S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 18, №2, стр. 1475–1481 (2021) УДК 519.172.2
DOI 10.33048/semi.2021.18.110 MSC 05C75

TIGHT DESCRIPTION OF FACES IN TORUS TRIANGULATIONS WITH MINIMUM DEGREE 5

O.V. BORODIN, A.O. IVANOVA

Abstract. The degree d of a vertex or face in a graph G is the number of incident edges. A face $f = v_1 \ldots v_d$ in a plane or torus graph G is of type $(k_1, k_2, \ldots, k_d)$ if $d(v_i) \leq k_i$ for each i. By δ we denote the minimum vertex-degree of G. In 1989, Borodin confirmed Kotzig's conjecture of 1963 that every plane graph with minimum degree δ equal to 5 has a $(5, 5, 7)$-face or a $(5, 6, 6)$-face, where all parameters are tight. It follows from the classical theorem of Lebesgue (1940) that every plane quadrangulation with $\delta \geq 3$ has a face of one of the types $(3, 3, 3, \infty)$, $(3, 3, 4, 11)$, $(3, 3, 5, 7)$, $(3, 4, 4, 5)$. Recently, we improved this description to the following one: "$(3, 3, 3, \infty)$, $(3, 3, 4, 9)$, $(3, 3, 5, 6)$, $(3, 4, 4, 5)$", where all parameters except possibly 9 are best possible and 9 cannot go down below 8. In 1995, Avgustinovich and Borodin proved that every torus quadrangulation with $\delta \geq 3$ has a face of one of the following types: $(3, 3, 3, \infty)$, $(3, 3, 4, 10)$, $(3, 3, 5, 7)$, $(3, 3, 6, 6)$, $(3, 4, 4, 6)$, $(4, 4, 4, 4)$, where all parameters are best possible. The purpose of our note is to prove that every torus triangulation with $\delta \geq 5$ has a face of one of the types $(5, 5, 8)$, $(5, 6, 7)$, or $(6, 6, 6)$, where all parameters are best possible.

Keywords: plane graph, torus, triangulation, quadrangulation, structure properties, 3-faces.

Borodin, O.V., Ivanova, A.O., Tight description of faces in torus triangulations with minimum degree 5.

The first author' work was supported by the Ministry of Science and Higher Education of the Russian Federation (project no. 0314-2019-0016). The second author's work was supported by the Ministry of Science and Higher Education of the Russian Federation (Grant No. FSRG-2020-0006).

Received October, 28, 2021, published December, 1, 2021.

1. Introduction

The *degree* d of a vertex or face in a plane or torus graph G is the number of incident edges. A k*-vertex* and k*-face* is one of degree k, a k^+*-vertex* has degree at least k, and so on. A face $f = v_1 \dots v_d$ in G is of *type* $(k_1, k_2, \dots, v_d)$, or a $(k_1, k_2, \dots, v_d)$*-face* if $d(v_i) \leq k_i$ whenever $1 \leq i \leq d$.

By δ and w denote the minimum vertex degree and smallest degree-sum of faces in G, respectively.

We now recall some results on the structure of faces in plane graph with $\delta \geq 3$, beginning with the fundamental theorem of Lebesgue [14] from 1940.

Theorem 1 (Lebesgue [14]). *Every plane graph with* $\delta \geq 3$ *has a face of one of the following types:*

$$(3,6,\infty),\ (3,7,41),\ (3,8,23),\ (3,9,17),\ (3,10,14),\ (3,11,13),$$
$$(4,4,\infty),\ (4,5,19),\ (4,6,11),\ (4,7,9),\ (5,5,9),\ (5,6,7),$$
$$(3,3,3,\infty),\ (3,3,4,11),\ (3,3,5,7),\ (3,4,4,5),\ (3,3,3,3,5).$$

The classical Theorem 1, along with other ideas in Lebesgue [14], has a lot of applications to plane graph coloring problems (the first example of such applications and recent surveys can be found in [5, 15]).

Some parameters of Lebesgue's Theorem were improved for several narrow classes of plane graphs.

In 1963, Kotzig [13] proved that every plane triangulation with $\delta = 5$ satisfies $w \leq 18$ and conjectured that $w \leq 17$ holds. In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form.

Theorem 2 (Borodin [2]). *Every plane graph with* $\delta = 5$ *has a* $(5,5,7)$*-face or a* $(5,6,6)$*-face, where all parameters are tight.*

Theorem 2 also confirmed a conjecture of Grübaum [10] from 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5-connected planar graph is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [16]).

Theorem 2 was extended to several classes of plane graphs over the last decades; see, for example, recent surveys [7, 12] and also [3–5, 11].

In particular, precise descriptions of the structure of faces were obtained for plane graphs with $\delta \geq 4$ and for triangulated plane graphs.

Theorem 3 (Borodin, Ivanova [6]). *Every plane graph with* $\delta \geq 4$ *has a 3-face of one of the following types:* $(4,4,\infty)$, $(4,5,14)$, $(4,6,10)$, $(4,7,7)$, $(5,5,7)$, $(5,6,6)$, *where all parameters are best possible.*

Theorem 4 (Borodin, Ivanova, Kostochka [9]). *Every plane triangulation has a face of one of the following types:* $(3,4,31)$, $(3,5,21)$, $(3,6,20)$, $(3,7,13)$, $(3,8,14)$, $(3,9,12)$, $(3,10,12)$, $(4,4,\infty)$, $(4,5,11)$, $(4,6,10)$, $(4,7,7)$, $(5,6,6)$, $(5,5,7)$, *where all parameters are tight.*

It follows from Theorem 1 that every plane quadrangulation with $\delta \geq 3$ has a face of one of the types $(3,3,3,\infty)$, $(3,3,4,11)$, $(3,3,5,7)$, $(3,4,4,5)$. Recently, we improved this result as follows.

Theorem 5 (Borodin, Ivanova [8]). *Every plane quadrangulation with* $\delta \geq 3$ *has a face of one of the types:* $(3,3,3,\infty)$, $(3,3,4,9)$, $(3,3,5,6)$, $(3,4,4,5)$, *where all parameters except possibly* 9 *are best possible.*

We believe that 9 above is also sharp and thus the whole description is tight. At least, we know that 9 cannot go down below 8.

In 1995, Avgustinovich and Borodin gave the following tight description of faces in quadrangulations of the torus.

Theorem 6 (Avgustinovich, Borodin [1]). *Every torus quadrangulation with* $\delta \geq 3$ *has a face of one of the following types:* $(3,3,3,\infty)$, $(3,3,4,10)$, $(3,3,5,7)$, $(3,3,6,6)$, $(3,4,4,6)$, $(4,4,4,4)$, *where all parameters are best possible.*

The purpose of this note is to prove the following tight describing of faces in torus triangulations with $\delta \geq 5$.

Theorem 7. *Every torus triangulation with* $\delta \geq 5$ *has a face of one of the following types:*
(Ta) $(5,5,8)$,
(Tb) $(5,6,7)$, *or*
(Tc) $(6,6,6)$,
where all parameters are best possible.

2. The tightness of Theorem 7

The tightness of (Ta) is confirmed by the torus graph in Fig. 1, since its every face is incident with a 5-vertex, 8-vertex and the third vertex of degree 5 or 8.

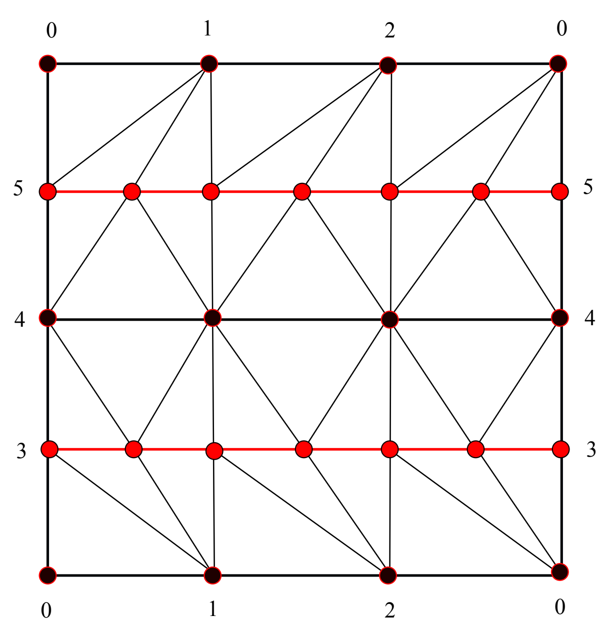

FIG. 1. A torus triangulation with all faces of type $(5^+,5^+,8^+)$.

Figure 2 represents a torus graph with seven pairwise adjacent 6-faces. Here, $0, 1, \dots, 6$ are not the vertices of a graph but just the points on the boundary of a plane pattern of the torus, so the points labeled the same should be glued to return to the torus. The dual of this graph has only 3-faces incident with three 6-vertices, which confirms the necessity and sharpness of T(c).

Now if we replace each 6-face in Fig. 2 by a construction shown in Fig. 3, we produce a torus triangulation in which every face $v_1v_2v_3$ satisfies $d(v_1) \geq 5$, $d(v_2) \geq 6$, and $d(v_3) \geq 7$. This confirms that the term T(b) is also best possible in Theorem 7.

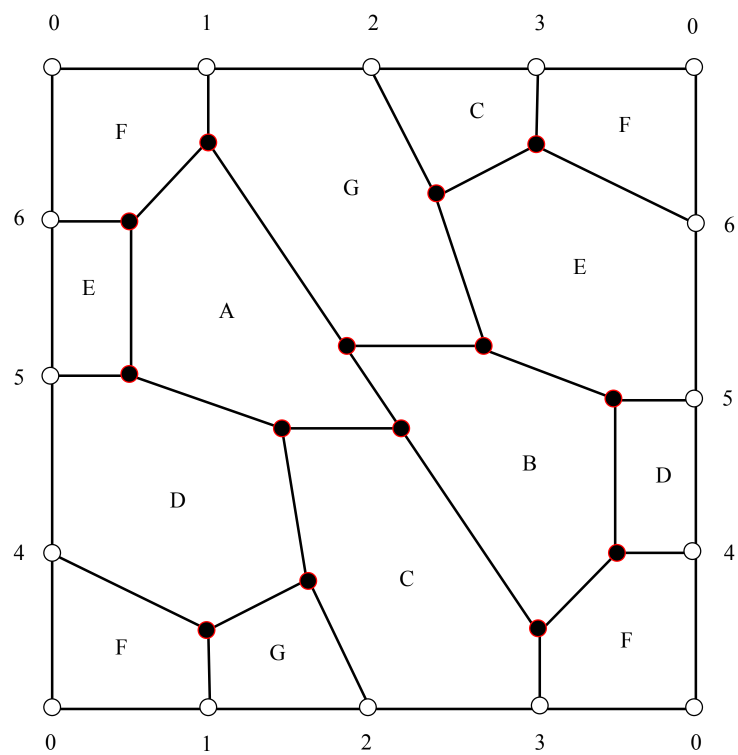

FIG. 2. Seven pairwise adjacent 6-faces on the torus.

3. Proving the existence of face-types in Theorem 7

Suppose T is a counterexample to Theorem 7.

Euler's formula $|V| - |E| + |F| = 0$ for the torus triangulation T implies

$$\sum_{v \in V} (d(v) - 6) = 0. \tag{1}$$

We assign a *charge* $\mu(v) = d(v) - 6$ to every vertex v, so only 5-vertices have a negative charge. Using the properties of T as a counterexample, we define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(v)$ is

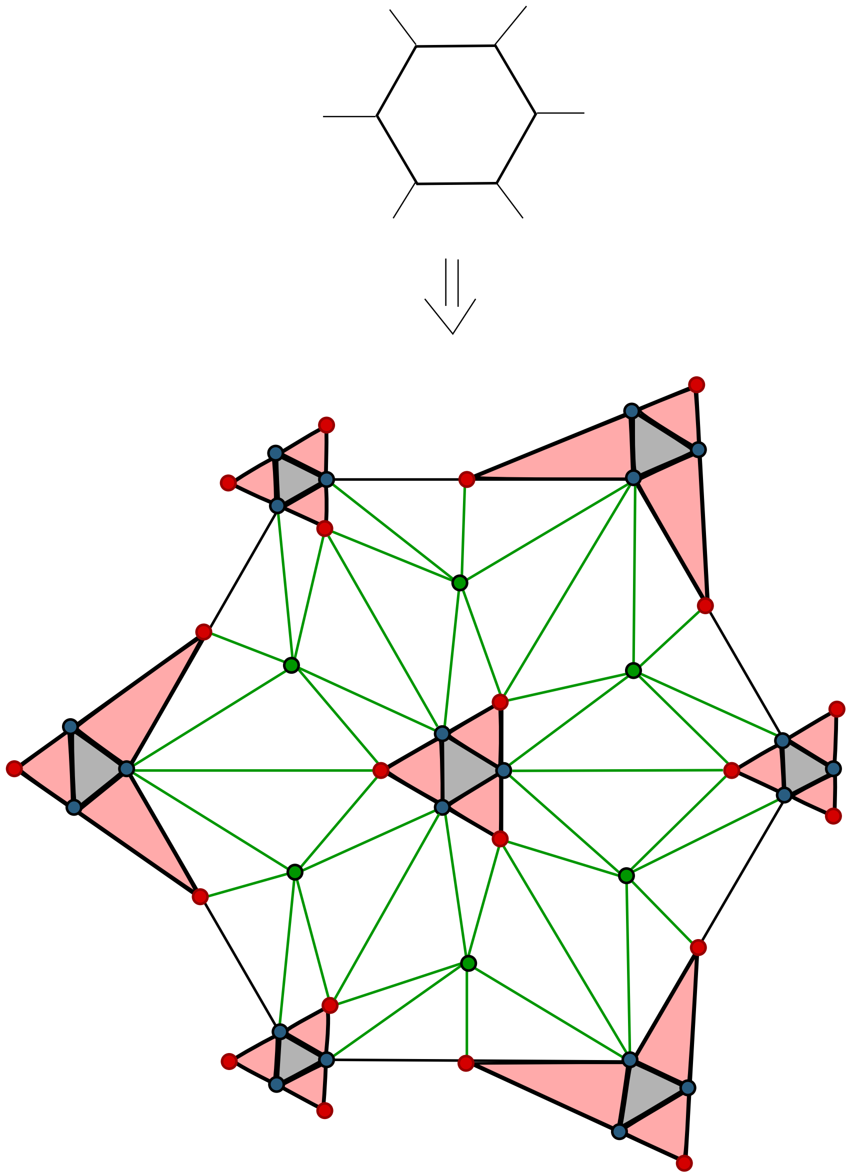

FIG. 3. A replacement for each 6-face in Figure 2 to produce a torus triangulation with all faces of type $(5^+, 6^+, 7^+)$.

non-negative whenever $v \in V$ and there is at least one vertex v with $\mu'(v) > 0$. This will contradict the fact that the sum of the new charges is, by (1), equal to 0.

We use the following rule of discharging.

R. Each 5-vertex receives $\frac{1}{3}$ from each 7^+-neighbor.

We now check that $\mu'(v) \geq 0$ by **R** for all $v \in V$, which implies due to (1) that vertices v with $\mu'(v) > 0$ cannot exist in T.

In what follows, by "non-(k, l, m)!" we mean a short-hand for "since T has no (k, l, m)-faces".

CASE 1. $d(v) \geq 10$. By $\mathbf{R}$, we have $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{3} = \frac{2(d(v)-9)}{3} > 0$, so T has no 10^+-vertices.

CASE 2. $d(v) = 9$. Since $\mu'(v) \geq 9-6-\frac{9}{3} = 0$, we can assume that each 9-vertex of T is completely surrounded by 5-vertices.

CASE 3. $d(v) = 8$. By by non-$(5, 5, 8)$!, our v has at most four 5-neighbors, so $\mu'(v) \geq 8 - 6 - 4 \times \frac{1}{3} > 0$, which means that T has no 8-vertices either.

CASE 4. $d(v) = 7$. Now v has at most three 5-neighbors by non-$(5, 5, 7)$!, which implies that $\mu'(v) \geq 7 - 6 - 3 \times \frac{1}{3} = 0$. Moreover, $\mu'(v) = 0$ implies that v has exactly three 5-neighbors.

CASE 5. $d(v) = 6$. Since v does not participate in $\mathbf{R}$, we have $\mu'(v) = \mu(v) = 0$.

CASE 6. $d(v) = 5$. Note that v has at most two 6^--neighbors by non-$(5, 6, 6)$!, so v has at least three 7^+-neighbors, which results in $\mu'(v) \geq 5 - 6 + 3 \times \frac{1}{3} \geq 0$ by $\mathbf{R}$.

To arrive at a final contradiction arising in connection with a 5-vertex v, it suffices to show that in fact either $\mu'(v) > 0$, which means that T has no 5-vertices, or v has a neighbor with a positive μ'.

The absence of 5-vertices in T would imply by Cases 1–4 that T entirely consists of 6-vertices, contrary to non-$(6, 6, 6)$!.

It remains to assume that our 5-vertex v has precisely two 6^--neighbors, say v_1 and v_3. However, then $d(v_4) \geq 8$ and $d(v_5) \geq 8$ by non-$(5, 6, 7)$!, so $\mu'(v_4) > 0$ as shown in Cases 1–3.

Thus we have proved $\mu'(v) \geq 0$ for every $v \in V$ and there is a vertex v with $\mu'(v) > 0$, which contradicts (1) and completes the proof of Theorem 7.

References

[1] S.V. Avgustinovich, O.V. Borodin, *Edge neighborhoods in normal maps*, in Korshunov, A.D. (ed.), *Operations research and discrete analysis*, Mathematics and its applications, **391**, 17–22, 1997. Zbl 0860.05022

[2] O.V. Borodin, *Solution of problems of Kotzig and Grünbaum concerning the isolation of cycles in planar graphs*, Math. Notes, **46**:5 (1989), 835–837. Zbl 0717.05034

[3] O.V. Borodin, *Triangulated 3-polytopes without faces of low weight*, Discrete Math., **186**:1-3 (1998), 281–285. Zbl 0956.52010

[4] O.V. Borodin, *Sharpening Lebesgue's theorem on the structure of lowest faces of convex polytopes*, Diskretn. Anal. Issled. Oper., Ser. 1, **9**:3 (2002), 29–39. Zbl 1015.52008

[5] O.V. Borodin, *Colorings of plane graphs: a survey*, Discrete Math., **313**:4 (2013), 517–539. Zbl 1259.05042

[6] O.V. Borodin, A.O. Ivanova, *Describing 3-faces in normal plane maps with minimum degree 4*, Discrete Math., **313**:23 (2013), 2841–2847. Zbl 1281.05054

[7] O.V. Borodin, A.O. Ivanova, *New results about the structure of plane graphs: A survey*, AIP Conference Proceedings, **1907**, 030051 (2017).

[8] O.V. Borodin, A.O. Ivanova, *An improvement of Lebesgue's description of edges in* 3*-polytopes and faces in plane quadrangulations*, Discrete Math., **342**:6 (2019), 1820–1827. Zbl 1414.05089

[9] O.V. Borodin, A.O. Ivanova, A.V. Kostochka, *Describing faces in plane triangulations*, Discrete Math., **319** (2014), 47–61. Zbl 1280.05027

[10] B. Grünbaum, *Polytopal graphs*, in *Studies in Graph Theory*, D.R. Fulkerson, ed., MAA Studies in Mathematics, **12** (1975), 201–224.

[11] M. Horňák, S. Jendrol', *Unavoidable sets of face types for planar maps*, Discuss. Math., Graph Theory, **16**:2 (1996), 123–141. Zbl 0877.05048

[12] S. Jendrol', H.-J. Voss, *Light subgraphs of graphs embedded in the plane. A survey*, Discrete Math., **313**:4 (2013), 406–421. Zbl 1259.05045

[13] A. Kotzig, *From the theory of Euler's polyhedrons*, Mat.-Fyz. Čas., **13** (1963), 20–30. Zbl 0134.19601

[14] H. Lebesgue, *Quelques conséquences simples de la formule d'Euler*, J. Math. Pures Appl., IX. Sér., **19** (1940), 27–43. Zbl 0024.28701

[15] O. Ore, M.D. Plummer, *Cyclic coloration of plane graphs*, Recent Progr. Comb., Proc. 3rd Waterloo Conf. 1968, (1969), 287–293. Zbl 0195.25701

[16] M.D. Plummer, *On the cyclic connectivity of planar graph*, Graph Theory Appl., Lect. Notes Math., **303** (1972), 235–242. Zbl 0247.05113

Oleg Veniaminovich Borodin
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: brdnoleg@math.nsc.ru

Anna Olegovna Ivanova
Ammosov North-Eastern Federal University,
48, Kulakovskogo str.,
Yakutsk, 677013, Russia
Email address: shmgnanna@mail.ru