negative integer then the inequality is valid only for $\frac{j_0}{m_0} \leq a < 1$.

Next, for $s = 0, 1, \ldots$ and $0 \leq \lambda < 1$, we denote by $C_b^{s,\lambda}(\mathbb{R}^n)$ the space of all s times continuously differentiable functions on $\mathbb{R}^n$ with finite norm

$$\|u\|_{C_b^{s,\lambda}(\mathbb{R}^n)} = \|u\|_{C_b^{s,0}(\mathbb{R}^n)} + \lambda \sum_{|\alpha| \leq s} [\partial^\alpha u]_{\lambda,\mathbb{R}^n},$$

where

$$\begin{array}{rcl} \|u\|_{C^{s,0}_b(\mathbb{R}^n)} & = & \displaystyle\sum_{|\alpha|\le s} \sup_{x\in\mathbb{R}^n} |\partial^\alpha u(x)|, \\ [u]_{\lambda,\mathbb{R}^n} & = & \displaystyle\sup_{\substack{x,y\in\mathbb{R}^n \\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\lambda}. \end{array}$$

If $0<\lambda<1$, these are the so-called Hölder spaces, see for instance [21, Ch. 1, § 1], [18, Ch. 1, § 1]. The normed spaces $C^{s,\lambda}_b(\mathbb{R}^n)$ with $s\in\mathbb{Z}_+$ and $\lambda\in[0,1)$ are known to be Banach spaces which admit the standard embedding theorems. As usual, $C^\infty_b(\mathbb{R}^n) = \cap_{s=0}^\infty C^{s,0}_b(\mathbb{R}^n)$ stands for the Fréchet space of C^∞-smooth functions endowed with the topology, induced by the family of seminorms $\{\|u\|_{C^{s,0}_b(\mathbb{R}^n)}\}_{s\in\mathbb{Z}_+}$.

We will use the symbol $L^p_{\Lambda^q}$ for the space of the differential forms u of degree q on $\mathbb{R}^n$ with components u_I in $L^p(\mathbb{R}^n)$. The space is endowed with the norm

$$\|u\|_{L^p_{\Lambda^q}} = \Big(\sum_{\#I=q}\int_{\mathbb{R}^n}|u_I(x)|^p dx\Big)^{1/p}.$$

In a similar way we designate the spaces of forms on $\mathbb{R}^n$ whose components are of Sobolev or Hölder class. We thus get $W^{s,p}_{\Lambda^q}$, $H^s_{\Lambda^q}$ and $C^{s,\lambda}_{b,\Lambda^q}$, respectively. By C^∞_{b,Λ^q} and $\mathcal{D}'_{\Lambda^q}$ are meant the spaces q-forms with coefficients of C^∞ or distribution class on $\mathbb{R}^n$.

In order to avoid fractional powers of the Laplace operator, for $j\in\mathbb{N}$ and a form u of degree q we set

$$\Delta^{j/2}u = \begin{cases} \Delta^{j/2}u, & j \text{ is even}, \\ (d_q\oplus d^*_{q-1})\Delta^{(j-1)/2}u, & j \text{ is odd}. \end{cases}$$

Then the integration by parts and (1.1) yield

$$\sum_{|\alpha|=j}\|\partial^\alpha u\|^2_{L^2_{\Lambda^q}} = \begin{cases} \|\Delta^{j/2}u\|^2_{L^2_{\Lambda^q}}, & j \text{ is even}, \\ \|\Delta^{j/2}u\|^2_{L^2_{\Lambda^{q+1}\oplus\Lambda^{q-1}}}, & j \text{ is odd}. \end{cases}$$

for each $u\in H^j_{\Lambda^q}$, $j\in\mathbb{Z}_+$.

Remark 2.1. *As all the norms on a finite dimensional space are equivalent, we see that there are positive constants c_1, c_2 such that*

$$c_1\|\Delta^{j/2}v\|^2_{L^2_{\Lambda^q}} \le \|D^j v\|_{L^2_{\Lambda^q}} \le c_2\|\Delta^{j/2}v\|^2_{L^2_{\Lambda^q}}$$

for all $v\in H^j_{\Lambda^q}$. Thus, in the special case $p=2$ we may always replace the norm $\|D^j u\|_{L^p_{\Lambda^q}}$ with the norm

$$\|(-\Delta)^{j/2}u\|_{L^2_{\Lambda^q}} := \|\nabla^j u\|_{L^2_{\Lambda^q}}.$$

Let us begin to comment problem (1.2) for different q.

Example 2.1. *For $q=0$ identifying the sections of the bundle Λ^0 with functions over $\mathbb{R}^n$ and the sections of the bundle Λ^1 with n-vector fields over $\mathbb{R}^n$, we may write $d_0=\nabla$, $d_0^*=-\mathrm{div}$. As $d_{-1}=0$, $d^*_{-1}=0$ then relations* (1.2) *reduce obviously to the Cauchy problem*

$$(2.5)\qquad \begin{cases} \partial_t u - \mu\Delta u + \mathcal{N}_0 u = f, & (x,t)\in\mathbb{R}^n\times(0,T), \\ u = u_0, & (x,t)\in\mathbb{R}^n\times\{0\} \end{cases}$$

Taking in account that the non-linearity is of type (1.3) *we see that*

$$\mathcal{N}_0 u = u \sum_{j=1} c_j \partial_j u$$

with some constants c_j*. Thus, for* $n = 1$ *we arrive to the Burgers' equation,* [3] *with* $\mathcal{N}_0 u = u' u$*. As it is known, it can be reduced to the heat equation by the Hopf-Cole transformation, see* [13] *and* [5] *and then it can be treated within the frame of linear theory.*

Example 2.2. *For* $q = n$ *identifying the sections of the bundle* Λ^n *with functions over* $\mathbb{R}^n$ *and the sections of the bundle* Λ^{n-1} *with* n*-vector fields over* $\mathbb{R}^n$*, we may write* $d_{n-1} = \operatorname{div}$, $d^*_{n-1} = -\nabla$*. As* $d_n = 0$, $d^*_n = 0$ *then relations* (1.2) *reduce to the following:*

$$\begin{cases} \partial_t u - \mu \Delta u + \operatorname{div} p + \mathcal{N}_0 u &= f, \quad (x,t) \in \mathbb{R}^n \times (0,T), \\ \nabla u &= 0, \quad (x,t) \in \mathbb{R}^n \times (0,T), \\ u &= u_0, \quad (x,t) \in \mathbb{R}^n \times \{0\} \end{cases}$$

Taking in account that the solutions of the gradient operator do not depend on the space variables x *we arrive at the following linear problem:*

$$\begin{cases} \frac{du(t)}{dt} + \operatorname{div} p(x,t) &= f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T), \\ u(0) &= u_0, \quad u_0 \in \mathbb{R}. \end{cases} \tag{2.6}$$

If we assume that u *is vanishing at the infinity with respect to the space variables then immediately* $u \equiv 0$ *and then* $u_0 = 0$ *and the remaining equation*

$$\operatorname{div} p(x,t) = f(x,t) \text{ in } \mathbb{R}^n \times (0,T)$$

is always uniquely solvable in reasonable function spaces under reasonable additional assumptions, cf. Proposition 2.2 below. However, without this assumption we can not achieve uniqueness even for $n = 1$.

Indeed, if $n = 1$ *and* $u_0 = 0$, $f = 0$ *then*

$$p(x,t) = -x \frac{du(t)}{dt} + a(t)$$

with an arbitrary function $a(t)$ *and an arbitrary function* $u(t)$ *satisfying* $u(0) = 0$.

As the case $n = 2$ is much more easy to handle proving existence/regularity theorems, we will be concentrated on $n \geq 3$. Thus, according Examples 2.1, 2.2 we can limit ourselves to the cases where $n \geq 3$ and $0 \leq q < n$.

Let us introduce the function spaces directly related to Navier-Stokes type equations (1.2). With this purpose we denote by $\mathcal{V}_{\Lambda^q}$ the subspace of C^∞_{0,Λ^q}, consisting of all differential form with the coefficients from $C^\infty_0(\mathbb{R}^n)$ satisfying $d^*_{q-1} u = 0$ in $\mathbb{R}^n$. In particular, $\mathcal{V}_{\Lambda^0} = C^\infty_0(\mathbb{R}^n)$, $\mathcal{V}_{\Lambda^n} = \{0\}$ and $\mathcal{V}_{\Lambda^1}$ can be easily identified with the space $\mathcal{V}$ of all divergence-free n-vector fields with components from $C^\infty_0(\mathbb{R}^n)$.

Fix a non-zero function $h_0 \in C^\infty_0(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} h_0(x)\, dx = 1. \tag{2.7}$$

The following two proposition are well known.

Proposition 2.1. *Let* $n \geq 2$ *and* $F \in \mathcal{D}'_{\Lambda^1}$*. The following conditions are equivalent:*

(1) *there is a function* $p \in \mathcal{D}'(\mathbb{R}^n)$ *with*

$$\nabla p = F \text{ in } \mathbb{R}^n; \tag{2.8}$$

(2) *F satisfies* $\operatorname{rot} F = 0$ *in the sense of distributions in* $\mathbb{R}^n$*;*
(3) *F satisfies* $\langle F, v\rangle_{\Lambda^1} = 0$ *for all* $v \in \mathcal{V}$*.*

If $F \in H^{s-1}_{\Lambda^1}$, $s \in \mathbb{Z}$, *then* $p \in H^s(\mathbb{R}^n)$ *for* $s < 0$ *and* $p \in H^s_{loc}(\mathbb{R}^n)$ *for* $s \geq 0$. *Under assumption* (2) *equation* (2.8) *has the unique solution satisfying*

$$\langle p, h_0\rangle = 0. \tag{2.9}$$

Proof. Follows from [6, Theorem 17′]. because $\mathbb{R}^n$ is a star-like domain. □

In the case where $1 < q \leq n$ we need some additional information on the behaviour of solutions to d-equations at the infinity.

Proposition 2.2. *Let* $n \geq 2$, $1 \leq q \leq n$ *and* $F \in \mathcal{D}'_{\Lambda^q}$. *The following conditions are equivalent:*

(1) *there is a form* $p \in \mathcal{D}'_{\Lambda^{q-1}}$ *with*

$$d_{q-1} p = F \ \textit{in}\ \mathbb{R}^n; \tag{2.10}$$

(2) *F satisfies* $d_q F = 0$ *in the sense of distributions in* $\mathbb{R}^n$*;*
(3) *F satisfies* $\langle F, v\rangle_{\Lambda^q} = 0$ *for all* $v \in \mathcal{V}_{\Lambda^q}$*.*

Under assumption (2) *equation* (2.10) *has a solution* $\tilde{p} \in \mathcal{D}'_{\Lambda^{q-1}}$ *satisfying*

$$d_{q-2} d^*_{q-2} \tilde{p} = 0 \ \textit{in}\ \mathbb{R}^n; \tag{2.11}$$

if $F \in H^{s-1}_{\Lambda^q}$, $s \in \mathbb{Z}$, *then* $\tilde{p} \in H^s_{\Lambda^{q-1}}$ *for* $s < 0$ *and* $\tilde{p} \in H^s_{loc,\Lambda^{q-1}}$ *for* $s \geq 0$. *Moreover, if* $n \geq 3$ *and* $F \in H^{s-1}_{\Lambda^q}$, $s > n/2$, *then under assumption* (2) *equation* (2.10) *has the unique solution* $p^{(0)} \in H^s_{loc,\Lambda^{q-1}}$ *satisfying*

$$d^*_{q-2} p^{(0)} = 0 \ \textit{in}\ \mathbb{R}^n; \tag{2.12}$$

$$\langle p^{(0)}_I, h_0\rangle = 0 \ \textit{for all}\ \#I = q-1, \tag{2.13}$$

$$\|p^{(0)}\|_{C_{b,\Lambda^{q-1}}} < +\infty. \tag{2.14}$$

Proof. We may follow the same scheme as the proof of Proposition 2.1. The equivalence of (1) and (3) follows from [6, Theorem 17′]. The equivalence of (1) and (2) is true because $\mathbb{R}^n$ is a star-like domain.

Moreover, as there is a distribution $G \in \mathcal{D}'_{\Lambda^{q-2}}$ with

$$\Delta G = d^*_{q-2} p \text{ in } \mathbb{R}^n$$

we see that the form $\tilde{p} = p - d_{q-2} G$ satisfies

$$d_{q-1}\tilde{p} = F,\ d_{q-2} d^*_{q-2}\tilde{p} = 0$$

in the sense of distributions in $\mathbb{R}^n$. In this case, if $F \in H^{s-1}_{\Lambda^q}$ then

$$\Delta \tilde{p} = d^*_{q-1} F \in H^{s-2}_{\Lambda^{q-1}}$$

and hence $\tilde{p} \in H^s_{\mathrm{loc},\Lambda^{q-1}}$ by the elliptic regularity, and even $\tilde{p} \in H^s_{\Lambda^{q-1}}$ for $s < 0$.

Let $\varphi_n(x)$ the standard two-sided fundamental solution of the convolution type to the Laplace operator in $\mathbb{R}^n$, $n \geq 3$,

$$\varphi_n(x) = \frac{1}{\sigma_n} \frac{|x|^{2-n}}{2-n},$$

where σ_n the area of the unit sphere in $\mathbb{R}^n$. As usual, we define the kernels

$$\varphi_n^{(q)}(x,y) = \sum_{\#I=q} \varphi_n(x-y)(\star dy_I)dx_I.$$

For $n \geq 3$ we consider the potentials

$$p_q^{(1)}(x) = \sum_{\#I=q-1} \int_{\mathbb{R}^n} F(y) \wedge d^*_{n-q}(y)\varphi_n^{(q-1)}(x,y).$$

First, we see that

$$\partial^\alpha |\varphi_n(x-y)| \leq c_\alpha \, |x-y|^{2-n-|\alpha|} \tag{2.15}$$

with a constant $c_\alpha > 0$ independent on x, y if $\alpha \in \mathbb{Z}_+$ and then we have

$$\sum_{\#I=q-1} \Big| \int_{|y-x|\geq 1} F(y) \wedge d^*_{n-q}(y)\varphi_n(x-y)(\star dy_I)dx_I \Big| \leq$$

$$C\|F\|_{L^2_{\Lambda^q}(\mathbb{R}^n \setminus B(x,1))} \| |x-y|^{1-n} \|_{L^2(\mathbb{R}^n \setminus B(x,1))} \leq c\|F\|_{L^2_{\Lambda^q}} \tag{2.16}$$

with a constant c independent on x, y because $n \geq 3$ (here $B(x,1)$ is the unit ball in $\mathbb{R}^n$ with the center ar x). Besides, as $s > n/2$, the Gagliardo-Nirenberg inequality (2.3) implies that the space $H^{s-1}(\mathbb{R}^n)$ is embedded continuously into L^p with some $p > n$ (for example, $p = \frac{2n}{n-2s+2}$ if $s < (n+1)/2$). Hence the dual number $p' = \frac{p}{p-1}$ is less than $\frac{n}{n-1}$ and hence

$$\sum_{\#I=q-1} \int_{|y-x|\leq 1} \Big| F(y) \wedge d^*_{n-q}(y)\varphi_n(x-y)(\star dy_I)dx_I \Big| \leq$$

$$\tilde{c}\|F\|_{L^p_{\Lambda^q}(B(x,1))} \| |x-y|^{1-n} \|_{L^{p'}(B(x,1))} \leq c\|F\|_{L^2_{\Lambda^q}} \tag{2.17}$$

with a constant c independent on x, y because $p'(n-1) < n$.

Thus, the potential $p^{(1)}(x)$ converges for each x absolutely and it defines a differential form with bounded coefficients over $\mathbb{R}^n$ such that

$$\|p^{(1)}\|_{C_{b,\Lambda^{q-1}}} \leq c_q \|F\|_{H^{s-1}_{\Lambda^q}} \tag{2.18}$$

with a constant $c_q > 0$ independent on F.

Next, as φ_n is the fundamental solution of the convolution type to the Laplace operator in $\mathbb{R}^n$, (1.1) implies that the distribution $p^{(1)} \in \mathcal{D}'_{\Lambda^{q-1}}$ satisfies

$$\Delta p^{(1)} = d^*_{q-1} F \text{ in } \mathbb{R}^n$$

in the sense of distributions. Hence, by the elliptic regularity, $p^{(1)} \in H^s_{loc,\Lambda^{q-1}}$ if $F \in H^{s-1}_{\Lambda^q}$, $s > n/2$. In particular, by the Sobolev Embedding theorem, we may consider the coefficients of the form $p^{(1)}$ as continuous functions over $\mathbb{R}^n$.

Next, the Norguet integral formula for the de Rham complex, see, for instance, [41, Corollary 2.5.6] suggests that $p^{(1)}$ satisfies (2.10) and (2.12) for closed forms F with sufficient decay at the infinity. Let us show that it is the case if $F \in H^{s-1}_{\Lambda^q}$, $s > n/2$.

Indeed, by (1.1),

$$d_q \Delta = \Delta d_q, \; d^*_q \Delta = \Delta d^*_q$$

and then

$$d_q \varphi_n^{(q)} = \varphi_n^{(q+1)} d_q, \; d^*_q \varphi_n^{(q)} = \varphi_n^{(q-1)} d^*_q, \tag{2.19}$$

on the forms having the coefficients with compact supports in $\mathbb{R}^n$.

Fix an arbitrary differential form $h \in D_{\Lambda^{q-2}}$. Then, by (1.1) and (2.19),

$$\langle p^{(1)}, d_{q-2}h\rangle_{\Lambda^{q-1}} = \int_{\mathbb{R}^n} F(y) \wedge d^*_{n-q}(y)(y) \int_{\mathbb{R}^n} \varphi_n^{(n-q+1)}(x,y) \wedge \star d_{q-2}(x)h(x) =$$

$$\int_{\mathbb{R}^n} F(y) \wedge d^*_{n-q}(y) d^*_{n-q+1}(y) \int_{\mathbb{R}^n} \varphi_n^{(n-q+2)}(x,y) \wedge \star h(x) = 0,$$

i.e. $p^{(1)}$ satisfies (2.12).

Next, for an arbitrary differential form $h \in D_{\Lambda^q}$ we have

$$\langle p^{(1)}, d^*_{q-1}h\rangle_{\Lambda^{q-1}} = \int_{\mathbb{R}^n} F(y) \wedge d^*_{n-q}(y) \int_{\mathbb{R}^n} \varphi_n^{(n-q+1)}(x,y) \wedge \star d^*_{q-1}(x)h(x) =$$

$$\int_{\mathbb{R}^n} F(y) \wedge d^*_{n-q}(y) d_{n-q}(y) \int_{\mathbb{R}^n} \varphi_n^{(n-q)}(x,y) \wedge \star h(x) =$$

$$\langle F, h\rangle_{\Lambda^q} - \int_{\mathbb{R}^n} F \wedge d_{n-q-1} d^*_{n-q-1} \varphi_n^{(n-q)}(\star h)$$

because φ_n is the fundamental solution to the Laplacian Δ.

However, $d_q F = 0$ and then, integrating by part we arrive at the following:

$$\int_{\mathbb{R}^n} F \wedge d_{n-q-1} d^*_{n-q-1} \varphi_n^{(n-q)}(\star h) = (-1)^q \lim_{R\to+\infty} \int_{|y|=R} F \wedge d^*_{n-q-1} \varphi_n^{(n-q)}(\star h).$$

If the last limit does not equal to zero then there is the positive limit

$$\lim_{R\to+\infty} \int_{|y|=R} |F \wedge d^*_{n-q-1} \varphi_n^{(n-q)}(\star h)|$$

and therefore the integral

$$\int_0^\infty \int_{|y|=R} |F \wedge d^*_{n-q-1} \varphi_n^{(n-q)}(\star h)| dR = \int_{\mathbb{R}^n} |F \wedge d^*_{n-q-1} \varphi_n^{(n-q)}(\star h)| dx$$

diverges. On the other hand, as $h \in D_{\Lambda^q}$, using (2.15) we see that

$$|d^*_{n-q-1} \varphi_n^{(n-q)}(\star h)(x)| \le C(h)|x|^{1-n}$$

with a positive constant $C(h)$ independent on x. Then arguing as at the beginning of the proof we conclude (see formulae (2.16), (2.17)) that this integral is convergent if $F \in H^{s-1}_{\Lambda^q}$, $s > n/2$. Thus $p^{(1)}$ satisfies (2.10).

Finally, if p and $\tilde{p}$ are bounded solutions to (2.10) satisfying (2.10) and (2.12) then the difference $(p - \tilde{p})$ is harmonic over $\mathbb{R}^n$, and therefore it is a constant in $\mathbb{R}^n$ by the Liouville Theorem. If in addition p and $\tilde{p}$ satisfy (2.13) then, according to (2.7),

$$0 = \langle p_I, h_0\rangle - \langle \tilde{p}_I, h_0\rangle = \langle p_I - \tilde{p}_I, h_0\rangle = (p_I - \tilde{p}_I) \int_{\mathbb{R}^n} h_0(x)\, dx = p_I - \tilde{p}_I,$$

for all I with $\#I = q-1$, i.e. p and $\tilde{p}$ coincide. Moreover, as $p^{(1)}$ satisfies to (2.10), (2.12) and (2.18) then the form

$$p^{(0)} = p^{(1)} - \sum_{\#I=q-1} \langle p_I^{(1)}, h_0\rangle\, dx_I$$

is solution to (2.10), (2.12) satisfying (2.13), (2.14) because $\langle p_I^{(1)}, h_0\rangle$ are constants. □

Next, for $s \in \mathbb{Z}_+$, the closure of $\mathcal{V}_{\Lambda^q}$ in the space $H^s_{\Lambda^q}$ will be denoted by $\mathbf{H}_{s,\Lambda^q}$. As usual, the duals $\mathbf{H}'_{s,\Lambda^q}$ of $\mathbf{H}_{s,\Lambda^q}$ can be identified with the completion of C^∞_{0,Λ^q} with respect to the norm

$$\|w\|_{\mathbf{H}'_{s,\Lambda^q}} = \sup_{\substack{v \in \mathcal{V}_{\Lambda^q} \\ v \neq 0}} \frac{|(w,v)_{L^2_{\Lambda^q}(\mathbb{R}^n)}|}{\|v\|_{H^s_{\Lambda^q}}}. \tag{2.20}$$

The characterization of the space $\mathbf{H}_1 = \mathbf{H}_{1,\Lambda^1}$ may be found in many books and papers, see for instance, [20, Lemma 3.1] or [43, §1.4] for $n = 3$.

Proposition 2.3. *Let $s \in \mathbb{Z}_+$, $0 \leq q \leq n$. The space $\mathbf{H}_{s,\Lambda^q}$ coincides with the subspace $\tilde{\mathbf{H}}_{s,\Lambda^q}$ of the space $H^s_{\Lambda^q}$, consisting of all the differential forms w satisfying $d^*_{q-1} w = 0$ in $\mathbb{R}^n$ in the sense of distributions theory.*

Proof. For $q = 0$ we have $d^*_{q-1} = 0$, $\mathcal{V}_{\Lambda^0} = C^\infty_0(\mathbb{R}^n)$ and then the statement holds because $C^\infty_0(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. As we mentioned above, for $q = n$ we have $\mathcal{V}_{\Lambda^n} = \{0\}$ and the operator d^*_{n-1} can be identified with the gradient operator. Hence $\mathbf{H}_{s,\Lambda^q} = \tilde{\mathbf{H}}_{s,\Lambda^q} = \{0\}$ because there are no constants in the space $H^s(\mathbb{R}^n)$ for $s \geq 0$. As we have mentioned above, for $q = 1$ the statement is known to be true.

Thus, it is left to consider $n \geq 3$ and q with $2 \leq q \leq n-1$.

By the definition, the space $\mathbf{H}_s \subset \tilde{\mathbf{H}}_s$ is a (closed) subspace $H^s_{\Lambda^q}$. Next, the differential operator d^*_{q-1} induces bounded linear operator in the Hilbert spaces:

$$d^*_{q-1} : H^s_{\Lambda^q} \to H^{s-1}_{\Lambda^{q-1}},\ s \in \mathbb{Z}_+.$$

Thus the space $\tilde{\mathbf{H}}_s$ is a (closed) subspace of $H^s_{\Lambda^q}$, representing the null spaces of the continuous operator d^*_{q-1}.

Let $w \in \tilde{\mathbf{H}}_{s,\Lambda^q}$, such that for all $v \in \mathcal{V}_{\Lambda^q}$ we have

$$0 = (w,v)_{H^s_{\Lambda^q}} = \langle L_s w, v\rangle_{s,\Lambda^q} \tag{2.21}$$

where $L_s w = \sum_{|\alpha|\leq s} (-1)^{|\alpha|} \partial^{2\alpha} w \in H^{-s}_{\Lambda^q}$ is form with the coefficients being distributions in $\mathbb{R}^n$. Besides, by the definition of the differential operator L_s we have

$$\langle L_s w, w\rangle_{s,\Lambda^q} = \|w\|^2_{H^s_{\Lambda^q}}. \tag{2.22}$$

If we treat the distribution $L_s w$ as a current over $\mathbb{R}^n$, using Propositions 2.1 and 2.2 we conclude that there is a differential form $p \in \mathcal{D}'_{\Lambda^{q-1}}$ such that

$$d_{q-1}\tilde{p} = L_s w,\ d_{q-2} d^*_{q-2}\tilde{p} = 0$$

in the sense of distributions in $\mathbb{R}^n$; $\tilde{p} \in H^{1-s}_{\Lambda^{q-1}}$ for $s \geq 2$ and $\tilde{p} \in H^{1-s}_{\mathrm{loc},\Lambda^{q-1}}$ for $0 \leq s \leq 1$.

On the other hand, as the scalar operators with constant coefficients commute, we see that

$$d^*_{q-1} L_s w = \sum_{j=1}^n \partial_j \sum_{|\alpha|\leq s} (-1)^{|\alpha|} \partial^{2\alpha} w_j = \sum_{|\alpha|\leq s} \partial^{2\alpha} d^*_{q-1} w = 0 \text{ in } \mathbb{R}^n$$

in the sense of distributions and then, by (1.1),

$$\Delta\tilde{p} = d^*_{q-1} d_{q-1}\tilde{p} = d^*_{q-1} d_{q-1} L_s w = 0 \text{ in } \mathbb{R}^n$$

in the sense of distributions. In particular, as Δ and L_s are elliptic differential operators with constant coefficients, we conclude that $\tilde{p} \in C^\infty_{\Lambda^{q-1}}$, $w \in C^\infty_{\Lambda^q}$, see [10].

Next, we note that for each $h \in C^{\infty}_{0,\Lambda^q}$ we have

$$h(x) = d_{q-1}h_1(x) + (h(x) - d_{q-1}h_1(x)) \text{ for all } x \in \mathbb{R}^n$$

where

$$h_1(x) = d^*_{q-1} \int_{\mathbb{R}^n} \varphi^{(q)}_n(x,y) \wedge h(y).$$

Hence (1.1) imply

$$d_q\, d_{q-1}h_1 = 0,\ d^*_{q-1}\, d_{q-1}h_1 = d^*_{q-1}h,\ d^*_{q-1}\, (h - d_{q-1}h_1) = 0 \text{ in } \mathbb{R}^n.$$

Keeping in the mind that $h \in C^{\infty}_{0,\Lambda^q}$ we see that $h_1 \in H^{s'}_{\Lambda^q}$, $s' \in \mathbb{Z}_+$ because $n \geq 3$ and (2.15). Hence, h_1 can be approximated in $H^s_{\Lambda^q}$, by a sequence $\{h_{1,j}\}$ from $C^{\infty}_{0,\Lambda^{q-1}}$. As $\tilde{p} \in C^{\infty}_{\mathrm{loc},\Lambda^{q-1}}$,

$$\langle d_{q-1}\tilde{p}, h\rangle_{s,\Lambda^q} = (d_{q-1}\tilde{p}, h)_{L^2_{\Lambda^q}} =$$

$$\lim_{j\to+\infty} (d_{q-1}\tilde{p}, d_{q-1}h_{1,j} + (h - d_{q-1}h_{1,j})_{L^2_{\Lambda^q}} =$$

$$\lim_{j\to+\infty} (d^*_{q-1}d_{q-1}\tilde{p}, h_{1,j})_{L^2_{\Lambda^{q-1}}} + \langle L_s w, h - d_{q-1}h_1\rangle_{s,\Lambda^q} = (w, h - d_{q-1}h_1)_{H^s_{\Lambda^{q-1}}}$$

because $d^*_{q-1}d_{q-1}\,\tilde{p} = 0$ in $\mathbb{R}^n$ and the definition of the operator L_s.

But, according to (1.1),

$$h - d_{q-1}h_1 = h - d_{q-1}d^*_{q-1}\varphi^{(q)}_n h = d^*_q d_q \varphi^{(q)}_n h.$$

Again, as $h \in C^{\infty}_{0,\Lambda^q}$, by (2.15), $d_q\varphi^{(q)}_n h \in H^{s'}_{\Lambda^{q+1}}$, $s' \in \mathbb{Z}_+$ if $n \geq 3$. Then $d_q\varphi^{(q)}_n h$ can by approximated in $H^{s+1}_{\Lambda^{q+1}}$ by a sequence of forms $\{g_j\}$ from $C^{\infty}_{0,\Lambda^{q+1}}$ and hence $h - d_{q-1}h_1$ can by approximated in $H^s_{\Lambda^q}$ by the sequence $\{d^*_q g_j\}$ from $\mathcal{V}_{0,\Lambda^q}$. In particular, this means that

$$\langle d_{q-1}\tilde{p}, h\rangle_{s,\Lambda^q} = (w, h - d_{q-1}h_1)_{H^s_{\Lambda^{q-1}}} = \lim_{j\to+\infty} (w, d^*_q g_j)_{H^s_{\Lambda^{q-1}}} = 0$$

because $w \in \tilde{\mathbf{H}}_{s,\Lambda^q}$ satisfies (2.21). Therefore

$$\|d_{q-1}\tilde{p}\|_{H^{-s}_{\Lambda^q}} = \sup_{\substack{h\in C^{\infty}_{0,\Lambda^q} \\ h\neq 0}} \frac{|\langle d_{q-1}\tilde{p}, h\rangle_{s,\Lambda^q}|}{\|h\|_{H^s_{\Lambda^q}}} = 0.$$

and $d_{q-1}\tilde{p} = L_s w = 0$ and (2.22) implies that $w = 0$; in particular, $\tilde{\mathbf{H}}_{s,\Lambda^q} = \mathbf{H}_{s,\Lambda^q}$, $s \in \mathbb{Z}_+$. □

We will also use the so-called Bochner spaces of functions of (x,t) in the strip $\mathbb{R}^n \times I$, where $I = [0,T]$. Namely, if $\mathcal{B}$ is a Banach space (possibly, a space of functions on $\mathbb{R}^n$) and $p \geq 1$, we denote by $L^p(I,\mathcal{B})$ the Banach space of all measurable mappings $u : I \to \mathcal{B}$ with the finite norm

$$\|u\|_{L^p(I,\mathcal{B})} := \|\|u(\cdot,t)\|_{\mathcal{B}}\|_{L^p(I)},$$

see for instance [43, Ch. III, § 1]. In the same line stays the space $C(I,\mathcal{B})$, i.e., it is the Banach space of all mappings $u : I \to \mathcal{B}$ with finite norm

$$\|u\|_{C(I,\mathcal{B})} := \sup_{t\in I} \|u(\cdot,t)\|_{\mathcal{B}}.$$

Let $\mathbf{P}_q$ be the Helmholtz-Leray type projection

$$\mathbf{P}_q : L^2_{\Lambda^q} \to \mathbf{H}_{0,\Lambda^q}. \tag{2.23}$$

Lemma 2.2. *For each $p > 1$ there is positive constant $D(p)$ such that*

$$\|\mathbf{P}_q v\|_{L^p_{\Lambda^q}} \leq D(p)\|v\|_{L^p_{\Lambda^q}} \text{ for all } v \in L^p_{\Lambda^q}. \tag{2.24}$$

Besides, if $v \in H^1_{\Lambda^q}$ then

$$\partial_j(\mathbf{P}_q v) = \mathbf{P}_{\Lambda^q}(\partial_j v) \text{ for each } 1 \leq j \leq n.$$

Moreover $\mathbf{P}_q$ maps $C(I, H^s_{\Lambda^q})$, $L^2(I, H^s_{\Lambda^q})$, $L^2(I, \mathbf{H}'_{1,\Lambda^q})$, $C(I, L^p_{\Lambda^q})$ continuously to themselves if $s \in \mathbb{Z}$, $p \in (1, +\infty)$.

Proof. As before, let φ_n stand for the standard fundamental solution of the Laplace operator in $\mathbb{R}^n$. By (1.1) we have

$$v = \varphi_n^{(q)}(d_q^* d_q + d_{q-1}d_{q-1}^*)v$$

for q-form v with the entries from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Moreover, (1.1) implies that

$$\mathbf{P}_q v = \varphi_n^{(q)}\, d_q^* d_q \text{ for all } v \in \mathcal{S}_{\Lambda^q}$$

From the viewpoint of the theory of pseudo-differential operators, it is a matrix Fourier multiplier (see, for instance, [42, Ch. I, and Ch. X1]) given by

$$\mathbf{P}_q v = \mathfrak{F}^{-1}\left(a_q(\zeta)\mathfrak{F}(v)\right),\ v \in L^2_{\Lambda^q}, \tag{2.25}$$

where $\mathfrak{F}(v)$ stands for the Fourier transform of the vector v, $\mathfrak{F}^{-1}(w)$ stands for the Fourier transform of the vector w, and $a_q(\zeta)$ can be identified with $(m(q) \times m(q))$-matrix

$$a_q(\zeta) = I_n - \Big(\frac{\sigma(d_q^* d_q)(\zeta)}{|\zeta|^2}\Big),\ \zeta \in \mathbb{R}^n \setminus \{0\}$$

where $\sigma(B)(\zeta)$ is the principal symbol of the differential operator B. As the projection $\mathbf{P}_q$ is a matrix Fourier multiplier in theory of pseudo-differential operators, then applying [42, Ch. X1, §1, Theorem 1.1]), we conclude that $\mathbf{P}_q$ maps $L^p_{\Lambda_q}$ continuously to itself for all $p > 1$. In particular, for each $p > 1$ there is positive constant $D(p)$ such that (2.24) is fulfilled.

Fix $v \in H^1_{\Lambda^q}$ and $j \in \mathbb{N}$, $1 \leq j \leq n$. By the properties of the Fourier transform (see, for instance, [42, Ch. I], we conclude that

$$\|\mathfrak{F}(v)\|_{L^2_{\Lambda^q}} = \|v\|_{L^2_{\Lambda^q}},\ \|\mathfrak{F}(\partial_j v)\|_{L^2_{\Lambda^q}} = \|\partial_j v\|_{L^2_{\Lambda^q}},$$

$$\mathfrak{F}(\partial_j v)(\zeta) = (\sqrt{-1}\zeta)\mathfrak{F}(v)(\zeta),$$

then both $(\sqrt{-1}\zeta)\mathfrak{F}(v)$ and $(\sqrt{-1}\zeta)a(\zeta)\mathfrak{F}(v)(\zeta)$ belong to $L^2_{\Lambda^q}$ because entries of the matrix a_q belong to $L^\infty(\mathbb{R}^n)$. Hence, according to (2.25),

$$\mathbf{P}_q(\partial_j v) = \mathfrak{F}^{-1}(a_q(\zeta)\sqrt{-1}\zeta\mathfrak{F}(v)(\zeta)) = \partial_j\mathfrak{F}^{-1}(a_q(\zeta)\mathfrak{F}(v)(\zeta)) = \partial_j\mathbf{P}_q(v).$$

This proves that $\mathbf{P}_q$ maps $C(I, H^s_{\Lambda^q})$, $L^2(I, H^s_{\Lambda^q})$ continuously to itself for $s \in \mathbb{Z}_+$. For negative s this fact follows from the definition of the duality between $H^s_{\Lambda^q}$ and $H^{-s}_{\Lambda^q}$, which was to be proved. □

After Leray [23, 24], a great attention was paid to weak solutions to the Navier-Stokes equations in cylinder domains in $\mathbb{R}^3 \times [0, \infty)$. Considering them in the Bochner spaces yields the classical existence theorem for the weak solutions. To formulate it we set $\mathbf{H}_s = \mathbf{H}_{s,\Lambda^1}$.

Theorem 2.1. *Let $k \in \mathbb{N}$ and $2k \leq n \leq 2(k+1)$. Given a pair $(f, u_0) \in L^2(I, \mathbf{H}_1') \times \mathbf{H}_0$, there exists a vector field $u \in L^\infty(I, \mathbf{H}_0) \cap L^2(I, \mathbf{H}_1)$ satisfying*

$$(2.26) \qquad \begin{cases} \frac{d}{dt}(u,v)_{L^2_{\Lambda^1}} + \mu \sum_{|\alpha|=1} (\partial^\alpha u, \partial^\alpha v)_{L^2_{\Lambda^1}} & = \langle f - u \cdot \nabla u, v \rangle_{\Lambda^q}, \\ u(\cdot, 0) & = u_0 \end{cases}$$

for all $v \in \mathbf{H}_k$. Moreover, $\partial_t u \in L^{\frac{4}{n+2-2k}}(I, \mathbf{H}_k')$.

Proof. See, for instance, [26, Ch. II, Theorem 6.1] or [43, Ch. III, Theorems 3.1, 3.3, 4.1], cf. the proof of Theorem 3.1 below). □

The key property of the non-linear term $\mathcal{N}_1 u = u \cdot \nabla u$ used in the proof of Theorem 2.1 is that

$$(2.27) \qquad (\mathcal{N}_1 u, u)_{L^2_{\Lambda^1}} = 0$$

if u satisfies $\operatorname{div} u = 0$ in $\mathbb{R}^n$ and decreases sufficiently at the infinity. For $q \neq 1$ and general non-linearity of type (1.3) property (2.27) is not necessarily true. For instance, if $n = 3$ and

$$\mathcal{N}_2 u = \star d_2 u \wedge u = u \operatorname{div} u$$

we easily have

$$(\mathcal{N}_1 u, u)_{L^2_{\Lambda^2}} = \int_{\mathbb{R}^n} (\operatorname{div} u)\, |u|^2 dx.$$

Thus, the standard arguments fail and we can not guarantee the Leray-Hopf weak solutions to (1.2) as in Theorem 2.1 in general (see, however, [33] for non-linearities including the last non-linearity as a summand, where Navier-Stokes type equations admit Leray-Hopf type weak solutions).

However we may achieve a Uniqueness Theorem for (1.2). For $n \geq 3$ the space $L^\infty(I, \mathbf{H}_0) \cap L^2(I, \mathbf{H}_1)$ is, perhaps, too large in order to achieve a uniqueness theorem even for the usual Navier-Stokes equations. The spaces $L^{\mathfrak{s}}(I, L^{\mathfrak{r}}(\mathbb{R}^n))$ with

$$\frac{2}{\mathfrak{s}} + \frac{n}{\mathfrak{r}} = 1, \quad 2 \leq \mathfrak{s} < \infty, \quad 2 \leq n < \mathfrak{r} \leq \infty$$

are well known to be uniqueness and regularity classes for the Navier-Stokes equations, see [34], [17], [38]. The limit case $\mathfrak{s} = \infty$, $n = \mathfrak{r} = 3$ was added to the list in [7] but we will not discuss it here. As the non-linearity (1.3) is more general than the standard one related to the Navier-Stokes equations, the assumptions appear to be stronger.

Theorem 2.2. *Let $n \geq 3$, $0 \leq q < n$. Then for each data $(f, u_0) \in L^2(I, \mathbf{H}_{1,\Lambda^q}') \times \mathbf{H}_{0,\Lambda^q}$, there is at most one form u in the space $L^\infty(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^2(I, L^\infty_{\Lambda^q}) \cap L^\infty(I, L^n_{\Lambda^q})$ satisfying the nonlinear Navier-Stokes type equations*

$$(2.28) \qquad \begin{cases} \frac{d}{dt}(u,v)_{L^2_{\Lambda^q}(\mathbb{R}^n)} + \mu \sum_{|\alpha|=1} (\partial^\alpha u, \partial^\alpha v)_{L^2_{\Lambda^q}(\mathbb{R}^n)} & = \langle f - \mathcal{N}_q u, v \rangle_{\Lambda^q}, \\ u(\cdot, 0) & = u_0 \end{cases}$$

for all $v \in \mathbf{H}_{k,\Lambda^q}$ where $\mathcal{N}_q u$ is given by (1.3).

Proof. It is almost literally the same as the proof of [18, Ch. 6, § 2, Theorem 1], [26, Theorem 6.9] or Theorem 3.4 and Remark 3.6 in [43] or Lemma 3.3 below. □

We now proceed with an Open Mapping Theorem for 1.2.

3. An open mapping theorem

This section is devoted to the so-called stability property for solutions to the Navier-Stokes type equations. One of the first statements of this kind was obtained by Ladyzhenskaya [18, Ch. 4, § 4, Theorem 11] in the case of the Navier-Stokes equations for flows in bounded domains in $\mathbb{R}^3$ with C^2 smooth boundaries.

In order to extend the property to the spaces of high smoothness, we consider a linearisation of problem (1.2). To formulate it we set

$$\mathbf{B}_q(w,u) = M_1^{(q)}(d_q w, u) + d_{q-1} M_2^{(q)}(w,u) + M_1^{(q)}(d_q u, w) + d_{q-1} M_2^{(q)}(u,w)$$

for q-forms u and w.

Namely, given differential q-forms f and w with sufficiently regular coefficients on $\mathbb{R}^n \times [0,T]$ and q-form u_0 on $\mathbb{R}^n$, find sufficiently regular q-form u and $(q-1)$-form p in the strip $\mathbb{R}^n \times [0,T]$ which satisfy

$$\text{(3.1)} \qquad \left\{ \begin{array}{rcll} \partial_t u - \mu \Delta u + \mathbf{B}_q(w,u) + d_{q-1} p & = & f, & (x,t) \in \mathbb{R}^n \times (0,T), \\ d^*_{q-1}\, u & = & 0, & (x,t) \in \mathbb{R}^n \times (0,T), \\ d^*_{q-2}\, p & = & 0, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u & = & u_0, & (x,t) \in \mathbb{R}^n \times \{0\}. \end{array} \right.$$

Here the third equation is introduced because of Proposition 2.2; for $q = 1$ it obviously disappears. Again, motivated by both the uniqueness theorem and the physical reasons related to the Navier-Stokes equations, one have to assume that the data and the solutions are essentially decreasing at the infinity. As the case $n = 2$ is much more easy to handle proving an existence theorem, we will be concentrated on $n \geq 3$.

Considering this problem in the Bochner spaces yields an existence theorem for the weak solutions to (3.1).

Theorem 3.1. *Let $n \geq 3$, $0 \leq q < n$, and suppose $w \in C(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^2(I, L^\infty_{\Lambda^q}) \cap L^\infty(I, L^n_{\Lambda^q})$. Given any pair $(f, u_0) \in L^2(I, \mathbf{H}'_{1,\Lambda^q}) \times \mathbf{H}_{0,\Lambda^q}$, there is a unique differential form $u \in C(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q})$ with $\partial_t u \in L^2(I, \mathbf{H}'_{1,\Lambda^q})$, satisfying*

$$\text{(3.2)} \qquad \left\{ \begin{array}{rcl} \dfrac{d}{dt}(u,v)_{L^2_{\Lambda^q}} + \mu \displaystyle\sum_{|\alpha|=1} (\partial^\alpha u, \partial^\alpha v)_{L^2_{\Lambda^q}} & = & \langle f - \mathbf{B}_q(w,u), v \rangle_{\Lambda^q}, \\ u(\cdot, 0) & = & u_0 \end{array} \right.$$

for all $v \in \mathbf{H}_{1,\Lambda^q}$.

Proof. It is similar to the proof of the uniqueness and existence theorem for the Stokes problem and the Navier-Stokes problem, see [44, § 2.3, § 2.4] (or [26, Ch. II, Theorem 6.1 and Theorem 6.9] or [43, Ch. III, Theorem 1.1, Theorem 3.1 and Theorem 3.4] for domains in $\mathbb{R}^3$). We shortly recall the arguments in the part we will use in order to obtain existence theorems related to (3.2) for more regular data and solutions. First we note that $\mathbf{H}_{1,\Lambda^q}$ is separable because it is a subspace of a separable space. According to Proposition 2.3, the space $\mathcal{V}_{\Lambda^q}$ is everywhere dense in $\mathbf{H}_{1,\Lambda^q}$. Pick a linearly independent countable system $\{b_j\}_{j\in\mathbb{N}} \subset \mathcal{V}_{\Lambda^q}$ such that its linear span $\mathcal{L}(\{b_j\}_{j\in\mathbb{N}})$ is everywhere dense in $\mathbf{H}_{1,\Lambda^q}$. As $\mathcal{V}_{\Lambda^q} \subset \mathbf{H}_{1,\Lambda^q} \subset \mathbf{H}_{0,\Lambda^q}$ $\mathcal{L}(\{b_j\}_{j\in\mathbb{N}})$ is everywhere dense in both $\mathbf{H}_{0,\Lambda^q}$ and $\mathbf{H}_{1,\Lambda^q}$, too. Then, keeping in the mind the Gram-Schmidt orthogonalization process, without loss of generality, we may assume that the system $\{b_j\}_{j\in\mathbb{N}}$ is a $L^2_{\Lambda^q}(\mathbb{R}^n)$-orthogonal basis $\mathbf{H}_{0,\Lambda^q}$.

Next, one defines the Faedo-Galerkin approximations in the usual way,

$$u_m = \sum_{i=1}^{m} g_i^{(m)}(t) b_i(x)$$

where the functions $g_i^{(m)}$ satisfy the following relations

$$(3.3) \quad (\partial_\tau u_m, b_j)_{L^2_{\Lambda^q}} + \mu \sum_{|\alpha|=1} (\partial^\alpha u_m, \partial^\alpha b_j)_{L^2_{\Lambda^q}} + (\mathbf{B}(w, u_m), b_j)_{L^2_{\Lambda^q}} = \langle f, b_j \rangle_{\Lambda^q},$$

$$u_m(x, 0) = u_{0,m}(x)$$

for all $0 \leq j \leq m$ with the initial datum $u_{0,m}$ from the linear span $\mathcal{L}(\{b_j\}_{j=1}^m)$ such that the sequence $\{u_{0,m}\}$ converges to u_0 in $\mathbf{H}_{0,\Lambda^q}$. For instance, as $\{u_{0,m}\}$ we may take the orthogonal projection onto the linear span $\mathcal{L}(\{b_j\}_{j=1}^m)$.

In this way (3.3) reduces to an initial problem for a $(m \times m)$-system of of ordinary differential equations with respect to the variable t with respect to the unknown coefficients $g_j^{(m)}$ on the interval $[0, T]$:

$$(3.4) \quad \begin{cases} \frac{d}{dt} g_j^{(m)}(t) + \sum_{i=1}^m \mathfrak{C}_{i,j}^{(m)}(t) g_i^{(m)}(t) = F_j(t), \\ g_j^{(m)}(0) = (u_{0,m}, b_j)_{L^2_{\Lambda^q}} = \mathfrak{a}_j^{(m)}, \end{cases}$$

where the scalar functions $F_j(t) = \langle f(\cdot, t), b_j \rangle_{\Lambda^q}$ belong to $L^2(0, T)$, and

$$(3.5) \quad \mathfrak{C}_{i,j}^{(m)}(t) = \mu \sum_{|\alpha|=1} (\partial^\alpha b_i, \partial^\alpha b_j)_{L^2_{\Lambda^q}} + (\mathbf{B}_q(w(\cdot, t), b_i), b_j)_{L^2_{\Lambda^q}}.$$

Integrating by parts, we see that

$$(3.6) \quad (d_{q-1} M_2^{(q)}(w, u), v)_{L^2_{\Lambda^q}} = (M_2^{(q)}(w, u), d_{q-1}^* v)_{L^2_{\Lambda^q}} = 0,$$

$$(M_1^{(q)}(d_q w, u), v)_{L^2_{\Lambda^q}} = (w, \tilde{M}_1^{(q)}(u, v))_{L^2_{\Lambda^q}}$$

for all $v \in \mathcal{V}_{\Lambda^q}$, where $\tilde{M}_1^{(q)}$ is a first order bilinear differential operator with constant coefficients. Therefore

$$(3.7) \quad (\mathbf{B}_q(w, u), v)_{L^2_{\Lambda^q}} = (M_1^{(q)}(d_q u, w), v)_{L^2_{\Lambda^q}} + (w, \tilde{M}_1^{(q)}(u, v))_{L^2_{\Lambda^q}}$$

for q-fors u, v, w with sufficiently regular coefficients, approximable by elements of $\mathcal{V}_{\Lambda^q}$ (see, for instance, [43, Ch.3, §3.1, formula (3.2)] for the Navier-Stokes equations). Thus, the components of the matrix $\mathfrak{C}^{(m)}(t)$ belong to $L^\infty(0, T)$ because $b_j, b_i \in \mathcal{V}_{\Lambda^q}$ and $w \in C(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^2(I, L^\infty_{\Lambda^q})$.

Let us denote by $g^{(m)}$, $F^{(m)}$ and $\mathfrak{a}^{(m)}$ the m-vectors constructed with the use of the components $g_j^{(m)}$, F_j and $\mathfrak{a}_j^{(m)}$, respectively, and by $\mathfrak{C}^{(m)}(t)$ the corresponding functional matrix constructed with the use of the components $\mathfrak{C}_{i,j}^{(m)}(t)$. Then (3.4) transforms to

$$\begin{cases} \frac{d}{dt} g^{(m)}(t) + \mathfrak{C}^{(m)}(t) g^{(m)}(t) = F^{(m)}(t), \\ g^{(m)}(0) = \mathfrak{a}^{(m)}. \end{cases}$$

and hence for each $m \in \mathbb{N}$ the system (3.3) admits a unique vector-solution $g^{(m)}$ on the interval $(0, T)$ given by

$$g^{(m)}(t) = \exp\Big(\int_0^t \mathfrak{C}^{(m)}(\tau) d\tau \Big) \int_0^t \exp\Big(- \int_0^\tau \mathfrak{C}^{(m)}(\tau') d\tau' \Big) F^{(m)}(\tau) d\tau,$$

where

$$\exp\Big(\int_0^t \mathfrak{C}^{(m)}(\tau)d\tau\Big) = \sum_{k=0}^{\infty}\frac{1}{k!}\Big(\int_0^t \mathfrak{C}^{(m)}(\tau)d\tau\Big)^k.$$

Since $w \in C(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^2(I, L^\infty_{\Lambda^q})$, formula (3.5) means that the entries of the matrix

$$\exp\Big(\int_0^t \mathfrak{C}^{(m)}(\tau)d\tau\Big)$$

belong actually to $C^{0,1}[0,T]$ and then the components of the vector $g^{(m)}$ belong to $C^{1/2}[0,T]$. In particular,

$$u_m \in L^2(I, \mathbf{H}_{s,\Lambda^q}) \cap C(I, \mathbf{H}_{s-1,\Lambda^q}),\ u_m' \in L^2(I, \mathbf{H}'_{1,\Lambda^q})$$

for each $s \in \mathbb{N}$.

In order to obtain a solution to (3.1) one usually appeals to a priori estimates. To obtain them, we invoke the following useful lemma by J.-L. Lions.

Lemma 3.1. *Let V, H and V' be Hilbert spaces such that V' is the dual to V and the embeddings $V \subset H \subset V'$ are continuous and everywhere dense. If $u \in L^2(I,V)$ and $\partial_t u \in L^2(I,V')$ then*

$$\frac{d}{dt}\|u(\cdot,t)\|^2_H = 2\,\langle \partial_t u, u\rangle_{\Lambda^q} \tag{3.8}$$

and u is equal almost everywhere to a continuous mapping from $[0,T]$ to H.

Proof. See [43, Ch. III, § 1, Lemma 1.2]. □

Thus, if we multiply the equation corresponding to index j in (3.3) by $g^{(m)}_j$ then, after the summation with respect to j, we obtain for all $\tau \in [0,T]$:

$$\frac{1}{2}\frac{d}{d\tau}\|u_m\|^2_{L^2_{\Lambda^q}} + \mu|\nabla u_m\|^2_{L^2_{\Lambda^q}} = \tag{3.9}$$

$$\langle f, u_m\rangle_{\Lambda^q} + (M^{(q)}_1(d_q u_m, w), u_m)_{L^2_{\Lambda^q}} + (w, \tilde{M}^{(q)}_1(u_m,u_m))_{L^2_{\Lambda^q}}$$

because of (3.8) and (3.7).

The following standard statement, where

$$\|(f,u_0)\|_{0,q,T} = \Big(\|u_0\|^2_{L^2_{\Lambda^q}} + \frac{2}{\mu}\|f\|^2_{L^2(I,\mathbf{H}'_{1\Lambda^q})} + \|f\|^2_{L^1(I,\mathbf{H}'_{1,\Lambda^q})}\Big)^{1/2},$$

$$\|u\|_{0,q,T} = \Big(\|u\|^2_{C(I,L^2_{\Lambda^q})} + \mu\|\nabla u\|^2_{L^2(I,L^2_{\Lambda^q})}\Big)^{1/2},$$

gives a basic a priori estimate for solutions to (3.2).

Lemma 3.2. *Let $n \geq 3$ and $w \in C(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^2(I, L^\infty_{\Lambda^q})$. If $u \in C(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q})$ and $(f,u_0) \in L^2(I, \mathbf{H}'_{1,\Lambda^q}) \times \mathbf{H}_{0,\Lambda^q}$ satisfy*

$$\left\{\begin{array}{rcl} \frac{1}{2}\frac{d}{d\tau}\|u(\cdot,\tau)\|^2_{L^2_{\Lambda^q}} + \mu\,\|\nabla u\|^2_{L^2_{\Lambda^q}} & = & \langle f,u\rangle_{\Lambda^q} + (\mathbf{B}_q(w,u),u))_{L^2_{\Lambda^q}} \\ u(\cdot,0) & = & u_0 \end{array}\right. \tag{3.10}$$

for all $t \in [0,T]$, then there is a constant $C_q > 0$ independent on u, w, t, μ such that

$$\begin{array}{rcl} \|u\|^2_{0,q,T} & \leq & \|(f,u_0)\|^2_{0,q,T}\Big(1 + 2\sqrt{2}\exp\Big(\frac{C_q}{\mu}\int_0^T \|w(\cdot,t)\|^2_{L^\infty_{\Lambda^q}}dt\Big) \\ & + & \frac{4}{\mu}\Big(\int_0^T \|w(\cdot,t)\|^2_{L^\infty_{\Lambda^q}}dt\Big)\exp\Big(\frac{C_q}{\mu}\int_0^T \|w(\cdot,t)\|^2_{L^\infty_{\Lambda^q}}dt\Big)\Big), \end{array} \tag{3.11}$$

Besides, if

$$\frac{2}{s}+\frac{n}{r}=\frac{n}{2} \text{ with } 2<r\leq\frac{2n}{n-2},\ 2\leq s<+\infty, \tag{3.12}$$

then there is a positive constant $c^{(\mu)}_{r,s}(w)$ independent on u and T such that

$$\|u\|_{L^s(I,L^r_{\Lambda^q})}\leq c^{(\mu)}_{r,s}(w)\|(f,u_0)\|_{0,\mu,T}.$$

Proof. It is similar to the proof of energy estimates for solutions to the Navier-Stokes equations, see [18, Ch. IV, § 3] or [43, Ch. III, Theorem 3.1].

The Hölder inequality, (2.1) and (3.7) imply

$$\begin{aligned} \left|\int_0^t (\mathbf{B}_q(w,u),u))_{L^2_{\Lambda^q}}ds\right| &\leq c_q\int_0^t\|\nabla u\|_{L^2_{\Lambda^q}}\|w\|_{L^\infty_{\Lambda^q}}\|u\|_{L^2_{\Lambda^q}}ds\\ &\leq \frac{\mu}{4}\int_0^t\|\nabla u\|^2_{L^2_{\Lambda^q}}ds+\frac{c_q^2}{\mu}\int_0^t\|w\|^2_{L^\infty_{\Lambda^q}}\|u\|^2_{L^2_{\Lambda^q}}ds \end{aligned} \tag{3.13}$$

with a positive constant c_q independent on u, v, t and μ.

On the other hand, by (2.20), we get

$$\begin{aligned} &2\left|\int_0^t\langle f(\cdot,s),u(\cdot,s)\rangle_{\Lambda^q}ds\right|\\ &\leq 2\int_0^t\|f(\cdot,s)\|_{\mathbf{H}'_{1,\Lambda^q}}\|u(\cdot,s)\|_{H^1_{\Lambda^q}}ds\\ &\leq 2\int_0^t\|f(\cdot,s)\|_{\mathbf{H}'_{1,\Lambda^q}}\Big(\|\nabla u(\cdot,s)\|_{L^2_{\Lambda^q}}+\|u(\cdot,s)\|_{L^2_{\Lambda^q}}\Big)ds\\ &\leq \int_0^t\Big(\frac{2}{\mu}\|f\|^2_{\mathbf{H}'_{1,\Lambda^q}}+\frac{\mu}{2}\|\nabla u\|^2_{L^2_{\Lambda^q}}+2\|f\|_{\mathbf{H}'_{1,\Lambda^q}}\|u(\cdot,s)\|_{L^2_{\Lambda^q}}\Big)ds \end{aligned} \tag{3.14}$$

for all $t\in[0,T]$. Integrating (3.10) with respect to τ over $[0,t]$ and taking both (3.13) and (3.14) into account yields

$$\|u(\cdot,t)\|^2_{L^2_{\Lambda^q}}+\mu\int_0^t\|\nabla u(\cdot,s)\|^2_{L^2_{\Lambda^q}}ds\ \leq\ \|u_0\|^2_{L^2_{\Lambda^q}}+ \tag{3.15}$$

$$\int_0^t\Big(\frac{2}{\mu}\|f\|^2_{\mathbf{H}'_{1,\Lambda^q}}+2\|f\|_{\mathbf{H}'_{1,\Lambda^q}}\|u\|_{L^2_{\Lambda^q}}+\frac{c_q^2}{\mu}\|w\|^2_{L^\infty_{\Lambda^q}(\mathbb{R}^n)}\|u\|^2_{L^2_{\Lambda^q}}\Big)ds.$$

Finally, on applying Lemma 2.1 with $\gamma=1/2$ and $Y(t)=\|u(\cdot,t)\|^2_{L^2_{\Lambda^q}}$ we readily obtain

$$\begin{aligned} \|u(\cdot,t)\|^2_{L^2_{\Lambda^q}} &\leq \Big(\Big(\|u_0\|^2_{L^2_{\Lambda^q}(\mathbb{R}^n)}+\frac{2}{\mu}\|f\|^2_{L^2([0,t],\mathbf{H}'_{1,\Lambda^q})}\Big)^{1/2}\exp\Big(\frac{c_q^2}{\mu}\int_0^t\|w\|^2_{L^\infty_{\Lambda^q}}ds\Big)\\ &+ \int_0^t\|f(\cdot,s)\|_{\mathbf{H}'_{1,\Lambda^q}}\exp\Big(\frac{1}{\mu}\int_s^t\|w(\|^2_{L^\infty_{\Lambda^q}}ds'\Big)ds\Big)^2\\ &\leq 2\,\|(f,u_0)\|^2_{0,q,T}\ \exp\Big(\frac{2c_q^2}{\mu}\int_0^T\|w\|^2_{L^\infty_{\Lambda^q}}ds\Big) \end{aligned}$$

for all $t\in[0,T]$. Estimate (3.11) follows from the latter inequality.

It is easy to see that

$$\|u\|_{L^p(I,L^2_{\Lambda^q})} \leq T^{1/p}\, \|u\|_{L^\infty(I,L^2_{\Lambda^q})} \tag{3.16}$$

holds for any $p \geq 1$, which accomplishes the energy estimate (3.11).

Finally, the last statement follows from Gagliardo-Nirenberg inequality (2.3) for $q_0 = r_0 = 2$, $j_0 = 0$, $m_0 = 1$, $p_0 = \frac{1}{r}$, $a = \frac{n(r-2)}{2\,r}$ with the exceptional case where $n = 2$, $r = +\infty$ and $a = 1$. □

Lemma 3.2 and (3.9) imply that the sequence $\{u_m\}$ is bounded in the space $C(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q})$. So, it bounded in $L^\infty(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q})$ and we can extract a subsequence that converges weakly-* in $L^\infty(I, \mathbf{H}_{0,\Lambda^q})$ and converges weakly in $L^2(I, \mathbf{H}_{1,\Lambda^q})$ to an element $u \in L^\infty(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q})$. For abuse of notation, we use the same designation $\{u_m\}$ for such a subsequence.

At this point, rather delicate arguments involving compact embedding theorems for the Bochner-Sobolev spaces on bounded domains show that the sequence $\{u_m\}$ may be considered as convergent in the space $L^2(I, L^2_{\Lambda^q}(\mathbb{R}^n))$, see [26, Ch. II, Theorem 6.1] or [43, Ch. III, Theorem 3.1]. This allows us to pass to the limit with respect to $m \to \infty$ in (3.3) and to conclude that the element u satisfies (3.2). We proceed with the uniqueness.

Lemma 3.3. *Let $n \geq 3$, $0 \leq q < n$, and $w \in L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^\infty(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, L^\infty_{\Lambda^q}) \cap L^\infty(I, L^n_{\Lambda^q})$. For each pair $(f, u_0) \in L^2(I, \mathbf{H}'_{1,\Lambda^q}) \times \mathbf{H}_{0,\Lambda^q}$ the linearised Navier-Stokes equations* (3.2) *have at most one solution in the space $L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^\infty(I, \mathbf{H}_{0,\Lambda^q})$.*

Proof. First, we note that (3.6), (3.7) and the Hölder inequality yield

$$|(\mathbf{B}_q(w, u), v)_{L^2_{\Lambda^q}}| \leq$$

$$c_q \Big(\|w\|_{L^n_{\Lambda^q}} \|\nabla u\|_{L^2_{\Lambda^q}} \|v\|_{L^{\frac{2n}{n-2}}_{\Lambda^q}} + \|w\|_{L^\infty_{\Lambda^q}} \|u\|_{L^2_{\Lambda^q}} \|\nabla v\|_{L^2_{\Lambda^q}} \Big)$$

for all $u \in \mathbf{H}_{1,\Lambda^q}$, $v \in \mathcal{V}_{\Lambda^q}$ with a constant c_q indepenent on u, w, v.

As $n \geq 3$, then by (2.3)

$$\|u\|_{L^{\frac{2n}{n-2}}_{\Lambda^q}} \leq C_1 \|\nabla u\|_{L^2_{\Lambda^q}} \tag{3.17}$$

with the Gagliardo-Nirenberg constant C_1, because

$$\frac{n-2}{2n} = \Big(\frac{1}{2} - \frac{1}{n}\Big)\alpha + \frac{1-\alpha}{2}, \text{ for } \alpha = 1 \text{ and } 1 - 0 - n/2 < 0.$$

On applying the Hölder inequality and (3.17), we readily conclude that for any $u \in L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^\infty(I, \mathbf{H}_{0,\Lambda^q})$, $w \in L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^\infty(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, L^\infty_{\Lambda^q}) \cap L^\infty(I, L^n_{\Lambda^q})$, we have

$$\|(\mathbf{B}_q(w, u)\|^2_{L^2(\mathbf{H}'_{1,\Lambda^q})} \leq \int_0^T \Big(\|w\|^2_{L^\infty_{\Lambda^q}} \|u\|^2_{L^2_{\Lambda^q}} + C_1 \|w\|^2_{L^n_{\Lambda^q}} \|\nabla u\|^2_{L^2_{\Lambda^q}} \Big) dt \leq$$

$$\|w\|^2_{L^2(I,L^\infty_{\Lambda^q})} \|u\|^2_{L^\infty(I,L^2_{\Lambda^q})} + C_1 \|w\|^2_{L^\infty(I,L^n_{\Lambda^q})} \|\nabla u\|^2_{L^2(I,L^2_{\Lambda^q})},$$

i.e., $\mathbf{B}_q(w, u) \in L^2(I, \mathbf{H}'_{1,\Lambda^q})$ and $\partial_t u \in L^2(I, \mathbf{H}'_{1,\Lambda^q})$, if u is a solution to problem (3.2).

Let now u' and u'' be any two solutions to (3.2) from the declared function space. Then the difference $u = u' - u''$ is a solution to (3.2) with zero data $(f, u_0) = (0, 0)$. Hence it follows that

$$\langle \partial_t u, u \rangle_{\Lambda^q} + \mu \, \|\nabla u\|^2_{L^2_{\Lambda^q}} = \langle \mathbf{B}_q(w,u), u \rangle_{\Lambda^q}.$$

Next, as $u \in L^2(I, \mathbf{H}_{1,\Lambda^q})$ and $\partial_t u \in L^2(I, \mathbf{H}'_{1,\Lambda^q})$, integrating the above equality with respect to t and using Lemma 3.1, we get

$$\|u(\cdot,t)\|^2_{L^2_{\Lambda^q}} + 2\mu \int_0^t \|\nabla u(\cdot,s)\|^2_{L^2_{\Lambda^q}} ds = 2 \int_0^t \langle \mathbf{B}_q(w,u), u \rangle_{\Lambda^q} \, ds$$

because $\mathbf{B}_q(w,u) \in L^2(I, \mathbf{H}'_{q,\Lambda^q})$ (and form w is assumed to belong to $L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap L^\infty(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, L^\infty_{\Lambda^q}) \cap L^\infty(I, L^n_{\Lambda^q})$, using (3.6) and (3.15) gives

$$\|u(\cdot,t)\|^2_{L^2_{\Lambda^q}} \le \frac{c_q^2}{\mu} \int_0^t \|w(\cdot,s)\|^2_{L^\infty_{\Lambda^q}} \|u(\cdot,s)\|^2_{L^2_{\Lambda^q}} \, ds.$$

Applying Gronwall's Lemma 2.1 to this inequality yields

$$0 \le \|u(\cdot,t)\|^2_{L^2_{\Lambda^q}} \le 0$$

for all $t \in [0,T]$, and so $u \equiv 0$, as desired. □

Finally, the form u belongs to $C(I, \mathbf{H}_{0,\Lambda^q})$, because $\partial_t u \in L^2(I, \mathbf{H}'_{1,\Lambda^q})$, $u \in L^2(I, \mathbf{H}_{1,\Lambda^q})$ and the embeddings $\mathbf{H}_{1,\Lambda^q} \subset \mathbf{H}_{0,\Lambda^q} \subset \mathbf{H}'_{1,\Lambda^q}$ are continuous and everywhere dense, i.e., the assumptions of Lemma 3.1 are fulfilled. □

Of course, similarly to the Navier-Stokes equations, Propositions 2.1 and 2.2 allow us to recover the "pressure" p as a form with distribution coefficients but we will do it for in the case of more regular solutions, only.

We are now in a position to introduce appropriate function spaces for solutions and for the data in order to obtain an open mapping theorem for regular solutions to the Navier-Stokes type equations. More precisely, as the principal differential part of the Navier-Stokes type equations is parabolic, we prefer to follow the dilation principle when introducing function spaces for the unknown "velocity" and given "exterior forces". Namely, for $s, k \in \mathbb{Z}_+$, we denote by $B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$ the set of all q-forms u in $C(I, \mathbf{H}_{k+2s,\Lambda^q}) \cap L^2(I, \mathbf{H}_{k+1+2s,\Lambda^q})$ such that

$$\partial_x^\alpha \partial_t^j u \in C(I, \mathbf{H}_{k+2s-|\alpha|-2j,\Lambda^q}) \cap L^2(I, \mathbf{H}_{k+1+2s-|\alpha|-2j,\Lambda^q})$$

provided $|\alpha| + 2j \le 2s$.

We endow the spaces $B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$ with the natural norms

$$\|u\|_{B^{k,2s,s}_{\mathrm{vel},\Lambda^q}} := \Big(\sum_{i=0}^{k} \sum_{|\alpha|+2j \le 2s} \|\partial_x^\alpha \partial_t^j u\|^2_{i,q,T} \Big)^{1/2},$$

where $\|u\|_{i,q,T} = \Big(\|\nabla^i u\|^2_{C(I, L^2_{\Lambda^q})} + \mu \|\nabla^{i+1} u\|^2_{L^2_{\Lambda^q}(I, L^2)} \Big)^{1/2}$.

Similarly, for $s, k \in \mathbb{Z}_+$, we define the space $B^{k,2s,s}_{\mathrm{for},\Lambda^q}$ to consist of all q-forms f in $C(I, H^{2s+k}_{\Lambda^q}) \cap L^2(I, H^{2s+k+1}_{\Lambda^q})$ with the property that

$$\partial_x^\alpha \partial_t^j f \in C(I, H^k_{\Lambda^q}) \cap L^2(I, H^{k+1}_{\Lambda^q})$$

provided $|\alpha| + 2j \le 2s$.

If $f \in B^{k,2s,s}_{\mathrm{for},\Lambda^q}$, then actually

$$\partial_x^\alpha \partial_t^j f \in C(I, H^{k+2(s-j)-|\alpha|}_{\Lambda^q}) \cap L^2(I, H^{k+1+2(s-j)-|\alpha|}_{\Lambda^q})$$

for all α and j satisfying $|\alpha| + 2j \le 2s$. We endow the spaces $B^{k,2s,s}_{\mathrm{for},\Lambda^q}$ with the natural norms

$$\|f\|_{B^{k,2s,s}_{\mathrm{for},\Lambda^q}} = \Big(\sum_{\substack{|\alpha|+2j\le 2s \\ 0\le i\le k}} \|\nabla^i \partial_x^\alpha \partial_t^j f\|^2_{C(I,L^2_{\Lambda^q})} + \|\nabla^{i+1} \partial_x^\alpha \partial_t^j f\|^2_{L^2(I,L^2_{\Lambda^q})} \Big)^{1/2}.$$

Finally, Proposition 2.2 suggests us defining the spaces $B^{k+1,2s,s}_{\mathrm{pre},\Lambda^{q-1}}$ for the "pressure" p to consist of all $(q-1)$-forms from the space $C(I, H^{2s+k+1}_{\mathrm{loc},\Lambda^{q-1}}) \cap L^2(I, H^{2s+k+2}_{\mathrm{loc},\Lambda^{q-1}})$ satisfying (2.13) for each $t \in [0,T]$,

$$d^*_{q-2} p = 0 \text{ in } \mathbb{R}^n \times [0,T], \tag{3.18}$$

$$\|p\|_{L^2(I,C_{b,\Lambda^{q-1}})} < +\infty \text{ for } 2s+k > n/2, \tag{3.19}$$

$$\|p\|_{L^2(I,C_{b,\Lambda^{q-1}})} + \|p\|_{C(I,C_{b,\Lambda^{q-1}})} < +\infty \text{ for } 2s+k > n/2+1, \tag{3.20}$$

and such that $d_{q-1} p \in B^{k,2s,s}_{\mathrm{for},\Lambda^{q-1}}$. Obviously, Proposition 2.2 allows us to equip the spaces with the norms $\|p\|_{B^{k+1,2s,s}_{\mathrm{pre},\Lambda^{q-1}}} =$

$$\begin{cases} \|d_{q-1}p\|_{B^{k,2s,s}_{\mathrm{for},\Lambda^q}}, & 2s+k \le n/2, \\ \|d_{q-1}p\|_{B^{k,2s,s}_{\mathrm{for},\Lambda^q}} + \|p\|_{L^2(I,C_{b,\Lambda^{q-1}})}, & n/2 < 2s+k \le n/2+1, \\ \|d_{q-1}p\|_{B^{k,2s,s}_{\mathrm{for},\Lambda^q}} + \|p\|_{L^2(I,C_{b,\Lambda^{q-1}})} + \|p\|_{C^2(I,C_{b,\Lambda^{q-1}})}, & n/2+1 < 2s+k. \end{cases}$$

It is easy to see that

$$B^{k,2s,s}_{\mathrm{vel},\Lambda^q},\ B^{k,2s,s}_{\mathrm{for},\Lambda^q},\ B^{k+1,2s,s}_{\mathrm{pre},\Lambda^{q-1}},$$

are Banach spaces. We proceed with two simple lemmata.

Lemma 3.4. *If $n \ge 3$ and $s \in \mathbb{N}$ then the following embedding are continuous:*

$$B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \hookrightarrow B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q},\ B^{k,2s,s}_{\mathrm{for},\Lambda^q} \hookrightarrow B^{k+2,2(s-1),s-1}_{\mathrm{for},\Lambda^q},$$

$$B^{k+1,2s,s}_{\mathrm{pre},\Lambda^{q-1}} \hookrightarrow B^{k+3,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}.$$

If, in addition, $k+2s > n/2-1$ then the embedding

$$B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \hookrightarrow L^2(I, L^\infty_{\Lambda^q}) \cap L^\infty(I, L^n_{\Lambda^q})$$

is continuous, too.

Proof. The continuity of the first three embeddings follows immediately from the definition of the spaces.

Next, by the definition, the space $B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$ is continuously embedded into the spaces $C(I, H^{k+2s}_{\Lambda^q})$ and $L^2(I, H^{k+2s+1}_{\Lambda^q})$.

Note that, by the Sobolev embedding theorem (see, for instance, [1, Ch. 4, Theorem 4.12]), for any $k', s' \in \mathbb{Z}_+$ and $\lambda \in (0,1)$ satisfying

$$k' - s' - \lambda > n/2, \tag{3.21}$$

there exists a constant $c(k,s,\lambda)$ depending on the parameters, such that

$$\|u\|_{C^{s',\lambda}_b(\mathbb{R}^n)} \le c(k',s',\lambda)\, \|u\|_{H^{k'}(\mathbb{R}^n)}$$

for all $u \in H^{k'}(\mathbb{R}^n)$.

Since $2s+k > n/2-1$ we see that $L^2(I, H^{k+2s+1}_{\Lambda^q})$ is continuously embedded to $L^2(I, L^\infty_{\Lambda^q})$.

If, in addition, $m' > n/2-1$ then by (2.3) and the hypothesis of the lemma,

$$\|u\|_{L^n_{\Lambda^q}} \le C_n \|\nabla^{m'} u\|^{\frac{n-2}{2m'}}_{L^2_{\Lambda^q}} \|u\|^{\frac{2m'-n+2}{2m'}}_{L^2_{\Lambda^q}} \tag{3.22}$$

with the Gagliardo-Nirenberg constant C_n, because

$$\frac{1}{n} = \Big(\frac{1}{2} - \frac{m'}{n}\Big)\alpha + \frac{1-\alpha}{2},\ \alpha = \frac{n-2}{2m'} \in (0,1).$$

Then, since $2s+k > n/2-1$, we see that $C(I, H^{k+2s}_{\Lambda^q})$ is continuously embedded to $L^\infty(I, L^n_{\Lambda^q})$. □

Lemma 3.5. *Suppose that $n \ge 3$, $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $2s+k > \frac{n}{2}-1$. As defined above, the mappings*

$$\begin{array}{rrcl}
d_{q-1}: & B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}} & \to & B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}, \\
\Delta: & B^{k,2s,s}_{\mathrm{vel},\Lambda^q} & \to & B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}, \\
\partial_t: & B^{k,2s,s}_{\mathrm{vel},\Lambda^q} & \to & B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}, \\
 & B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \ni u(x,t) & \to & u(x,0) \in \mathbf{H}_{k+2s,\Lambda^q}, \\
\partial^\beta_x: & B^{k,2(s-1),s-1}_{\mathrm{vel},\Lambda^q} & \to & B^{k-|\beta|,2(s-1),s-1}_{\mathrm{for},\Lambda^q},\ |\beta| \le k, \\
\partial^\beta_x: & B^{k,2(s-1),s-1}_{\mathrm{vel},\Lambda^q} & \to & B^{k-|\beta|,2(s-1),s-1}_{\mathrm{vel},\Lambda^q}(I,\mathbb{R}^n),\ |\beta| \le k, \\
\partial^\beta_x: & B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q} & \to & B^{k-|\beta|,2(s-1),s-1}_{\mathrm{for},\Lambda^q},\ |\beta| \le k, \\
\mathcal{N}_q: & B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q} & \to & B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}, \\
\mathcal{N}_q: & B^{k,2s,s}_{\mathrm{vel},\Lambda^q} & \to & B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}.
\end{array}$$

are continuous. Besides, if $w \in B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q}$ then the mappings

$$\begin{array}{rrcl}
\mathbf{B}_q(w,\cdot): & B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q} & \to & B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}, \\
\mathbf{B}_q(w,\cdot): & B^{k,2s,s}_{\mathrm{vel},\Lambda^q} & \to & B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q},
\end{array}$$

are continuous, too, and for all $u, w \in B^{k+2,2(s-1),(s-1)}_{\mathrm{vel},\Lambda^q}$,

$$\|\mathbf{B}_q(w,u)\|_{B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}} \le c^{(q)}_{s,k} \|w\|_{B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q}} \|u\|_{B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q}}, \tag{3.23}$$

with positive constants $c^{(q)}_{s,k}$ independent on u, w.

Proof. Indeed, the first seven linear operators are continuous by the very definition of the function spaces.

We begin with $s = 1$. By the definition, the space $B^{k+2,0,0}_{\mathrm{vel},\Lambda^q}$ is continuously embedded into the spaces $C(I, H^{k+2}_{\Lambda^q})$ and $L^2(I, H^{k+3}_{\Lambda^q})$.

Applying (3.22) with $m' = k+2 > n/2-1$, we obtain

$$\|\mathbf{B}_q(w,u)\|^2_{L^2_{\Lambda^q}} \le \|w\|^2_{L^n_{\Lambda^q}} \|\nabla u\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}} + \|\nabla w\|^{\frac{2n}{n-2}}_{L^2_{\Lambda^q}} \|u\|^2_{L^n_{\Lambda^q}} \le \tag{3.24}$$

$$c\Big(\|\nabla^2 u\|^2_{L^2_{\Lambda^q}} \|\nabla^{k+2} w\|^{\frac{n-2}{k+2}}_{L^2_{\Lambda^q}} \|w\|^{\frac{2k-n+6}{k+2}}_{L^2_{\Lambda^q}} + \|\nabla^2 w\|^2_{L^2_{\Lambda^q}} \|\nabla^{k+2} u\|^{\frac{n-2}{k+2}}_{L^2_{\Lambda^q}} \|u\|^{\frac{2k-n+6}{k+2}}_{L^2_{\Lambda^q}}\Big),$$

where the constant c being independent of u and w, and so

$$\|\mathbf{B}_q(w,u)\|^2_{C(I,L^2_{\Lambda^q})} \leq c\Big(\|w\|^2_{C(I,H^{k+2}_{\Lambda^q})}\|\nabla^2 u\|^2_{C(I,L^2_{\Lambda^q})} + \|\nabla^2 w\|^2_{C(I,L^2_{\Lambda^q})}\|u\|^2_{C(I,H^{k+2}_{\Lambda^q})}\Big). \tag{3.25}$$

By Leibniz rule,

$$\partial_j \mathbf{B}_q(w,u) = \mathbf{B}_q(\partial_j w,u) + \mathbf{B}_q(w,\partial_j u). \tag{3.26}$$

As $n \geq 3$, using (3.22) we also get

$$\|\nabla \mathbf{B}_q(w,u)\|^2_{L^2_{\Lambda^q}} \leq c\Big(\|w\|^2_{L^n_{\Lambda^q}}\|\nabla^2 u\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}} + 2\,\|\nabla w\|^2_{L^n_{\Lambda^q}}\|\nabla u\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}} + \|\nabla^2 w\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}}\|u\|^2_{L^n_{\Lambda^q}}\Big) \leq c\Big(\|w\|^2_{H^{k+2}_{\Lambda^q}}\|\nabla^3 u\|^2_{L^2(\mathbb{R}^n)} + 2\,\|w\|^2_{H^{k+3}_{\Lambda^q}}\|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla^3 w\|^2_{L^2_{\Lambda^q}}\|u\|^2_{H^{k+2}_{\Lambda^q}}\Big) \tag{3.27}$$

with a constant c independent of u and w. On combining (3.24) and (3.27) we deduce that, for $n \geq 3$,

$$\|\mathbf{B}_q(w,u)\|^2_{L^2(I,H^1_{\Lambda^q})} \leq c\Big(\|w\|^2_{C(I,H^{k+2}_{\Lambda^q})}\|\nabla^3 u\|^2_{L^2(I,L^2_{\Lambda^q})} + 2\,\|w\|^2_{L^2(I,H^{k+3}_{\Lambda^q})}\|\nabla^2 u\|^2_{C(I,L^2_{\Lambda^q})} + \|\nabla^3 w\|^2_{L^2(I,L^2_{\Lambda^q})}\|u\|^2_{C(I,H^{k+2}_{\Lambda^q})}\Big). \tag{3.28}$$

Inequalities (3.25), (3.28) provide that the operator $\mathbf{B}_q(w,\cdot)$ maps $B^{k+2,0,0}_{\mathrm{vel},\Lambda^q}$ continuously to $B^{0,0,0}_{\mathrm{for},\Lambda^q}$ if $k > n/2-3$.

If $|\alpha| = k' \leq k_0$, $k_0 = k$ or $k_0 = k+1$, then, similarly to (3.26) we get

$$\partial^\alpha_x \mathbf{B}_q(w,u) = \sum_{\beta+\gamma=\alpha} c_{\beta,\gamma}\,\mathbf{B}_q(\partial^\beta_x w, \partial^\gamma_x u). \tag{3.29}$$

Next, for any $0 \leq k' \leq k_0$, similarly to (3.27), using the Hölder inequality with a number $p = p(k',l) > 1$ we obtain

$$\|\nabla^{k'}\mathbf{B}_q(w,u)\|^2_{L^2_{\Lambda^q}} \leq \sum_{l=0}^{k'} c_{k',l}\Big(\|\nabla^l w\|^2_{L^{\frac{2p}{p-1}}_{\Lambda^q}}\|\nabla^{k'+1-l}u\|^2_{L^{2p}_{\Lambda^q}} + \|\nabla^{k'+1-l}w\|^2_{L^{\frac{2p}{p-1}}_{\Lambda^q}}\|\nabla^l u\|^2_{L^{2p}_{\Lambda^q}}\Big)$$

the coefficients $c_{\beta,\gamma}$ and $c_{k',l}$ being of binomial type.

If $0 \leq k' \leq k_0$, then we take $p = p(k',0) = \frac{n}{n-2}$ and use (2.1), (3.22) with $m' = k+2$, to obtain

$$\|\nabla^{k'+1}u\|^2_{L^{2p}_{\Lambda^q}}\|w\|^2_{L^{\frac{2p}{p-1}}_{\Lambda^q}} = \|\nabla^{k'+1}u\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}}\|w\|^2_{L^n_{\Lambda^q}} \leq C_{k',k}\|\nabla^{k'+2}u\|^2_{L^2_{\Lambda^q}}\|\nabla^{k+2}w\|^{\frac{n-2}{2(k+2)}}_{L^2_{\Lambda^q}}\|w\|^{\frac{2k-n+6}{2(k+2)}}_{L^2_{\Lambda^q}} \leq C_{k',k}\|\nabla^{k'+2}u\|^2_{L^2_{\Lambda^q}}\|w\|^2_{H^{k+2}_{\Lambda^q}} \tag{3.30}$$

with positive constants $C_{k',k}$ independent on u, w.

If $1 \leq l \leq k' \leq k_0$ then we may apply (2.3) to each factor in the typical summand

$$\|\nabla^{k'+1-l}u\|^2_{L^{2p}_{\Lambda^q}}\|\nabla^l w\|^2_{L^{\frac{2p}{p-1}}_{\Lambda^q}}$$

with entries $p = p(k', l)$, $\alpha_j = \alpha_j^{(l)}$, satisfying

$$\left\{\begin{array}{l} \frac{1}{2p} = \frac{k'-l+1}{n} + \left(\frac{1}{2} - \frac{k_0+2}{n}\right)\alpha_1 + \frac{1-\alpha_1}{2}, \\ \frac{p-1}{2p} = \frac{l}{n} + \left(\frac{1}{2} - \frac{k+2}{n}\right)\alpha_2 + \frac{1-\alpha_2}{2}, \\ \frac{k'-l+1}{k_0+2} \leq \alpha_1 < 1, \frac{l}{k+2} \leq \alpha_2 < 1. \end{array}\right. \tag{3.31}$$

Relations (3.31) are actually equivalent to the following:

$$\frac{k'-l+1}{k_0+2} \leq \alpha_1 = \frac{n}{k_0+2}\Big(\frac{1}{2} - \frac{1}{2p} + \frac{1+k'-l}{n}\Big) < 1 \tag{3.32}$$

$$\frac{l}{k+2} \leq \alpha_2 = \frac{n}{k+2}\Big(\frac{1}{2p} + \frac{l}{n}\Big) < 1. \tag{3.33}$$

The lower bounds are always true if $p > 1$ and so, these inequalities are reduced to

$$\frac{1}{2} + \frac{k'-l-(k_0+1)}{n} < \frac{1}{2p} < \frac{k+2-l}{n},\ p > 1.$$

The segment for $\frac{1}{2p}$ is not empty because

$$\frac{1}{2} + \frac{k'-l-1-k_0}{n} < \frac{k+2-l}{n}$$

provided by the assumptions $k+3 > n/2$, $0 \leq k' \leq k_0$. Moreover, as

$$\frac{1}{2} + \frac{k'-l-k_0-1}{n} < \frac{1}{2},\ \frac{k+2-l}{n} > 0,$$

we see that there is a proper $q > 1$ to achieve (3.32), (3.33).

Then, similarly to (3.30), and using Young's inequality (2.1),

$$\|\nabla^{k'+1-l}u\|^2_{L^{2p}(\mathbb{R}^n)}\|\nabla^l w\|^2_{L^{\frac{2p}{p-1}}_{\Lambda^q}} \leq \tag{3.34}$$

$$C^{(l,k)}_{k',k_0}\|\nabla^{k_0+2}u\|^{2\alpha_1}_{L^2_{\Lambda^q}}\|\nabla^{k+2}w\|^{2\alpha_2}_{L^2_{\Lambda^q}}\|u\|^{2(1-\alpha_1)}_{L^2_{\Lambda^q}}\|w\|^{2(1-\alpha_2)}_{L^2_{\Lambda^q}} \leq$$
$$C^{(l,k)}_{k',k_0}\|u\|^2_{H^{k_0+2}_{\Lambda^q}}\|w\|^2_{H^{k+2}_{\Lambda^q}}$$

with positive constants $C^{(l,k)}_{k',k_0}$ independent on u, w.

Hence, (3.30), (3.34) yield

$$\|\mathbf{B}_q(w,u)\|^2_{C(I,H^k_{\Lambda^q})} \leq \tag{3.35}$$

$$c\Big(\|u\|^2_{C(I,H^{k+2}_{\Lambda^q})}\|w\|^2_{C(I,H^{k+2}_{\Lambda^q})} + \|w\|^2_{C(I,H^{k+2}_{\Lambda^q})}\|u\|^2_{C(I,H^{k+2}_{\Lambda^q})}\Big),$$

$$\|\mathbf{B}_q(w,u)\|^2_{L^2_T(H^{k+1}_{\Lambda^q})} \leq \tag{3.36}$$

$$c\Big(\|u\|^2_{C_T(H^{k+2}(_{\Lambda^q})}\|w\|^2_{L_T(H^{k+3}_{\Lambda^q})} + \|w\|^2_{C_T(H^{k+2}_{\Lambda^q})}\|u\|^2_{L_T(H^{k+3}_{\Lambda^q})}\Big)$$

with a positive constant c independent on u, w.

Now (3.35), (3.36) imply that the mapping $\mathbf{B}_q(w,\cdot)$ maps $B^{k+2,0,0}_{\mathrm{vel},\Lambda^q}$ continuously to $B^{k,0,0}_{\mathrm{for},\Lambda^q}$ for any $k > n/2-3$ if $n \geq 3$ and bound (3.23) hold true for $s = 1$ because of (2.1).

Next, we argue by the induction. Assume that for some $s' \geq 1$ the mapping $\mathbf{B}_q(w,\cdot)$ maps the space $B^{k+2,2(s'-1),s'-1}_{\mathrm{vel},\Lambda^q}$ to $B^{k,2(s'-1),s'-1}_{\mathrm{for},\Lambda^q}$ continuously for any $k > n/2-2s'-1$ and bound (3.23) holds true for $s = s'$. Then the space $B^{k+2,2s',s'}_{\mathrm{vel},\Lambda^q}$

is embedded continuously to the space $B^{k+4,2(s'-1),s'-1}_{\mathrm{vel},\Lambda^q}$ and, by the inductive assumption, $\mathbf{B}_q(w,\cdot)$ maps $B^{k+4,2(s'-1),s'-1}_{\mathrm{vel},\Lambda^q}$ continuously to $B^{k+2,2(s'-1),s'-1}_{\mathrm{for},\Lambda^q}$ for any $(k+2) > n/2-2s'-1$ or, the same, $k > n/2-2(s'+1)-1$. Moreover, bound (3.23) holds true for $s = s'$ and with $k+2$ instead of k.

It is left to check the behaviour of the partial derivatives $\partial_t^{s'}\partial_x^\alpha \mathbf{B}_q(w,u)$ with $|\alpha| \le k+1$. By the very definition, of space $B^{k+2,2s',s'}_{\mathrm{vel},\Lambda^q}$, the partial derivatives $\partial_t^i u$, $\partial_t^i w$ belong to $B^{k+2,2(s'-i),(s'-i)}_{\mathrm{vel},\Lambda^q}$, $C(I, H^{k+2+2(s'-i)}_{\Lambda^q})$ and $L^2(I, H^{k+3+2(s'-i)}_{\Lambda^q})$.

By the Leibniz rule,

$$\partial_t \mathbf{B}_q(w,u) = \mathbf{B}_q(\partial_t w, u) + \mathbf{B}_q(w, \partial_t u).$$

Then for acceptable $\alpha \in \mathbb{Z}_+$ and i, similarly to (3.29), we get with binomial type coefficients $c_{\beta,\gamma}$ and C^l_j

$$\partial_x^\alpha \partial_t^i \mathbf{B}_q(w,u) = \sum_{\beta+\gamma=\alpha} \sum_{l=0}^{i} c_{\beta,\gamma} C^l_i \, \mathbf{B}_q(\partial_x^\beta \partial_t^l w, \partial_x^\gamma \partial_t^{i-l} u). \tag{3.37}$$

Similarly to (3.30), if $0 \le k' \le k+1$ then we take $p = p^{(s')}_{i,0} = \frac{n}{n-2}$ and use (2.1), (3.22) with $m' = 2s'+k+2$, to obtain

$$\begin{gathered} \|\partial_t^{s'} \nabla^{k'+1} u\|^2_{L^{2p}_{\Lambda^q}} \|w\|^2_{L^{\frac{2p}{p-1}}_{\Lambda^q}} = \|\partial_t^{s'} \nabla^{k'+1} u\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}} \|w\|^2_{L^n_{\Lambda^q}} \le \\ C^{k,s'}_{k',0} \|\partial_t^{s'} \nabla^{k'+2} u\|^2_{L^2_{\Lambda^q}} \|\nabla^{2s'+k+2} w\|^{\frac{n-2}{2(2s'+k+2)}}_{L^2_{\Lambda^q}} \|w\|^{\frac{2(2s'+k)-n+6}{2(2s'+k+2)}}_{L^2_{\Lambda^q}} \le \\ C^{k,s'}_{k',0} \|\partial_t^{s'} \nabla^{k'+2} u\|^2_{L^2_{\Lambda^q}} \|w\|^2_{H^{2s'+k+2}_{\Lambda^q}} \end{gathered} \tag{3.38}$$

with positive constants $C^{k,s'}_{k',0}$ independent on u, w.

Again, similarly to (3.34), If $1 \le j \le k' \le k_0$ then we may apply (2.3) to each factor in the typical summand

$$\|\partial_t^{s'-i} \nabla^{k'+1-j} u\|^2_{L^{2p}_{\Lambda^q}} \|\partial_t^i \nabla^j w\|^2_{L^{\frac{2p}{p-1}}_{\Lambda^q}}$$

with entries satisfying

$$\left\{ \begin{array}{l} \frac{1}{2p} = \frac{k'-j+1}{n} + \Big(\frac{1}{2} - \frac{k_0+2+2i}{n}\Big)\alpha_1 + \frac{1-\alpha_1}{2}, \\ \frac{p-1}{2p} = \frac{j}{n} + \Big(\frac{1}{2} - \frac{k+2+2(s'-i)}{n}\Big)\alpha_2 + \frac{1-\alpha_2}{2}, \\ \frac{k'-j+1}{k_0+2+2i} \le \alpha_1 < 1, \ \frac{j}{k+2+2(s'-i)} \le \alpha_2 < 1. \end{array} \right. \tag{3.39}$$

Relations (3.39) are actually equivalent to the following:

$$\frac{k'-j+1}{k_0+2+2i} \le \alpha_1 = \frac{n}{k_0+2+2i}\Big(\frac{1}{2} - \frac{1}{2p} + \frac{1+k'-j}{n}\Big) < 1 \tag{3.40}$$

$$\frac{j}{k+2+2(s'-i)} \le \alpha_2 = \frac{n}{k+2+2(s'-i)}\Big(\frac{1}{2p} + \frac{j}{n}\Big) < 1. \tag{3.41}$$

The lower bounds are always true if $q > 1$ and so, these inequalities are reduced to

$$\frac{1}{2} + \frac{k'-j-(k_0+1+2i)}{n} < \frac{1}{2p} < \frac{k+2+2(s'-i)-j}{n}, \ p > 1.$$

The segment for $\frac{1}{2q_j}$ is not empty because

$$\frac{1}{2}+\frac{k'-j-1-k_0-2i}{n}<\frac{k+2-j+2(s'-i)}{n}$$

provided by the assumptions $k>n/2-2(s'+1)-1$, $0\leq k'\leq k_0$. Moreover, as

$$\frac{1}{2}+\frac{k'-j-(k_0+1+2i)}{n}<\frac{1}{2},\ \frac{k+2+2(s'-i)-j}{n}>0,$$

we see that there is a proper $q>1$ to achieve (3.40), (3.41).

Then, similarly to (3.30), and using Young's inequality (2.1),

$$\|\partial_t^{s'-i}\nabla^{k'+1-j}u\|_{L^{2p}_{\Lambda^q}}\|\partial_t^i\nabla^j w\|_{L^{\frac{2p}{p-1}}_{\Lambda^q}}\leq \tag{3.42}$$

$$c\|\partial_t^{s'-i}\nabla^{k_0+2+2i}u\|^{\alpha_1}_{L^2_{\Lambda^q}}\|\partial_t^i\nabla^{k+2+2(s'-i)}w\|^{\alpha_2}_{L^2_{\Lambda^q}}\|\partial_t^{s'-i}u\|^{1-\alpha_1}_{L^2_{\Lambda^q}}\|\partial_t^i w\|^{1-\alpha_2}_{L^2_{\Lambda^q}}\leq$$

$$c\|\partial_t^{s'-i}u\|_{H^{k_0+2+2i}_{\Lambda^q}}\|\partial_t^i w\|_{H^{k+2+2(s'-i)}_{\Lambda^q}}$$

with positive constants c independent on u,w.

Hence, (3.38), (3.42) yield

$$\|\partial_t^{s'}\mathbf{B}_q(w,u)\|^2_{C(I,H^k_{\Lambda^q})}\leq \tag{3.43}$$

$$c\sum_{i=0}^{s'}\|\partial_t^{s'-i}u\|^2_{C(I,H^{k+2+2i}_{\Lambda^q})}\|\partial_t^i w\|^2_{C(I,H^{k+2+2(s'-i)}_{\Lambda^q})}+$$

$$c\sum_{i=0}^{s'}\|\partial_t^{s'-i}w\|^2_{C(I,H^{k+2+2i}_{\Lambda^q})}\|\partial_t^i u\|^2_{C(I,H^{k+2+2(s'-i)}_{\Lambda^q})},$$

$$\|\partial_t^{s'}\mathbf{B}_q(w,u)\|^2_{L^2(I,H^{k+1}_{\Lambda^q})}\leq \tag{3.44}$$

$$c\sum_{i=0}^{s'}\|\partial_t^{s'-i}u\|^2_{L^2(I,H^{k+3+2i}_{\Lambda^q})}\|\partial_t^i w\|^2_{C(I,H^{k+2+2(s'-i)}_{\Lambda^q})}+$$

$$c\sum_{i=0}^{s'}\|\partial_t^{s'-i}w\|^2_{L^2(I,H^{k+3+2i}_{\Lambda^q})}\|\partial_t^i u\|^2_{C(I,H^{k+3+2(s'-i)}_{\Lambda^q})}$$

with a positive constant c independent on u,v.

Now (3.43), (3.44) imply that the mapping $\mathbf{B}_q(w,\cdot)$ maps $B^{k+2,2s',s'}_{\mathrm{vel},\Lambda^q}$ continuously to $B^{k,2s',s'}_{\mathrm{for},\Lambda^q}$ if $n\geq 3$. Moreover, by (2.1), bound (3.23) holds true for $s=s'+1$. This finishes the proof of inequality (3.23) and the continuity of operator $\mathbf{B}_q(w,\cdot):B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q}\to B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}$, for $n\geq 3$ and for all $k\in\mathbb{Z}_+$ and $s\in\mathbb{N}$ satisfying $2s+k>n/2-1$.

The boundedness of the operator $\mathbf{B}_q(w,\cdot):B^{k,2s,s}_{\mathrm{vel},\Lambda^q}\to B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}$ now follows from Lemma 3.4.

Now, since the bilinear form $\mathbf{B}_q$ is symmetric and $\mathbf{B}_q(u,u)=2\mathcal{N}_q(u)$, we easily obtain

$$\mathcal{N}_q(u)-\mathcal{N}_q(u_0)=\mathbf{B}_q(u_0,u-u_0)+(1/2)\,\mathbf{B}_q(u-u_0,u-u_0). \tag{3.45}$$

Therefore, by the continuity of the mapping $B_q(w,\cdot)$,

$$\|\mathcal{N}_q(u)-\mathcal{N}_q(u_0)\|_{B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}}\leq\frac{1}{2}c(k,s)\|u-u_0\|^2_{B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q}}+$$

$$c(k,s)\|u_0\|_{B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q}}\|u-u_0\|_{B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q}}$$

with a positive constant $c(k,s)$ independent of u and u_0, i.e., the nonlinear operator $\mathcal{N}_q$ maps $B^{k+2,2(s-1),s-1}_{\mathrm{vel},\Lambda^q}$ continuously into $B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}$ and $B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$ continuously into $B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}$. □

Proposition 3.1. *Let $n \geq 3$, $1 \leq q < n$, $2s+k > n/2$ and $F \in B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}$ satisfies $\langle F, v\rangle_{\Lambda^q} = 0$ for all $v \in \mathcal{V}_{\Lambda^q}$. Then there is a unique form $p^{(0)}$ from the space $B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}$, satisfying*

$$d_{q-1}p^{(0)} = F \ \text{in}\ \mathbb{R}^n \times [0,T]. \tag{3.46}$$

Proof. Indeed, the space $B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q}$ is embedded continuously to $C(I, H^{k+2s-2}_{\Lambda^q}) \cap L^2(I, H^{k+2s-1}_{\Lambda^q})$. Then for almost all $t \in [0,T]$ the unique solution $p^{(0)}(\cdot,t)$ to (3.46), satisfying (2.13), (2.14), (2.18) and (3.18) was constructed in Proposition 2.2. Then (2.18) for $p^{(0)}$ yields

$$\|p^{(0)}\|_{L^2(I,C_{b,\Lambda^{q-1}})} \leq c_q\|F\|_{L^2(I,H^{2s+k-1}_{\Lambda^q})} \text{ for } 2s+k > n/2,$$
$$\|p^{(0)}|_{C(I,C_{b,\Lambda^{q-1}})} \leq c_q\|F\|_{C(I,H^{2s+k-2}_{\Lambda^q})} \text{ for } 2s+k > n/2+1.$$

Finally, by the elliptic regularity, $p^{(0)} \in C(I, H^{2s+k-1}_{\mathrm{loc},\Lambda^{q-1}}) \cap L^2(I, H^{2s+k}_{\mathrm{loc},\Lambda^{q-1}})$. □

Now we arrive at the principal theorem related to linearizations of the Navier-Stokes type Equations associated with the de Rham complex.

Theorem 3.2. *Let $n \geq 3$, $0 \leq q < n$, $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $2s+k > n/2$, and $w \in B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$. Then (3.1) induces a bijective continuous linear mapping*

$$\mathcal{A}^{(q)}_w : B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \times B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}} \to B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{2s+k,\Lambda^q}. \tag{3.47}$$

which admits a continuous inverse $(\mathcal{A}^{(q)}_w)^{-1}$.

Proof. It follows the same scheme as the proof of similar regularity theorems for Stokes and Navier-Stokes equations, see, for instance, [18], [43].

We begin with a simple lemma.

Lemma 3.6. *If $n \geq 3$, $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $2s+k > n/2$ then (3.47) is an injective continuous linear mapping.*

Proof. Indeed, the continuity of $\mathcal{A}^{(q)}_w$ follows from Lemma 3.5. Let

$$\begin{aligned} (u,p) &\in B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \times B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}, \\ \mathcal{A}^{(q)}_w(u,p) = (f,u_0) &\in B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{k+2s,\Lambda^q}. \end{aligned}$$

The integration by parts with the use of (1.1) yields

$$-(\Delta u, u)_{L^2_{\Lambda^q}} = \|\nabla u(\cdot,t)\|^2_{L^2_{\Lambda^q}}. \tag{3.48}$$

As $d^*_{q-1} u = 0$ in $\mathbb{R}^n \times [0,T]$, we see that

$$(d_{q-1}p, u)_{L^2_{\Lambda^q}} = (p, d^*_{q-1} u)_{L^2_{\Lambda^{q-1}}} = 0. \tag{3.49}$$

As $2s+k+1 > n/2$, Lemma 3.4 implies that the space $B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$ is continuously embedded into $L^2(I, L^\infty_{\Lambda^q}) \cap L^\infty(I, L^n_{\Lambda^q})$. Then formulas (3.8), (3.6), (3.48) and (3.49) readily imply that u is the unique weak solution to (3.1) granted by Theorem 3.1, i.e., (3.2) is fulfilled. In particular, if $(f, u_0) = 0$ then $u \equiv 0$ and p satisfies $d_{q-1} p = 0$ in $\mathbb{R}^n \times [0,T]$. By the definition of the space $B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}$ the form p satisfies also (2.13) for each $t \in [0,T]$, (3.18) and (3.19). Hence $p(\cdot,t) = 0$ for almost all $t \in [0,T]$ by Proposition 3.1 and then the operator $\mathcal{A}^{(q)}_w$ is injective. □

Let us continue with the proof of the surjectivity.

Lemma 3.7. *Let $n \geq 3$, $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $2s + k > n/2$ and $w \in B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$. Then for each $(f, u_0) \in B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{2s+k,\Lambda^q}$ there is a solution $u \in B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$ to (3.2).*

Proof. Let (f, u_0) be arbitrary data in $B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{2s+k,\Lambda^q}$ and and let $\{u_m\}$ be the sequence of the corresponding Faedo-Galerkin approximations, constructed in the proof of Theorem 3.1. The scalar functions $F_i(t) = \langle f(\cdot,t), b_i \rangle_{\Lambda^q}$ belong to $C^{s-1}[0,T] \cap H^s[0,T]$ and the components $\mathfrak{C}^{(m)}_{i,j}(t)$ belong to $C^s[0,T] \cap H^{s+1}[0,T]$, see (3.5). Since $w \in B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$, formula (3.5) means that the entries of the matrix $\exp\left(\int_0^t \mathfrak{C}^{(m)}(\tau) d\tau\right)$ belong actually to $C^{s+1}[0,T] \cap H^{s+2}[0,T]$ and then the components of the vector $g^{(m)}$ belong to $C^s[0,T] \cap H^{s+1}[0,T]$.

Let us begin with $s = 1$ and $k \in \mathbb{Z}_+$ satisfying $k > n/2 - 2$. If we multiply the equation corresponding to index j in (3.3) by $\frac{dg^{(m)}_j}{d\tau}$ then, after the summation with respect to j, we obtain for all $\tau \in [0,T]$:

$$
\|\partial_\tau u_m\|^2_{L^2_{\Lambda^q}} + \frac{\mu}{2} \frac{d}{\partial\tau} \|\nabla u_m\|^2_{L^2_{\Lambda^q}} = (f, \partial_\tau u_m)_{L^2_{\Lambda^q}} + (\mathbf{B}_q(u_m, w), \partial_\tau u_m)_{L^2_{\Lambda^q}}. \tag{3.50}
$$

After the integration with respect to $\tau \in [0,t]$ and the application of the Hölder inequalities with $q_1 = \infty$, $q_2 = 2$, $q_3 = 2$ and $p_1 = \frac{2n}{n-2}$, $p_2 = n$, $p_3 = 2$ with the use of (3.6), (3.7) we arrive at the following:

$$
\begin{aligned}
\|\partial_\tau u_m\|^2_{L^2([0,t],L^2_{\Lambda^q})} + \mu \|\nabla u_m(\cdot,t)\|^2_{L^2_{\Lambda^q}} \leq 2\|\nabla u_{0m}\|^2_{L^2_{\Lambda^q}} + \\
2\|f\|^2_{L^2(I,L^2_{\Lambda^q})} + c_q \int_0^t (\|w\|^2_{L^\infty_{\Lambda^q}} \|\nabla u_m\|^2_{L^2_{\Lambda^q}} + \|u_m\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}} \|\nabla w\|^2_{L^n_{\Lambda^q}}) \, d\tau \leq \\
2\|\nabla u_{0m}\|^2_{L^2_{\Lambda^q}} + 2\|f\|^2_{L^2(I,L^2_{\Lambda^q})} + \\
c_q \int_0^t \left(\|w\|^2_{H^{k+2}_{\Lambda^q}} + \|\nabla^{k+2} w\|^{\frac{n-2}{k+1}}_{L^2_{\Lambda^q}} \|\nabla w\|^{\frac{2k-n+4}{k+1}}_{L^2_{\Lambda^q}}) \|\nabla u_m\|^2_{L^2_{\Lambda^q}} \right) d\tau,
\end{aligned} \tag{3.51}
$$

with a constant $c_q > 0$ independent on w and m, the last bound being a consequence of the Sobolev Embedding Theorems and Gagliardo-Nirenberg inequalities (3.17), (3.22) with $m' = k + 1 > n/2 - 1$.

As $u_{0,m}$ we may take the orthogonal projection on the linear span $\mathcal{L}(\{b_j\}_{j \in \mathbb{N}})$ in $\mathbf{H}_{1,\Lambda^q}$ achieving

$$
\lim_{m \to +\infty} \|u_0 - u_{0,m}\|_{H^1_{\Lambda^q}} = 0, \|u_{0,m}\|_{H^1_{\Lambda^q}} \leq \|u_0\|_{H^1_{\Lambda^q}},
$$

cf. [43, formula (3.87)].

On the other hand, $k > n/2 - 2$ provides $\frac{n-2}{k+1} < 2$ and then

$$
\|\nabla^{k+2} w\|^{\frac{n-2}{k+1}}_{L^2_{\Lambda^q}} \|\nabla w\|^{\frac{2k-n+4}{k+1}}_{L^2_{\Lambda^q}} \leq \frac{k+1}{n-2} \|\nabla^{k+2} w\|^2_{L^2_{\Lambda^q}} + \frac{k+1}{2k-n+4} \|\nabla w\|^2_{L^2_{\Lambda^q}} \tag{3.52}
$$

by Young's inequality (2.1).

Since $w \in C(I, H^{k+2}(\mathbb{R}^n))$, inequality (3.51) implies that the sequence $\{u_m\}$ is bounded in the space $C(I, H^1_{\Lambda^q}) \cap L^2(I, H^1_{\Lambda^q})$ and the sequence $\{\partial_t u_m\}$ is bounded in the space $L^2(I, L^2_{\Lambda^q})$. In particular, we may extract a subsequence $\{u_{m'}\}$, $\{\partial_t u_{m'}\}$ such that

1) $\{u_{m'}\}$ converges $*$-weakly in $L^\infty(I, L^2_{\Lambda^q})$ to an element $u \in L^\infty(I, \mathbf{H}_{0,\Lambda^q})$,

2) for each $1 \le j \le n$, the sequence of forms $\{\partial_j u_{m'}\}$ converges $*$-weakly in $C(I, \mathbf{H}_{0,\Lambda^q})$ to an element $u^{(j)} \in C(I, \mathbf{H}_{0,\Lambda^q})$,

3) $\{u_{m'}\}$ converges weakly in $L^2(I, \mathbf{H}_{1,\Lambda^q})$ to an element $u \in L^2(I, \mathbf{H}_{1,\Lambda^q})$,

4) $\{\partial_t u_{m'}\}$ converges weakly in $L^2(I, \mathbf{H}_{0,\Lambda^q})$ to an element $\tilde{u} \in L^2(I, \mathbf{H}_{0,\Lambda^q})$.

By the very construction and Theorem 3.1, the form u is the unique solution to (3.2) from the space $C(I, \mathbf{H}_{1,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q})$ with $\partial_t u = \tilde{u} \in L^2(I, \mathbf{H}_{0,\Lambda^q})$, $\partial_j u = u^{(j)} \in C(I, \mathbf{H}_{0,\Lambda^q})$.

Moreover, similarly to (3.24), using the Hölder inequality, (3.6), (3.7) and (2.3), we obtain

$$\|\mathbf{P}_q \mathbf{B}_q(w,u)\|^2_{L^2_{\Lambda^q}} \le \|\mathbf{B}_q(w,u)\|^2_{L^2_{\Lambda^q}} \le$$
$$c_q\Big(\|\nabla u\|^2_{L^2_{\Lambda^q}} \|w\|^2_{L^\infty_{\Lambda^q}} + \|\nabla w\|^2_{L^n_{\Lambda^q}} \|u\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}}\Big) \le$$
$$c_q\Big(\|\nabla u\|^2_{L^2_{\Lambda^q}} \|w\|^2_{H^{k+2}_{\Lambda^q}} + \|\nabla^{k+2} w\|^{\frac{n-2}{k+1}}_{L^2_{\Lambda^q}} \|\nabla w\|^{\frac{2k-n+4}{k+1}}_{L^2_{\Lambda^q}} \|\nabla u\|^2_{L^2_{\Lambda^q}}\Big),$$

with a constant c_q independent on u, w and then, by (3.52),

(3.53)
$$\|\mathbf{P}_q \mathbf{B}_q(w,u)\|^2_{L^2(I,L^2_{\Lambda^q})} \le \|\mathbf{B}_q(w,u)\|^2_{L^2(I,L^2_{\Lambda^q})} \le c_q \|u\|^2_{C(I,H^1_{\Lambda^q})} \|w\|^2_{L^2(I,H^{k+2}_{\Lambda^q})},$$

(3.54)
$$\|\mathbf{P}_q \mathbf{B}_q(w,u)\|^2_{C(I,L^2_{\Lambda^q})} \le \|\mathbf{B}_q(w,u)\|^2_{C(I,L^2(\mathbb{R}^n))} \le c_q \|u\|^2_{C(I,H^1_{\Lambda^q})} \|w\|^2_{C(I,H^{k+2}_{\Lambda^q})},$$

with a constant $c_q > 0$ independent on u, w. Thus, the forms $\mathbf{P}_q \mathbf{B}_q(w,u)$ and $\mathbf{B}_q(w,u)$ belong to $L^2(I, L^2_{\Lambda^q}) \cap C(I, L^2_{\Lambda^q})$.

Actually, (3.2) imply that, in the sense of distributions,

$$\mu \Delta u = \partial_t u + \mathbf{P}_q(\mathbf{B}_q(w,u) - f) \text{ in } \mathbb{R}^n \times (0,T). \tag{3.55}$$

Then, by Lemma 2.2 the form Δu belongs to $L^2(I, \mathbf{H}_{0,\Lambda^q})$.

As it is known, one of the equivalent norms on $H^s_{\Lambda^q}$ is the norm

$$(\|1 + |\zeta|^2)^{s/2} \mathfrak{F}(u)\|_{L^2_{\Lambda^q}}. \tag{3.56}$$

As $u, \Delta u \in L^2(I, \mathbf{H}_{0,\Lambda^q})$, we see that $(\|1 + |\zeta|^2)\mathfrak{F}(u)\|_{L^2(I,L^2_{\Lambda^q})}$ is finite and then $u \in L^2(I, \mathbf{H}_{2,\Lambda^q}) \cap C(I, \mathbf{H}_{1,\Lambda^q})$.

Thus, we constructed a unique solution $u \in L^2(I, \mathbf{H}_{2,\Lambda^q}) \cap C(I, H_{1,\Lambda^q})$ to (3.2) if $u_0 \in \mathbf{H}_{1,\Lambda^q}$, $f \in L^2(I, L^2_{\Lambda^q})$. But we have actually at least $u_0 \in \mathbf{H}_{2,\Lambda^q}$, $f \in L^2(I, H^1_{\Lambda^q})$. Moreover, for each $1 \le j \le n$, using the Sobolev Embeddings and (3.22) with $m' = k+1$, (3.52), we obtain with a constant $c_q > 0$ independent on u, w,

$$\|\mathbf{B}_q(\partial_j w, u)\|^2_{L^2_{\Lambda^q}} \le c_q\Big(\|\partial_j w\|^2_{L^\infty_{\Lambda^q}} \|\nabla u\|^2_{L^2_{\Lambda^q}} + \|\partial_j \nabla w\|^2_{L^n_{\Lambda^q}} \|\nabla u\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}}\Big) \le$$
$$c_q \|w\|^2_{H^{k+3}_{\Lambda^q}} \|u\|^2_{H^1_{\Lambda^q}},$$
$$\|\mathbf{B}_q(\partial_j w, u)\|^2_{L^2(I,L^2_{\Lambda^q})} \le c \|w\|^2_{L^2(I,H^{k+3}_{\Lambda^q})} \|u\|^2_{C(I,H^1_{\Lambda^q})},$$

i.e. $\mathbf{B}_q(\partial_j w, u) \in L^2(I, L^2_{\Lambda^q})$.

Next, by Lemma 3.5, for each $1 \leq j \leq n$ we have $(\partial_j f, \partial_j u_0) \in L^2(I, H^2_{\Lambda^q}) \times \mathbf{H}_{1,\Lambda^q}$, Therefore, by the already proved part of the lemma, there is a unique solution $u^{(j)} \in C(I, \mathbf{H}_{1,\Lambda^q}) \cap L^2(I, \mathbf{H}_{2,\Lambda^q})$ to

(3.57)
$$\begin{cases} \frac{d(u^{(j)}, v)_{L^2_{\Lambda^q}}}{d\tau} + \mu \sum_{|\alpha|=1} (\partial^\alpha u^{(j)}, \partial^\alpha v)_{L^2_{\Lambda^q}} = \langle \partial_j f - \mathbf{B}_q(\partial_j w, u) - \mathbf{B}_q(w, u^{(j)}), v\rangle_{\Lambda^q}, \\ u^{(j)}(\cdot, 0) = \partial_j u_0 \end{cases}$$

for each $v \in \mathbf{H}_{1,\Lambda^q}$. On the other hand, $\partial_j u \in L^2(I, \mathbf{H}_{1,\Lambda^q}) \cap C(I, \mathbf{H}_{0,\Lambda^q})$ satisfies (3.57), too. Hence the uniqueness provides that $\partial_j u = u^{(j)}$ for each $1 \leq j \leq n$. Combining this with the fact that (3.56) define the equivalent norms on the Sobolev scale $H^s_{\Lambda^q}$ we conclude that $u \in C(I, \mathbf{H}_{2,\Lambda^q}) \cap L^2(I, \mathbf{H}_{3,\Lambda^q})$.

Next, by the Sobolev Embedding Theorem and inequality (3.22) with $m' = k+1$,

$$\|\nabla \mathbf{P}_q \mathbf{B}_q(w, u)\|^2_{L^2(I, L^2_{\Lambda^q})} \leq \|\nabla \mathbf{B}_q(w, u)\|^2_{L^2(I, L^2_{\Lambda^q})} \leq$$

$$c_q \int_0^T \Big(\|\nabla^2 u\|^2_{L^2_{\Lambda^q}} \|w\|^2_{L^\infty_{\Lambda^q}} + 2\|\nabla w\|^2_{L^n_{\Lambda^q}} \|\nabla u\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}} + \|\nabla^2 w\|^2_{L^n_{\Lambda^q}} \|u\|^2_{L^{\frac{2n}{n-2}}_{\Lambda^q}} \Big) d\tau \leq$$

$$c_q \|u\|^2_{L^2(I, H^2_{\Lambda^q})} \|w\|^2_{C(I, H^{k+2}_{\Lambda^q})} + c\|u\|^2_{C(I, H^1_{\Lambda^q})} \|w\|^2_{L^2(I, H^{k+3}_{\Lambda^q})},$$

with a constant c_q independent on u, w, i.e., because of (3.53), (3.54), the forms $\mathbf{P}_q \mathbf{B}_q(w, u)$ and $\mathbf{B}_q(w, u)$ belong to $L^2(I, H^1_{\Lambda^q}) \cap C(I, L^2_{\Lambda^q})$. Then it follows from (3.55) that $\partial_t u \in C(I, \mathbf{H}_{0,\Lambda^q}) \cap L^2(I, \mathbf{H}_{1,\Lambda^q})$ if $f \in C(I, L^2_{\Lambda^q}) \cap L^2(I, H^1_{\Lambda^q}) = B^{0,0,0}_{\mathrm{for},\Lambda^q}$ and $u_0 \in \mathbf{H}_{2,\Lambda^q}$.

Thus, for each pair $(f, u_0) \in B^{0,0,0}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{2,\Lambda^q}$ and each $w \in B^{k,2,1}_{\mathrm{vel},\Lambda^q}$, $k > n/2 - 2$, there is a unique solution $u \in B^{0,2,1}_{\mathrm{vel},\Lambda^q}$ to (3.2).

If $k = 0$ then the proof of the lemma for $s = 1$ is complete (however this is possible for $n = 3$ only because $k > n/2 - 2$). For $k \geq 1$ we argue by the induction with respect to $k' \in \mathbb{Z}_+$, $0 \leq k' \leq k - 1$.

Assume that for each pair $(f, u_0) \in B^{k',0,0}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{2+k',\Lambda^q}$ and $w \in B^{k,2,1}_{\mathrm{vel},\Lambda^q}$, $k > n/2 - 2$, there is a unique solution $u \in B^{k',2,1}_{\mathrm{vel},\Lambda^q}$ to (3.2). We have to prove that the solution u belongs to $B^{k'+1,2,1}_{\mathrm{vel},\Lambda^q}$ if $(f, u_0) \in B^{k'+1,0,0}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{3+k',\Lambda^q}$.

Indeed, by Lemma 3.5, for each $1 \leq j \leq n$ we have

$$(\partial_j f, \partial_j u_0) \in B^{k',0,0}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{2+k',\Lambda^q},\ \partial_j w \in B^{k-1,2,1}_{\mathrm{vel},\Lambda^q}$$

and, by Lemma 3.5, we have $\mathbf{B}_q(\partial_j w, u) \in B^{k',0,0}_{\mathrm{for},\Lambda^q}$. Then, by the inductive assumption there is a unique differential form $u^{(j)} \in B^{k',2,1}_{\mathrm{vel},\Lambda^q}$ satisfying (3.57). Moreover, (3.35), (3.35), (3.36) and Lemmata 2.2, 3.5 imply that the forms $\mathbf{P}_q \mathbf{B}_q(w, u)$, $\mathbf{B}_q(w, u)$ belong to the space $C(I, \mathbf{H}_{k'-1,\Lambda^q}) \cap L^2(I, \mathbf{H}_{k',\Lambda^q})$. Now, formula (3.55) yields $\partial_t u \in C(I, \mathbf{H}_{k'-1,\Lambda^q}) \cap L^2(I, \mathbf{H}_{k',\Lambda^q})$. Thus, we conclude that $u \in B^{k'+1,2,1}_{\mathrm{vel},\Lambda^q}$.

Finally, we invoke an induction with respect to s. With this purpose, assume that the statement of the lemma is true for $s = s_0$ and each $k \in \mathbb{Z}_+$ satisfying $2s_0 + k > n/2$. Let us prove it for $s = s_0 + 1$ and any $k \in \mathbb{Z}_+$ satisfying $2(s_0 + 1) + k > n/2$. More precisely, we have to show that each data $(f, u_0) \in B^{k,2s_0,s_0}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{2(s_0+1)+k,\Lambda^q}$ and $w \in B^{k_0,2(s_0+1),s_0+1}_{\mathrm{vel},\Lambda^q}$ admits a unique solution $u \in B^{k_0,2(s_0+1),s_0+1}_{\mathrm{vel},\Lambda^q}$ to (3.2).

According to Lemma 3.4 we have $(f,u_0) \in B^{k+2,2(s_0-1),s_0-1}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{2s_0+k+2,\Lambda^q}$ and $w \in B^{k+2,2s_0,s_0}_{\mathrm{vel},\Lambda^q}$. Then, by the inductive assumption, there is a unique solution $u \in B^{k+2,2s_0,s_0}_{\mathrm{vel},\Lambda^q}$ to (3.2). Again, (3.35), (3.36), and Lemmata 2.2, 3.5 imply that the forms $\mathbf{P}_q\mathbf{B}_q(w,u), \mathbf{B}_q(w,u)$ belong to $B^{k,2s_0,s_0}_{\mathrm{for},\Lambda^q}$ and then (3.55) yields $\partial_t u \in B^{k,2s_0,s_0}_{\mathrm{for},\Lambda^q}$, too. Summarizing, we conclude that u belongs to the space $B^{k,2(s_0+1),s_0+1}_{\mathrm{vel},\Lambda^q}$, that was to be proved. □

At this point it follows from Proposition 3.1 that there is a unique function $p \in B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}$ such that

$$d_{q-1}p = (I - \mathbf{P}_q)(f - \mathbf{B}_q(w,u)) \text{ in } \mathbb{R}^n \times [0,T]. \tag{3.58}$$

Adding (3.55) to (3.58) we conclude that the pair

$$(u,p) \in B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \times B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}$$

is the unique solution to (3.1) related to the datum $(f,u_0) \in B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^{q-1}} \times \mathbf{H}_{2s+k,\Lambda^{q-1}}$. This implies the surjectivity of the mapping $\mathcal{A}^{(q)}_w$.

Finally, as the mapping $\mathcal{A}^{(q)}_w$ is bijective and continuous, the continuity of the inverse $(\mathcal{A}^q_w)^{-1}$ follows from the inverse mapping theorem for Banach spaces. □

Since problem (3.1) is a linearisation of the Navier-Stokes equations type at an arbitrary q-form w, it follows from Theorem 3.2 that the corresponding nonlinear mapping given by the Navier-Stokes type equations is locally invertible. The implicit function theory for Banach spaces even implies that the local inverse mappings can be obtained from the contraction principle of Banach. In this way we obtain what we shall call the open mapping theorem for problem (1.2).

Theorem 3.3. *Let $n \geq 3$, $0 \leq q < n$, $s \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, $2s+k > n/2$. Then (1.2) induces an injective continuous nonlinear mapping*

$$\mathcal{A}^{(q)} : B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \times B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}} \to B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{2s+k,\Lambda^q} \tag{3.59}$$

which is moreover open.

Proof. Indeed, the continuity of the mapping $\mathcal{A}^{(q)}$ is clear from Lemma 3.5. Moreover, suppose that

$$\begin{array}{rcl} (u,p) & \in & B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \times B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}, \\ \mathcal{A}^{(q)}(u,p) = (f,u_0) & \in & B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{k+2s,\Lambda^q}. \end{array}$$

As in the proof of Theorem 3.2, formulas (3.8), (3.6), (3.48) and (3.49) imply that (2.26) is fulfilled, i.e., u is a weak solution to equations (1.2).

As $2s+k+1 > n/2$, by Lemma 3.4 the space $B^{k,2s,s}_{\mathrm{vel},\Lambda^q}$ is continuously embedded into $L^2(I, L^\infty_{\Lambda^q}) \cap L^\infty(I, L^n_{\Lambda^q})$. Hence, Theorem 2.2 shows that if (u',p') and (u'',p'') belong to $B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \times B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}$ and $\mathcal{A}^{(q)}(u',p') = \mathcal{A}^{(q)}(u'',p'')$ then $u' = u''$ and

$$d_{q-1}(p'-p'')(\cdot,t) = 0, \ d^*_{q-2}(p'-p'')(\cdot,t) = 0$$

for all $t \in [0,T]$. Then $p' = p''$ according to Proposition 3.1 if $1 \leq q < n$. Of course, for $q = 0$ we do not need any p. So, the operator $\mathcal{A}^{(q)}$ of (3.59) is injective for any q, $0 \leq q < n$.

Finally, equality (3.45) makes it evident that the Frechét derivative $(\mathcal{A}^{(q)}_{(w,p_0)})'$ of the nonlinear mapping $\mathcal{A}^{(q)}$ at an arbitrary point

$$(w, p_0) \in B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \times B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}$$

coincides with the continuous linear mapping $\mathcal{A}^{(q)}_w$ of (3.47). By Theorem 3.2, $\mathcal{A}^{(q)}_w$ is an invertible continuous linear mapping from the space $B^{k,2s,s}_{\mathrm{vel},\Lambda^q} \times B^{k+1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}}$ to $B^{k,2(s-1),s-1}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{k+2s,\Lambda^q}$. Both the openness of the mapping $\mathcal{A}^{(q)}$ and the continuity of its local inverse mapping now follow from the implicit function theorem for Banach spaces, see for instance [12, Theorem 5.2.3, p. 101]. □

In particular, for $q = 1$ we obtain an open mapping theorem for the classical Navier-Stokes equations over the constructed scale of the Bochner-Sobolev type spaces.

Theorem 3.3 suggests a clear direction for the development of the topic, in which one takes into account the following property of the so-called clopen (closed and open) sets.

Corollary 3.1. *Let $n \geq 3$, $0 \leq q < n$, $s \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, $2s + k > n/2$. The range of the mapping* (3.59) *is closed if and only if it coincides with the whole destination space.*

Proof. Since the destination space is convex, it is connected. As is known, the only clopen sets in a connected topological vector space are the empty set and the space itself. Hence, the range of the mapping $\mathcal{A}^{(q)}$ is closed if and only if it coincides with the whole destination space. □

The following statement echoes the idea of using the properness property to study nonlinear operator equations, see for instance [40].

Corollary 3.2. *Let $n \geq 3$, $0 \leq q < n$, $s \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, $2s + k > n/2$. The range of the mapping* (3.59) *is closed if and only if pre-image of precompact sets under this map are bounded.*

Finally, we set

$$C^{\infty}_{\mathrm{pre},\Lambda^{q-1}} = \cap_{s=1}^{\infty} B^{1,2(s-1),s-1}_{\mathrm{pre},\Lambda^{q-1}},\ C^{\infty}_{\mathrm{vel},\Lambda^q} = \cap_{s=1}^{\infty} B^{0,2s,s}_{\mathrm{vel},\Lambda^q},$$

$$\mathbf{H}_{\infty,\Lambda^q} = \cap_{s=1}^{\infty} \mathbf{H}_{2s,\Lambda^q},\ C^{\infty}_{\mathrm{for},\Lambda^q} = \cap_{s=0}^{\infty} B^{0,2s,s}_{\mathrm{for},\Lambda^q}.$$

Corollary 3.3. *Let $n \geq 3$. Equations* (1.2) *induce an injective continuous nonlinear mapping*

$$\mathcal{A}^{(q)} : C^{\infty}_{\mathrm{vel},\Lambda^q} \times C^{\infty}_{\mathrm{pre},\Lambda^{q-1}} \to C^{\infty}_{\mathrm{for},\Lambda^q} \times \mathbf{H}_{\infty,\Lambda^q}$$

which is moreover open.

Proof. It follows immediately from Theorem 3.3. □

We finish the paper by mentioning a familiar example by P. Fatou (1922). He constructed a holomorphic mapping $f(z)$ of $\mathbb{C}^2$ whose Jacobi matrix $f'(z)$ has a constant determinant different from zero. The mapping f is a homeomorphism onto the image, however, the image of f leaves out a closed subset of $\mathbb{C}^2$ with nonempty interior. This shows that nonlinear mappings may behave rather intricately.

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