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ON COMPRESSED ZERO-DIVISOR GRAPHS OF FINITE COMMUTATIVE LOCAL RINGS

E.V. ZHURAVLEV, O.A. FILINA

Abstract. We describe the compressed zero-divisor graphs of a commutative finite local rings R of characteristic p with Jacobson radical J such that $J^4 = (0)$, $F = R/J \cong GF(p^r)$ and $\dim_F J/J^2 = 2$, $\dim_F J^2/J^3 = 2$, $\dim_F J^3 = 1$ or $\dim_F J/J^2 = 3$, $\dim_F J^2/J^3 = 1$, $\dim_F J^3 = 1$.

Keywords: finite ring, local ring, zero-divisor graph.

1. Introduction

All rings in this paper are considered to be finite, associative and commutative rings with identity. By $J = J(R)$ and R^* denote the Jacobson radical and the group of unit elements of R respectively. Also let F denote a finite field $GF(q)$ of order $q = p^r$ where p is prime.

By $\Gamma(R)$ denote a zero-divisor graph of the ring R, i.e. a graph with elements of R as its vertices and pairs (x, y) (where x may be equal to y) such that $xy = 0$ as its edges (see [1, 2]).

Let S be a commutative semigroup with zero. For every $x \in S$ consider

$$\mathrm{Ann}(x) = \{y \in S \mid xy = 0\}.$$

We introduce the following equivalence relation:

$$\text{for all } x, y \in S \quad x \sim y \Leftrightarrow \mathrm{Ann}(x) = \mathrm{Ann}(y).$$

By $[x]$ denote the equivalence class of $x \in S$ and by $S/\sim$ denote the quotient set with respect to said equivalence relation.

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Note that $\sim$ is in fact a congruence relation on S: indeed, for all $x_1, x_2, y_1, y_2 \in S$ if $x_1 \sim x_2$ and $y_1 \sim y_2$ then $x_1x_2 \sim y_1y_2$. Hence we can consider $S/\sim$ a semigroup with the operation $[x][y] = [xy]$.

Consider a graph $\Gamma(S/\sim)$ with elements of $S/\sim$ as its vertices and with pairs of vertices $([x], [y])$ where $[x]$ may be equal to $[y]$, such that $[x][y] = [0]$ (or equivalently such that $xy = 0$) as its edges. Such graphs are called compressed zero-divisor graphs (see [3, 4]). If $[x] \in \Gamma(S/\sim)$, then either $ab = 0$ for all $a, b \in [x]$ or $ab \neq 0$ for all $a, b \in [x]$. Therefore every vertex $[x]$ with a loop in $\Gamma(S/\sim)$ is a complete subgraph in $\Gamma(S)$, such that all of its vertices have loops, while a vertex $[x]$ without a loop in $\Gamma(S/\sim)$ is an empty subgraph of $\Gamma(S)$, i.e. a set of isolated vertices.

The ring R is said to be local if $R/J = F$ is a field. All zero-divisors of a local ring form a radical J, that is a nilpotent ideal of nilpotency index K, $K \in \mathbb{N}$ (the set of natural numbers), and every element of the local ring is either a unit element or a nilpotent element. It is known that the finite commutative ring with identity is a direct sum of local rings (see [5, 6]).

Proposition 1. [5, 6, 7]*Let R be a local ring of characteristic p, $R/J = GF(p^r) = F$ and K be the nilpotency index of J. Then*

1. *J^k/J^{k+1} is a finite-dimensional vector space over F for all $1 \leq k \leq K-1$,*
2. *there exists $n \in \mathbb{N}$ such that $|R| = p^{nr}$ and $|J| = p^{(n-1)r}$.*

Zero-divisor graphs were first introduced by D.F. Andrerson, P.S. Livingston and I. Beck in [1, 2]. The research on that topic is connected to the problem of describing the rings, such that their zero-divisor graphs satisfy certain conditions. For instance, one can find complete descriptions of rings with planar zero-divisor graphs [8, 9, 10, 11], eulerian zero-divisor graphs [12], and finite rings with complete bipartite zero-divisor graphs. Additionally, in [14, 15], a description of ring varieties, such that finite rings with isomorphic zero-divisor graphs are isomorphic, while in [16] a study of commutative finite rings with compressed zero divisor graphs of order 2 has been carried out and all graphs of order 3 that are zero-divisor graphs for some finite ring were found.

These results show that the problem of presenting zero-divisor graphs of finite rings is still relevant. In [3, 4, 17] one can find a description of such graphs for commutative local rings R of order p^{nr}, where $nr \leq 5$. The authors of these papers used the classification of finite rings from [18, 19, 20, 21]. More specifically, Tadayoyonfar and Ashrafi used their knowledge of the rings' multiplication table as well as the presentation of a zero-divisor graph as a union and sum of empty and complete graphs (see [17]), while Bloomfield (see [4]), presented the geomtetric representation of the graphs $\Gamma(R/\sim)$ for such rings.

Currently, local rings of order p^{nr}, $nr > 5$ are classified only in some special cases. For instance, in [22, 23] all local rings R of characteristic p of all the following types

1. $\dim_F J/J^2 = 2,\ \dim_F J^2/J^3 = 2,\ \dim_F J^3 = 1,\ J^4 = (0)$;
2. $\dim_F J/J^2 = 3,\ \dim_F J^2/J^3 = 1,\ \dim_F J^3 = 1,\ J^4 = (0)$,

are described up to isomorphism. These are the rings of order p^{6r}. Moreover, in [24], the compressed graphs of rings of characteristic 2 with extra conditions $\dim_F J/J^2 = 2$, $\dim_F J^2/J^3 = 2$, $\dim_F J^3 = 1$, $J^4 = (0)$ are considered. The aim of this paper is to continue the research, started in [24]. For both types of rings and for any p we wish to show the equivalence classes for the relation $\sim$ and to present the geometric representations for graphs $\Gamma(R/\sim)$.

2. Preliminaries

Let R be a local ring of characteristic p and $\dim_F J/J^2 = 3$, $\dim_F J^2/J^3 = 1$, $\dim_F J^3 = 1$, $J^4 = 0$. Then R can be presented as a sum

$$R = F \oplus Fu_1 \oplus Fu_2 \oplus Fu_3 \oplus Fv \oplus Fw$$

and

$$J = Fu_1 \oplus Fu_2 \oplus Fu_3 \oplus Fv \oplus Fw,$$

where $\{u_1, u_2, u_3, v, w\}$ is a basis of J over F. Note that $u_1, u_2, u_3 \in J \setminus J^2$, $v \in J^2 \setminus J^3$, $w \in J^3$ (see [22]). Since $u_iu_j \in J^2$ and $u_iv \in J^3$, we have that

$$u_iu_j = a_{ij}v + b_{ij}w \quad \text{и} \quad u_iv = c_{i1}w$$

for some a_{ij}, b_{ij}, $c_{i1} \in F$, $i, j = \overline{1,3}$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \ C = \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix}$$

be the multiplication matrices of he ring R and let δ_1 be some element of $F^* \setminus F^{*2}$. All pairwise non-isomorphic local rings that are considered here can be defined by the following matrices (see [22]):

(1) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$;

(2) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$;

(3) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, where $p = 2$;

(4) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, where $d \in \{1, \delta_1\}$.

Let R be a local ring of characteristic p and $\dim_F J/J^2 = 2$, $\dim_F J^2/J^3 = 2$, $\dim_F J^3 = 1$, $J^4 = 0$. Then R also can be presented as a sum

$$R = F \oplus Fu_1 \oplus Fu_2 \oplus Fv_1 \oplus Fv_2 \oplus Fw$$

and

$$J = Fu_1 \oplus Fu_2 \oplus Fv_1 \oplus Fv_2 \oplus Fw,$$

where $\{u_1, u_2, v_1, v_2, w\}$ is a basis of J over F. Note that $u_1, u_2 \in J \setminus J^2$, $v_1, v_2 \in J^2 \setminus J^3$, $w \in J^3$ (see [22, 23]). Since $u_iu_j \in J^2$ and $u_iv_j \in J^3$, we have that

$$u_iu_j = a_{ij}^{(1)}v_1 + a_{ij}^{(2)}v_2 + b_{ij}w \quad \text{и} \quad u_iv_j = c_{ij}w$$

for some $a_{ij}^{(1)}$, $a_{ij}^{(2)}$, b_{ij}, $c_{ij} \in F$, $i, j = \overline{1,2}$.

Let

$$A_1 = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{pmatrix}, \ A_2 = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \ C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

be the multiplication matrices of R.

Now we introduce some auxiliary definitions and provide the list of all pairwise non-isomorphic rings, that are considered in our case (see [22, 23]).

Let

$$M_1 = \left\{z \in F^* \ \mid \ \forall x \in F \ z(1+3\delta_1 x^2) - x\delta_1(3+\delta_1 x^2) \neq 0\right\}.$$

Now consider the set of functions

$$\mathcal{K}_1 = \left\{\varphi^{\pm}_{a,c} : M_1 \to F\right\}, \quad \varphi^{\pm}_{a,c}(z) = \frac{\pm az(a^2+3\delta_1 c^2) - c\delta_1(3a^2+\delta_1 c^2)}{a(a^2+3\delta_1 c^2) \mp cz(3a^2+\delta_1 c^2)},$$

where $a = 0$, $c = 1$ or $a = 1$, $c \in F$.

This set is a group with respect to the binary operation $(\phi_1 \circ \phi_2)(z) = \phi_1(\phi_2(z))$ $(\phi_1, \phi_2 \in \mathcal{K}_1)$. This group is acting on the set M_1 (see [22]). The set M_1 splits into non-intersecting orbits, we denote the set of their representatives by $\mathcal{K}_1 \setminus M_1$.

Let δ_2 be an element of F, such that $\forall x \in F$ $\delta_2 \neq x + x^2$. Also let $\mu = a^2c + ac^2 + c^3(1+\delta_2)$, $\eta = a^3 + ac^2\delta_2 + c^3\delta_2$,

$$M_2 = \{z \in F \,|\, \forall s \in \{0,1\} \ \forall a, c \in F, \ a \neq 0 \text{ or } c \neq 0, \ (\eta(1+s) + \mu\delta_2)z + \eta \neq 0\}.$$

Again, consider a set of functions

$$\mathcal{K}_2 = \{\varphi_{s,a,c} : \ M_2 \to F\}, \quad \varphi_{s,a,c}(z) = \frac{(\eta+\mu s)z + \mu}{(\eta(1+s)+\mu\delta_2)z + \eta},$$

where $s \in \{0,1\}$, $a, c \in F$, $a \neq 0$ or $c \neq 0$. This set also forms the group acting on M_2 with respect to the binary operation $(\phi_1 \circ \phi_2)(z) = \phi_1(\phi_2(z))$ $(\phi_1, \phi_2 \in \mathcal{K}_2)$ (see [23]). By $\mathcal{K}_2 \setminus M_2$ we again denote the set of orbit representatives.

For $p = 2$ all pairwise non-isomorphic local rings are defined by the following quadruples of matrices (see [23]):

(1) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;

(2) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;

(3) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

(4) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

(5) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & \delta_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ \delta_2 & 0 \end{pmatrix}$;

(6) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & \delta_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & z \\ 1+\delta_2 z & 1 \end{pmatrix}$,

where $z \in \mathcal{K}_2 \backslash M_2$;

(7) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$,

(8) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,

(9) $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & z \\ 1 & 1 \end{pmatrix}$,

where z is an element of F such that $z+1 \notin (F^*)^3$, $z \neq 1$.

While for $p \neq 2$ all pairwise non-isomorphic local rings are defined by the following quadruples of matrices (see [22]):

(1) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$

(2) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$

(3) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, for $p = 3$;

(4) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$

(5) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \delta_1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & \delta_1 \end{pmatrix};$

(6) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix};$

(7) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix},$

where $\zeta \neq \pm 1, \dfrac{1-\zeta}{1+\zeta} \notin (F^*)^3$;

(8) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \delta_1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & \xi \\ \xi & \delta_1 \end{pmatrix},$

where $\xi \in \mathcal{K}_1 \setminus M_1$;

(9) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

3. Zero-divisor graphs

We recall some useful properties of $\sim$ - classes (i.e. the equivalence classes of $\sim$ defined earlier) (see. [4]).

Proposition 2. *Let R be a local ring and let $A \subseteq R$. If $x \in \mathrm{Ann}(A)$, then $[x] \in \mathrm{Ann}(A)$. In other words, an annulator of any subset of R is a union of $\sim$ – classes.*

Proposition 3. *Let R be a local ring and also let $J^K = 0$, $J^{K-1} \neq 0$, $K \in \mathbb{N}$.*

(1) *If $x \in J^{K-1} \setminus \{0\}$, then $[x] = \mathrm{Ann}(J) \setminus \{0\}$.*
(2) *If for $x \in J \setminus \{0\}$, $\alpha \in R^*$, $y \in \mathrm{Ann}(J)$ the element $\alpha x + y \neq 0$, then $\mathrm{Ann}(x) = \mathrm{Ann}(\alpha x + y)$.*

We now consider a compressed graph $\Gamma(R/{\sim})$ of a local ring R and introduce some additional notation regarding the way groups of similar vertices are presented in the following pictures. If $\Gamma(R/{\sim})$ contains a complete subgraph with loops, we will picture this subgraph as a single vertex with a loop. If a set of vertices of $\Gamma(R/{\sim})$ contains a subset of pairwise non-adjacent vertices that have no loops we will picture that subgraph as a single vertex without a loop. In all other cases we will picture the subgraph as a single vertex with a dashed loop. For every such "vertex" we will provide a condition for its elements being adjacent (or having a loop) at the bottom of the picture. If a group of vertices, that is depicted per rules described above is connected by an edge to some vertex $[x]$ of $\Gamma(R/\sim)$ it means that every vertex from this group is adjacent to $[x]$. Similarly if such group is connected by a dashed edge then some vertex from this group is adjacent to $[x]$, when a condition presented at the bottom of a picture is satisfied.

It is obvious that $[0] = \{0\}$ and $[1] = R^*$ for every local ring R. Therefore, for the sake of brevity, we will not include vertices $[0]$ and $[1]$ into a geometric representation of a graph $\Gamma(R/\sim)$. Note that $[0]$ is adjacent to every other vertex of $\Gamma(R/\sim)$, while $[1]$ is only adjacent to $[0]$.

Theorem 1. *Let R be a local ring of characteristic p and*

$$J^4 = 0, \dim_F J/J^2 = 3, \dim_F J^2/J^3 = 1, \dim_F J^3 = 1,$$

where $F = R/J = GF(p^r)$. Then, in addition to $[0] = \{0\}$ and $[1] = R^*$, $R/\sim$ is defined by one of the following sets:

(1) $[u_1] = F^*u_1 + Fu_2 + Fu_3 + Fv + Fw$, $[v] = F^*v + Fu_2 + Fu_3 + Fw$, $[w] = (Fu_2 + Fu_3 + Fw)\backslash\{0\}$;

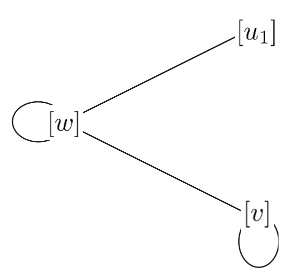
(2) $[u_1+m_iu_2] = F^*(u_1+m_iu_2)+Fu_3+Fv+Fw$, $[u_2+n_iv] = F^*(u_2+n_iv)+Fu_3 + Fw$, $[v] = F^*v + Fu_3 + Fw$, $[w] = (Fu_3 + Fw)\backslash\{0\}$;

(3) $[u_1 + s_iu_2 + l_ju_3] = F^*(u_1 + s_iu_2 + l_ju_3) + Fv + Fw$ *for all* $s_i, l_j \in F$, $[u_2+n_iu_3+k_jv] = F^*(u_2+n_iu_3+k_jv)+Fw$ *for all* $n_i, k_j \in F$, $[u_3+m_iv] = F^*(u_3 + m_iv) + Fw$ *for all* $m_i \in F$, $[v] = F^*v + Fw$, $[w] = F^*w$;

(4) $[u_1 + s_iu_2 + l_ju_3] = F^*(u_1 + s_iu_2 + l_ju_3) + Fv + Fw$ *for all* $s_i, l_j \in F$, $[u_2+n_iu_3+k_jv] = F^*(u_2+n_iu_3+k_jv)+Fw$ *for all* $n_i, k_j \in F$, $[u_3+m_iv] = F^*(u_3 + m_iv) + Fw$ *for all* $m_i \in F$, $[v] = F^*v + Fw$, $[w] = F^*w$.

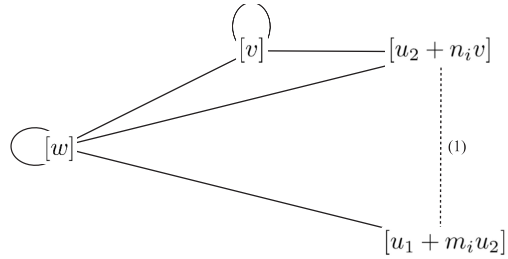
For every case presented above, the geometric representation of the graph $\Gamma(R/\sim)$ (excluding the vertices $[0]$ and $[1]$) looks as follows:

(1)



pic. 1

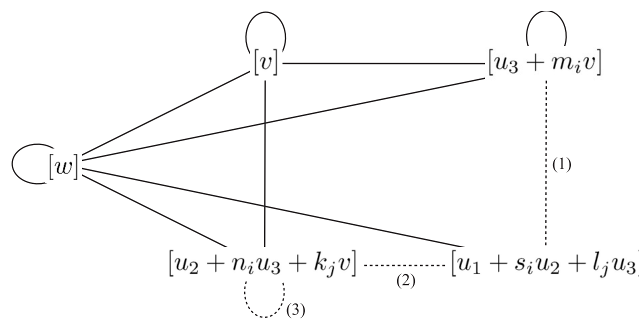
(2)



(1) *if* $m_i + n_j = 0$.

pic. 2

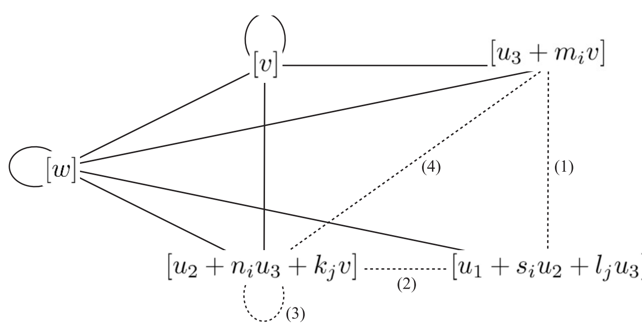
(3)



1) *if* $m_i + s_j = 0$;
2) *if* $k_j + l_\beta + n_i s_\alpha = 0$.
3) *if* $n_\alpha + n_\beta = 0$.

pic. 3

(4)



1) *if* $m_i + dl_j = 0$;
2) *if* $k_j + s_\alpha + dl_\beta n_i = 0$;
3) *if* $dn_i n_j + 1 = 0$;
4) *if* $n_i = 0$.

pic. 4

Proof. To find the graph $\Gamma(R/\sim)$ of a ring R, we use the following algorithm:

(1) Present a splitting of R into $\sim$ - classes.
(2) Find the annulator for every $\sim$ - class, while noting that annulators for all elements from the same $\sim$ - class are equal and that annulators of elements from different $\sim$ - classes are different. Beacuse of this, by Proposition 3, for example, instead of the set $F^*u_1 + Fv + Fw$ we can consider the set $u_1 + Fv + Fw$.
(3) Note that the order of a union of considered classes is equal to the order of the ring, i.e. it is equal to q^6.
(4) Find the splitting of annulators of $\sim$ - classes into equivalence classes (such a splitting exists by Proposition 2) and provide a geometric representation of the graph.

Let α_1, α_2, α_3, β, γ, α_1', α_2', α_3', β', γ', denote the elements of F and let x and x' denote the elements of the radical J. For instance, for the ring $R = F \oplus Fu_1 \oplus Fu_2 \oplus Fu_3 \oplus Fv \oplus Fw$:

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \beta v + \gamma w;$$

$$x' = \alpha'_1 u_1 + \alpha'_2 u_2 + \alpha'_3 u_3 + \beta' v + \gamma' w.$$

We now consider each case in more detail:

Case 1. Let

$$\begin{pmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} w, \ \begin{pmatrix} u_1v \\ u_2v \\ u_3v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} w.$$

We will show that

$$R = [1] \cup [u_1] \cup [v] \cup [w] \cup [0],$$
$$[u_1] = F^* u_1 + F u_2 + F u_3 + F v + F w,$$
$$[v] = F^* v + F u_2 + F u_3 + F w,$$
$$[w] = (F u_2 + F u_3 + F w) \backslash \{0\}.$$

In that case

$$xx' = 0 \Leftrightarrow \beta\alpha'_1 + \alpha_1\beta' = 0, \ \alpha_1\alpha'_1 = 0.$$

If $x \in u_1 + F u_2 + F u_3 + F v + F w$, then $\alpha_1 = 1$ and

$$xx' = 0 \Leftrightarrow \beta\alpha'_1 + \beta' = 0, \ \alpha'_1 = 0.$$

From this we get that $\alpha'_1 = 0$, $\beta' = 0$. Therefore

$$\mathrm{Ann}(x) = F u_2 + F u_3 + F w.$$

If $x \in v + F u_2 + F u_3 + F w$, то $\alpha_1 = 0$, $\beta = 1$ and

$$xx' = 0 \Leftrightarrow \alpha'_1 = 0.$$

Therefore

$$\mathrm{Ann}(x) = F u_2 + F u_3 + F v + F w.$$

If $x \in (F u_2 + F u_3 + F w) \backslash \{0\}$, then $\alpha_1 = \beta = 0$, $\alpha_2 \neq 0$ or $\alpha_3 \neq 0$ or $\gamma \neq 0$, and $xx' = 0$ for all $x' \in J$. Hence

$$\mathrm{Ann}(x) = J.$$

Moving on, we get that

$$|R^*| + |F^* u_1 + F u_2 + F u_3 + F v + F w| + |F^* v + F u_2 + F u_3 + F w| +$$
$$+ |(F u_2 + F u_3 + F w) \backslash \{0\}| + |\{0\}| = (q^6 - q^5) + (q-1)q^4 + (q-1)q^3 + q^3 - 1 + 1 = q^6.$$

Thus we obtain that

$$R = [1] \cup [u_1] \cup [v] \cup [w] \cup [0],$$
$$\mathrm{Ann}[u_1] = [w] \cup [0],$$
$$\mathrm{Ann}[v] = [w] \cup [v] \cup [0],$$
$$\mathrm{Ann}[w] = J.$$

The geometric representation of the graph can be seen on picture 1.

Case 2. Let

$$\begin{pmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} w, \ \begin{pmatrix} u_1v \\ u_2v \\ u_3v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} w.$$

We want to show that

$$R = [1] \bigcup_{m_i \in F} [u_1 + m_i u_2] \bigcup_{n_i \in F} [u_2 + n_i v] \cup [v] \cup [w] \cup [0],$$

$$[u_1+m_iu_2]=F^*(u_1+m_iu_2)+Fu_3+Fv+Fw,$$
$$[u_2+n_iv]=F^*(u_2+n_iv)+Fu_3+Fw,$$
$$[v]=F^*v+Fu_3+Fw,$$
$$[w]=(Fu_3+Fw)\backslash\{0\}.$$

In that case

$$xx'=0\Leftrightarrow \alpha_2\alpha_2'+\beta\alpha_1'+\alpha_1\beta'=0,\ \alpha_1\alpha_1'=0.$$

If $x\in u_1+m_iu_2+Fu_3+Fv+Fw$ for some $m_i\in F$, $i\in\{1,\dots,q\}$, then $\alpha_1=1$, $\alpha_2=m_i$ and

$$xx'=0\Leftrightarrow \alpha_1'=0,\ \beta'+\beta\alpha_1'+m_i\alpha_2'=0.$$

From this we get that, $\alpha_1'=0$, $\beta'=-m_i\alpha_2'$. Therefore

$$\mathrm{Ann}(x)=F(u_2-m_iv)+Fu_3+Fw.$$

If $x\in u_2+n_iv+Fu_3+Fw$ for some $n_i\in F$, $i\in\{1,\dots,q\}$, then $\alpha_1=0$, $\alpha_2=1$, $\beta=n_i$ and

$$xx'=0\Leftrightarrow \alpha_2'+n_i\alpha_1'=0.$$

From this we obtain that, $\alpha_2'=-n_i\alpha_1'$. Hence

$$\mathrm{Ann}(x)=F(u_1-n_iu_2)+Fu_3+Fv+Fw.$$

If $x\in v+Fu_3+Fw$, then $\alpha_1=0$, $\alpha_2=0$, $\beta=1$ and

$$xx'=0\Leftrightarrow \alpha_1'=0.$$

So we get that

$$\mathrm{Ann}(x)=Fu_2+Fu_3+Fv+Fw.$$

If $x\in(Fu_3+Fw)\backslash\{0\}$, then $\alpha_1=\alpha_2=\beta=0$, $\alpha_3\neq 0$ or $\gamma\neq 0$, and $xx'=0$ for all $x'\in J$. Hence $\mathrm{Ann}(x)=J$.

Moving on, we get that

$$|R^*|+\left|\bigcup_{m_i\in F}(F^*(u_1+m_iu_2)+Fu_3+Fv+Fw)\right|+$$
$$+\left|\bigcup_{n_i\in F}(F^*(u_2+n_iv)+Fu_3+Fw)\right|+|F^*v+Fu_3+Fw|+|(Fu_3+Fw)\backslash\{0\}|+|\{0\}|=$$
$$=(q^6-q^5)+(q-1)q^4+(q-1)q^3+(q-1)q^2+q^2-1+1=q^6.$$

Thus

$$R=[1]\bigcup_{m_i\in F}[u_1+m_iu_2]\bigcup_{n_i\in F}[u_2+n_iv]\cup[v]\cup[w]\cup[0]$$

and for all $m_i,n_i\in F$, $i\in\{1,\dots,q\}$ it is true that

$$\mathrm{Ann}[u_1+m_iu_2]=[u_2-m_iv]\cup[w]\cup[0],$$
$$\mathrm{Ann}[u_2+n_iv]=[u_1-n_iu_2]\cup[v]\cup[w]\cup[0],$$
$$\mathrm{Ann}[v]=\bigcup_{n_i\in F}[u_2+n_iv]\cup[v]\cup[w]\cup[0],$$
$$\mathrm{Ann}[w]=J.$$

The geometric representation of the graph can be seen on picture 2.

Case 3. Let $p = 2$,

$$\begin{pmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} w, \ \begin{pmatrix} u_1v \\ u_2v \\ u_3v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} w.$$

We will show that

$$R = [1] \bigcup_{s_i, l_j \in F} [u_1 + s_iu_2 + l_ju_3] \bigcup_{n_i, k_j \in F} [u_2 + n_iu_3 + k_jv] \bigcup_{m_i \in F} [u_3 + m_iv] \cup [v] \cup [w] \cup [0],$$

$$[u_1 + s_iu_2 + l_ju_3] = F^*(u_1 + s_iu_2 + l_ju_3) + Fv + Fw,$$
$$[u_2 + n_iu_3 + k_jv] = F^*(u_2 + n_iu_3 + k_jv) + Fw,$$
$$[u_3 + m_iv] = F^*(u_3 + m_iv) + Fw,$$
$$[v] = F^*v + Fw,$$
$$[w] = F^*w.$$

In that case

$$xx' = 0 \Leftrightarrow \alpha_2\alpha_3' + \alpha_3\alpha_2' + \beta\alpha_1' + \alpha_1\beta' = 0, \alpha_1\alpha_1' = 0.$$

If $x \in u_1 + s_iu_2 + l_ju_3 + Fv + Fw$ for some $s_i, l_j \in F$, then $\alpha_1 = 1$, $\alpha_2 = s_i$, $\alpha_3 = l_j$ and

$$xx' = 0 \Leftrightarrow \alpha_1' = 0, \beta' + l_j\alpha_2' + s_i\alpha_3' = 0.$$

Therefore $\alpha_1' = 0$, $\beta' = l_j\alpha_2' + s_i\alpha_3'$. Hence

$$\mathrm{Ann}(x) = F(u_2 + l_jv) + F(u_3 + s_iv) + Fw.$$

If $x \in u_2 + n_iu_3 + k_jv + Fw$ for some $n_i, k_j \in F$, $i, j \in \{1, \dots, q\}$, then $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = n_i$, $\beta = k_j$ and

$$xx' = 0 \Leftrightarrow \alpha_3' + k_j\alpha_1' + n_i\alpha_2' = 0.$$

From this we get that $\alpha_3' = k_j\alpha_1' + n_i\alpha_2'$. Therefore

$$\mathrm{Ann}(x) = F(u_1 + k_ju_3) + F(u_2 + n_iu_3) + Fv + Fw.$$

If $x \in u_3 + m_iv + Fw$ for some $m_i \in F$, $i \in \{1, \dots, q\}$, then $\alpha_1 = 0, \alpha_2 = 0$, $\alpha_3 = 1, \beta = m_i$ and

$$xx' = 0 \Leftrightarrow \alpha_2' + m_i\alpha_1' = 0.$$

So we get that $\alpha_2' = m_i\alpha_1'$. Hence

$$\mathrm{Ann}(x) = F(u_1 + m_iu_2) + Fu_3 + Fv + Fw.$$

If $x \in v + Fw$, then $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \beta = 1$ and

$$xx' = 0 \Leftrightarrow \alpha_1' = 0.$$

Therefore

$$\mathrm{Ann}(x) = Fu_2 + Fu_3 + Fv + Fw.$$

If $x = w$, then $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\beta = 0$ and $xx' = 0$ for all $x' \in J$. From this it is clear that, $\mathrm{Ann}(x) = J$.

Moving on, we get that

$$|R^*| + \left|\bigcup_{s_i,l_i\in F}(F^*(u_1+s_iu_2+l_ju_3)+Fv+Fw)\right| +$$
$$+\left|\bigcup_{n_i,k_j\in F}(F^*(u_2+n_iu_3+k_jv)+Fw)\right| + \left|\bigcup_{m_i\in F}(F^*(u_3+m_iv)+Fw)\right| +$$
$$+|F^*v+Fw|+|F^*w|+|\{0\}| =$$
$$=(q^6-q^5)+(q-1)q^4+(q-1)q^3+(q-1)q^2+(q-1)q+q-1+1=q^6.$$

So we can see that

$$R=[1]\bigcup_{s_i,l_j\in F}[u_1+s_iu_2+l_ju_3]\bigcup_{n_i,k_j\in F}[u_2+n_iu_3+k_jv]\bigcup_{m_i\in F}[u_3+m_iv]\cup[v]\cup[w]\cup[0]$$

and that for all $s_i, l_j, n_i, k_j, m_i \in F$, $i, j \in \{1, \ldots, q\}$ it is true that

$$\text{Ann}[u_1+s_iu_2+l_ju_3] = \bigcup_{n_\alpha\in F}[u_2+n_\alpha u_3+(l_j+n_\alpha s_i)v]\cup[u_3+s_iv]\cup[w]\cup[0],$$
$$\text{Ann}[u_2+n_iu_3+k_jv] = \bigcup_{s_\alpha\in F}[u_1+s_\alpha u_2+(k_j+n_is_\alpha)u_3]\bigcup_{k_\beta\in F}[u_2+n_iu_3+k_\beta v]\cup[v]\cup[w]\cup[0],$$
$$\text{Ann}[u_3+m_iv] = \bigcup_{l_\alpha\in F}[u_1+m_iu_2+l_\alpha u_3]\bigcup_{m_\alpha\in F}[u_3+m_\alpha v]\cup[v]\cup[w]\cup[0],$$
$$\text{Ann}[v] = \bigcup_{n_\alpha,k_\beta\in F}[u_2+n_\alpha u_3+k_\beta v]\bigcup_{m_\alpha\in F}[u_3+m_\alpha v]\cup[v]\cup[w]\cup[0],$$
$$\text{Ann}[w]=J.$$

The geometric representation of the graph can be seen on picture 3.

Case 4. Let

$$\begin{pmatrix}u_1u_1 & u_1u_2 & u_1u_3\\ u_2u_1 & u_2u_2 & u_2u_3\\ u_3u_1 & u_3u_2 & u_3u_3\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}v + \begin{pmatrix}0&0&0\\0&1&0\\0&0&d\end{pmatrix}w,\ \begin{pmatrix}u_1v\\u_2v\\u_3v\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}w.$$

We shall prove that

$$R=[1]\bigcup_{s_i,l_j\in F}[u_1+s_iu_2+l_ju_3]\bigcup_{n_i,k_j\in F}[u_2+n_iu_3+k_jv]\bigcup_{m_i\in F}[u_3+m_iv]\cup[v]\cup[w]\cup[0],$$
$$[u_1+s_iu_2+l_ju_3]=F^*(u_1+s_iu_2+l_ju_3)+Fv+Fw,$$
$$[u_2+n_iu_3+k_jv]=F^*(u_2+n_iu_3+k_jv)+Fw,$$
$$[u_3+m_iv]=F^*(u_3+m_iv)+Fw,$$
$$[v]=F^*v+Fw,$$
$$[w]=F^*w.$$

In that case we have that

$$xx'=0\Leftrightarrow \alpha_2\alpha_2'+d\alpha_3\alpha_3'+\alpha_1\beta'+\beta\alpha_1'=0, \alpha_1\alpha_1'=0.$$

If $x \in u_1 + s_i u_2 + l_j u_3 + Fv + Fw$ for some $s_i, l_j \in F$, $i, j \in \{1, \dots, q\}$, то $\alpha_1 = 1, \alpha_2 = s_i, \alpha_3 = l_j$ and

$$xx' = 0 \Leftrightarrow \alpha_1' = 0, \beta' + \beta\alpha_1' + s_i\alpha_2' + dl_j\alpha_3' = 0.$$

From this we get that, $\alpha_1' = 0, \beta' = -(s_i\alpha_2' + dl_j\alpha_3')$. Therefore,

$$\mathrm{Ann}(x) = F(u_2 - s_i v) + F(u_3 - dl_j v) + Fw.$$

If $x \in u_2 + n_i u_3 + k_j v + Fw$ for some $n_i, k_j \in F$, $i, j \in \{1, \dots, q\}$, then $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = n_i$, $\beta = k_j$ and

$$xx' = 0 \Leftrightarrow \alpha_2' + k_j\alpha_1' + dn_i\alpha_3' = 0.$$

Thus $\alpha_2' = -(k_j\alpha_1' + dn_i\alpha_3')$. Therefore

$$\mathrm{Ann}(x) = F(u_1 - k_j u_2) + F(u_3 - dn_i u_2) + Fv + Fw.$$

If $x \in u_3 + m_i v + Fw$ for some $m_i \in F$, $i \in \{1, \dots, q\}$, then $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 1$, $\beta = m_i$ and

$$xx' = 0 \Leftrightarrow d\alpha_3' + m_i\alpha_1' = 0.$$

Thus $\alpha_3' = -\frac{m_i}{d}\alpha_1'$. Hence

$$\mathrm{Ann}(x) = F(u_1 - \frac{m_i}{d}u_3) + Fu_2 + Fv + Fw.$$

If $x \in v + Fw$, then $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \beta = 1$ and

$$xx' = 0 \Leftrightarrow \alpha_1' = 0.$$

So we obtain that

$$\mathrm{Ann}(x) = Fu_2 + Fu_3 + Fv + Fw.$$

If $x = w$, то $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\beta = 0$ and $xx' = 0$ for all $x' \in J$. Therefore

$$\mathrm{Ann}(x) = J.$$

Moving on, we get that

$$|R^*| + \left|\bigcup_{s_i, l_j \in F} (F^*(u_1 + s_i u_2 + l_j u_3) + Fv + Fw)\right| +$$
$$+ \left|\bigcup_{n_i, k_j \in F} (F^*(u_2 + n_i u_3 + k_j v) + Fw)\right| + \left|\bigcup_{m_i \in F} (F^*(u_3 + m_i v) + Fw)\right| +$$
$$+ |F^*v + Fw| + |F^*w| + |\{0\}| =$$
$$= (q^6 - q^5) + (q-1)q^4 + (q-1)q^3 + (q-1)q^2 + (q-1)q + q - 1 + 1 = q^6.$$

Thus

$$R = [1] \bigcup_{s_i, l_j \in F} [u_1 + s_i u_2 + l_j u_3] \bigcup_{n_i, k_j \in F} [u_2 + n_i u_3 + k_j v] \bigcup_{m_i \in F} [u_3 + m_i v] \cup [v] \cup [w] \cup [0]$$

and for all $s_i, l_j, n_i, k_j, m_i \in F$, $i, j \in \{1, \dots, q\}$ it is true that

$$\mathrm{Ann}[u_1 + s_i u_2 + l_j u_3] = \bigcup_{n_\alpha \in F} [u_2 + n_\alpha u_3 - (s_i + dl_j n_\alpha)v] \cup [u_3 - dl_j v] \cup [w] \cup [0],$$

$$\mathrm{Ann}[u_2 + k_j v] = \bigcup_{l_\beta \in F} [u_1 - k_j u_2 + l_\beta u_3] \bigcup_{m_\alpha \in F} [u_3 + m_\alpha v] \cup [v] \cup [w] \cup [0],$$

$$\mathrm{Ann}[u_2+n_iu_3+k_jv]=\bigcup_{l_\beta\in F}[u_1-(k_j+dl_\beta n_i)u_2+l_\beta u_3]$$
$$\bigcup_{k_\alpha\in F}[u_2-\frac{1}{dn_i}u_3+k_\alpha v]\cup[v]\cup[w]\cup[0],\ \text{где } n_i\neq 0,$$

$$\mathrm{Ann}[u_3+m_iv]=\bigcup_{s_\alpha\in F}[u_1+s_\alpha u_2-\frac{m_i}{d}u_3]\bigcup_{k_\alpha\in F}[u_2+k_\alpha v]\cup[v]\cup[w]\cup[0],$$

$$\mathrm{Ann}[v]=\bigcup_{n_\alpha,k_\beta\in F}[u_2+n_\alpha u_3+k_\beta v]\bigcup_{m_\alpha\in F}[u_3+m_\alpha v]\cup[v]\cup[w]\cup[0],$$

$$\mathrm{Ann}[w]=J.$$

The geometric representation of the graph can be seen on picture 4.

□

Theorem 2. *Let R be a local ring of characteristic p such that*

$$J^4=0,\dim_F J/J^2=2,\dim_F J^2/J^3=2,\dim_F J^3=1,$$

where $F=R/J=GF(p^r)$.

If $p=2$, then, in addition to $[0]=\{0\}$ and $[1]=R^*$, $R/\sim$ is defined by one of the following sets:

(1) $[u_1]=F^*u_1+Fu_2+Fv_1+Fv_2+Fw$,
$[u_2]=(Fu_2+Fv_2)\backslash\{0\}+Fv_1+Fw$,
$[w]=(Fv_1+Fw)\backslash\{0\}$;

(2) $[u_1]=F^*u_1+Fu_2+Fv_1+Fv_2+Fw$,
$[u_2]=F^*u_2+Fv_1+Fv_2+Fw$,
$[v_2]=F^*v_2+Fv_1+Fw$,
$[w]=(Fv_1+Fw)\backslash\{0\}$;

(3) $[u_1+n_iu_2]=F^*(u_1+n_iu_2)+Fv_1+Fv_2+Fw$ *for all* $n_i\in F$,
$[u_2+k_iv_2]=F^*(u_2+k_iv_2)+Fv_1+Fw$ *for all* $k_i\in F$,
$[m_iv_1+v_2]=F^*(m_iv_1+v_2)+Fw$ *for all* $m_i\in F$,
$[v_1]=F^*v_1+Fw$,
$[w]=F^*w$;

(4) $[u_1+n_iu_2]=F^*(u_1+n_iu_2)+Fv_1+Fv_2+Fw$ *for all* $n_i\in F$,
$[u_2+k_iv_2]=F^*(u_2+k_iv_2)+Fv_1+Fw$ *for all* $k_i\in F$,
$[m_iv_1+v_2]=F^*(m_iv_1+v_2)+Fw$ *for all* $m_i\in F$,
$[v_1]=F^*v_1+Fw$,
$[w]=F^*w$;

(5) $[u_1+n_iu_2]=F^*(u_1+n_iu_2)+Fv_1+Fv_2+Fw$ *for all* $n_i\in F$,
$[u_2]=F^*u_2+Fv_1+Fv_2+Fw$,
$[v_1+m_iv_2]=F^*(v_1+m_iv_2)+Fw$ *for all* $m_i\in F$,
$[v_2]=F^*v_2+Fw$,
$[w]=F^*w$;

(6) $[n_iu_1+u_2]=F^*(n_iu_1+u_2)+Fv_1+Fv_2+Fw$ *for all* $n_i\in F$,
$[u_1]=F^*u_1+Fv_1+Fv_2+Fw$,
$[m_iv_1+v_2]=F^*(m_iv_1+v_2)+Fw$ *for all* $m_i\in F$,
$[v_1]=F^*v_1+Fw$,
$[w]=F^*w$;

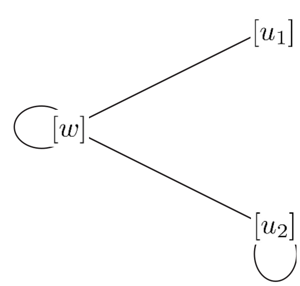
(7) $[n_iu_1 + u_2] = F^*(n_iu_1 + u_2) + Fv_1 + Fv_2 + Fw$ *for all* $n_i \in F$, $n_i \neq 0, 1$,
$[u_1] = F^*u_1 + Fv_1 + Fv_2 + Fw$,
$[u_2 + k_iv_2] = F^*(u_2 + k_iv_2) + Fv_1 + Fw$ *for all* $k_i \in F$,
$[u_1 + u_2 + l_iv_1] = F^*(u_1 + u_2 + l_iv_2) + F(v_1 + v_2) + Fw$ *for all* $l_i \in F$,
$[v_1 + m_iv_2] = F^*(v_1 + m_iv_2) + Fw$ *for all* $m_i \in F$,
$[v_2] = F^*v_2 + Fw$,
$[w] = F^*w$;

(8) $[u_1] = \{\alpha_1u_1 + \alpha_2u_2 + \beta_1v_1 + \beta_2v_2 + \gamma w \mid \alpha_i, \beta_i, \gamma \in F, \alpha_1 \neq 0, \alpha_1 \neq \alpha_2\}$,
$[u_2] = F^*u_2 + Fv_1 + Fv_2 + Fw$,
$[u_1 + u_2 + v_1] = \{\alpha_1(u_1+u_2) + \beta_1v_1 + \beta_2v_2 + \gamma w | \alpha_1, \beta_i, \gamma \in F, \alpha_1 \neq 0, \beta_1 \neq \beta_2\}$,
$[u_1 + u_2] = F^*(u_1 + u_2) + F(v_1 + v_2) + Fw$,
$[v_1] = \{\beta_1v_1 + \beta_2v_2 + \gamma w \mid \beta_i, \gamma \in F, \beta_1 \neq \beta_2\}$,
$[w] = (F(v_1 + v_2) + Fw) \setminus \{0\}$, $i = \overline{1, 2}$;

(9) $[n_iu_1 + u_2] = F^*(n_iu_1 + u_2) + Fv_1 + Fv_2 + Fw$ *for all* $n_i \in F$, $n_i \neq 0, 1$,
$[u_1] = F^*u_1 + Fv_1 + Fv_2 + Fw$,
$[u_2 + k_iv_2] = F^*(u_2 + k_iv_2) + Fv_1 + Fw$ *for all* $k_i \in F$,
$[u_1 + u_2 + l_iv_1] = F^*(u_1 + u_2 + l_iv_2) + F(v_1 + v_2) + Fw$ *for all* $l_i \in F$,
$[m_iv_1 + v_2] = F^*(m_iv_1 + v_2) + Fw$ *for all* $m_i \in F$,
$[v_1] = F^*v_1 + Fw$,
$[w] = F^*w$.

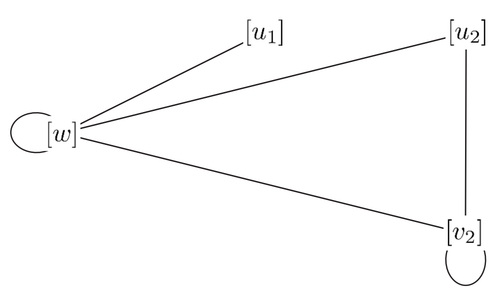
For every case presented above, the geometric representation of a graph $\Gamma(S/\sim)$ *(excluding the vertices* $[0]$ *u* $[1]$*) looks as follows:*

(1)



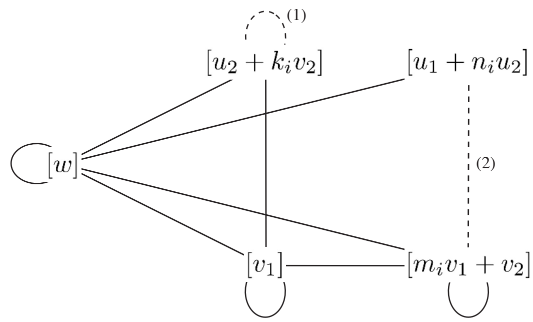
pic. 5

(2)



pic. 6

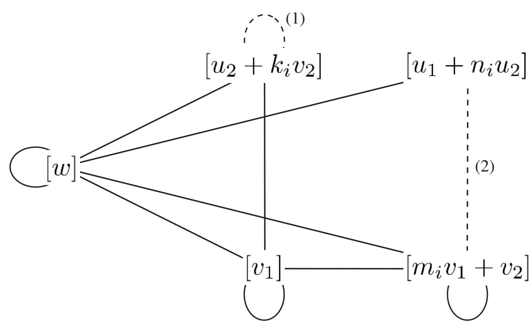
(3)



(1) *if* $k_i = k_j$;
(2) *if* $m_i = n_j$.

pic. 7

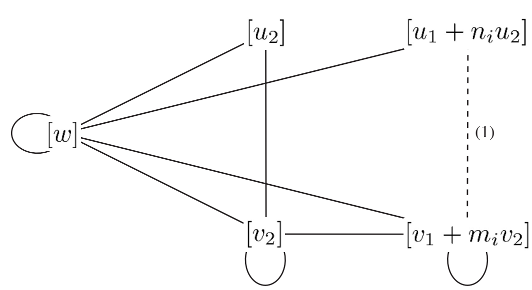
(4)



(1) *if* $k_i + k_j + 1 = 0$;
(2) *if* $m_i = n_j$.

pic. 8

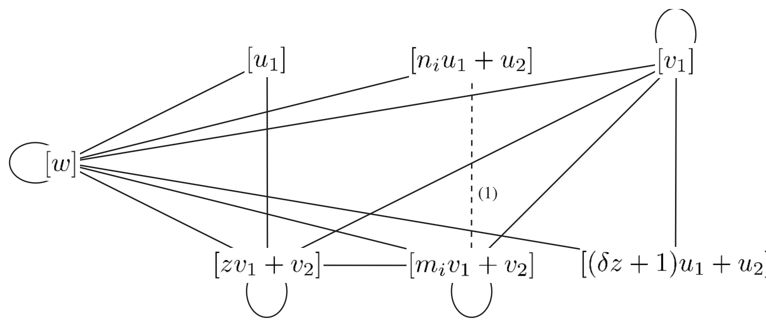
(5)



(1) *if* $m_i = \delta n_j$.

pic. 9

(6)

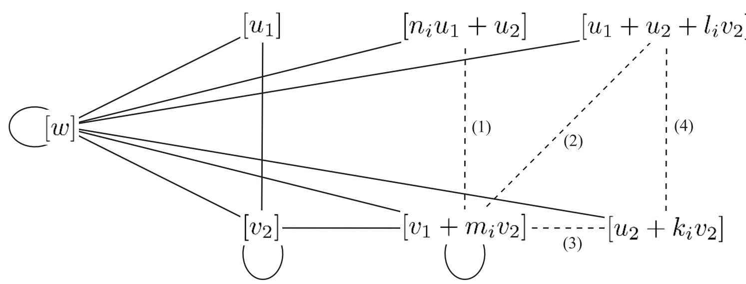


$m_i \neq z$, $n_i \neq \delta z + 1$.

(1) *if* $m_i n_j + z n_j + (\delta z + 1) m_i + 1 = 0$.

pic. 10

(7)



$n_i \neq 0, 1$.

(1) *if* $m_i = n_j + 1$;

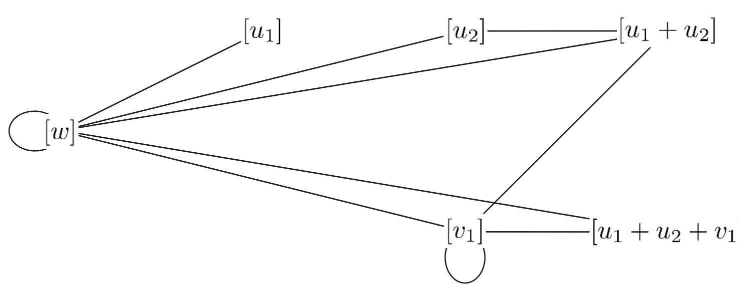
(2) *if* $m_i = 0$;

(3) *if* $m_i = 1$;

(4) *if* $l_i = k_j$.

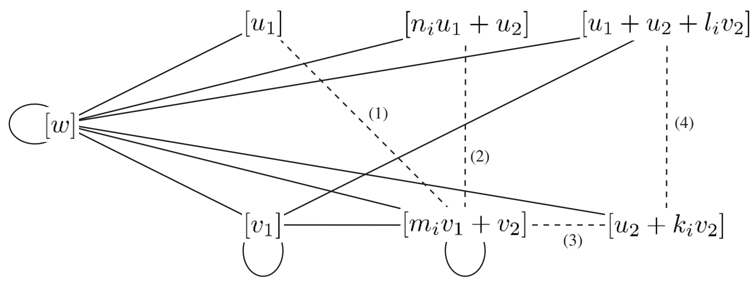
pic. 11

(8)



pic. 12

(9)



$n_i \neq 0, 1$.
(1) *if* $m_i = z$;
(2) *if* $1 + m_i + m_in_j + n_jz = 0$, $m_i \neq 1, z$;
(3) *if* $m_i = 1$;
(4) *if* $l_i = (1 + z)k_j$.

pic. 13

If $p \neq 2$, *then, in addition to* $[0] = \{0\}$ *and* $[1] = R^*$, $R/\sim$ *is defined by one of the following sets:*

(1) $[u_1] = F^*u_1 + Fu_2 + Fv_1 + Fv_2 + Fw$,
$[u_2] = (Fu_2 + Fv_1) \setminus \{0\} + Fv_2 + Fw$,
$[w] = (Fv_2 + Fw) \setminus \{0\}$;

(2) $[u_1] = F^*u_1 + Fu_2 + Fv_1 + Fv_2 + Fw$, $[u_2] = F^*u_2 + Fv_1 + Fv_2 + Fw$,
$[v_1] = F^*v_1 + Fv_2 + Fw$, $[w] = (Fv_2 + Fw) \setminus \{0\}$;

(3) $[u_1 + n_iu_2] = F^*(u_1 + n_iu_2) + Fv_1 + Fv_2 + Fw$ *for all* $n_i \in F$,
$[u_2 + k_iv_1] = F^*(u_2 + k_iv_1) + Fv_2 + Fw$ *for all* $k_i \in F$,
$[v_1 + m_iv_2] = F^*(v_1 + m_iv_2) + Fw$ *for all* $m_i \in F$,
$[v_2] = F^*v_2 + Fw$, $[w] = F^*w$;

(4) $[u_1 + n_iu_2] = F^*(u_1 + n_iu_2) + Fv_1 + Fv_2 + Fw$, $n_i \neq \pm 1$ *for all* $n_i \in F$,
$[u_1 + u_2 + k_iv_2] = F^*(u_1 + u_2 + k_iv_2) + F(v_1 + v_2) + Fw$ *for all* $k_i \in F$,
$[u_1 - u_2 + l_iv_2] = F^*(u_1 - u_2 + l_iv_2) + F(v_1 - v_2) + Fw$ *for all* $l_i \in F$,
$[u_2] = F^*u_2 + Fv_1 + Fv_2 + Fw$,
$[m_iv_1 + v_2] = F^*(m_iv_1 + v_2) + Fw$ *for all* $m_i \in F$,
$[v_1] = F^*v_1 + Fw$,
$[w] = F^*w$;

(5) $[u_1 + n_iu_2] = F^*(u_1 + n_iu_2) + Fv_1 + Fv_2 + Fw$ *for all* $n_i \in F$,
$[u_2] = F^*u_2 + Fv_1 + Fv_2 + Fw$,
$[m_iv_1 + v_2] = F^*(m_iv_1 + v_2) + Fw$ *for all* $m_i \in F$,
$[v_1] = F^*v_1 + Fw$,
$[w] = F^*w$;

(6) $[u_1] = \{\alpha_1u_1 + \alpha_2u_2 + \beta_1v_1 + \beta_2v_2 + \gamma w | \alpha_1 \neq \alpha_2, \alpha_1 \neq -\alpha_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in F\}$,
$[u_1 + u_2] = F^*(u_1 + u_2) + Fv_1 + Fv_2 + Fw$,
$[u_1 - u_2 + v_1] = \{\alpha_1(u_1 - u_2) + \beta_1v_1 + \beta_2v_2 + \gamma w | \alpha_1 \neq 0, \beta_1 \neq -\beta_2, \alpha_1, \beta_1, \beta_2, \gamma \in F\}$,
$[u_1 - u_2] = F^*(u_1 - u_2) + F(v_1 - v_2) + Fw$,
$[v_1] = \{\beta_1v_1 + \beta_2v_2 + \gamma w | \beta_1 \neq -\beta_2, \beta_1, \beta_2, \gamma \in F\}$,
$[w] = (F(v_1 - v_2) + Fw) \setminus \{0\}$;

(7) $[u_1 + k_iu_2] = F^*(u_1 + k_iu_2) + Fv_1 + Fv_2 + Fw$ *for all* $k_i \in F$, $k_i \neq \pm 1$,
$[u_1 + u_2 + n_iv_2] = F^*(u_1 + u_2 + n_iv_2) + F(v_1 + v_2) + Fw$ *for all* $n_i \in F$,

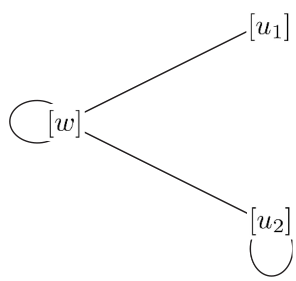
$[u_1 - u_2 + l_i v_2] = F^*(u_1 - u_2 + l_i v_2) + F(v_1 - v_2) + Fw$ *for all* $l_i \in F$,
$[u_2] = F^* u_2 + Fv_1 + Fv_2 + Fw$,
$[v_1 + m_i v_2] = F^*(v_1 + m_i v_2) + Fw$ *for all* $m_i \in F$,
$[v_2] = F^* v_2 + Fw$,
$[w] = F^* w$;

(8) $[u_1 + n_i u_2] = F^*(u_1 + n_i u_2) + Fv_1 + Fv_2 + Fw$ *for all* $n_i \in F$,
$[u_2] = F^* u_2 + Fv_1 + Fv_2 + Fw$,
$[v_1 + m_i v_2] = F^*(v_1 + m_i v_2) + Fw$ *for all* $m_i \in F$,
$[v_2] = F^* v_2 + Fw$,
$[w] = F^* w$;

(9) $[u_1 + m_i u_2] = F^*(u_1 + m_i u_2) + Fv_1 + Fv_2 + Fw$ *for all* $m_i \in F$,
$[u_2 + n_i v_1] = F^*(u_2 + n_i v_1) + Fv_2 + Fw$ *for all* $n_i \in F$,
$[v_1 + k_i v_2] = F^*(v_1 + k_i v_2) + Fw$ *for all* $k_i \in F$,
$[v_2] = F^* v_2 + Fw$,
$[w] = F^* w$.

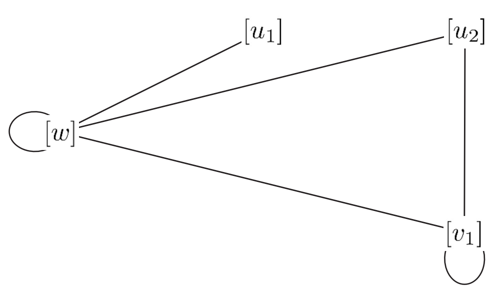
For every case presented above, the geometric representation of the graph $\Gamma(R/\sim)$ *(excluding the vertices* $[0]$ *u* $[1]$*) looks as follows:*

(1)



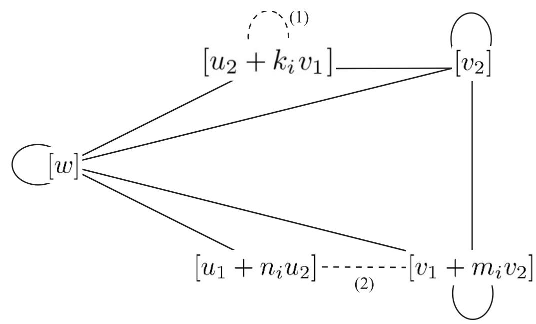
pic. 14

(2)



pic. 15

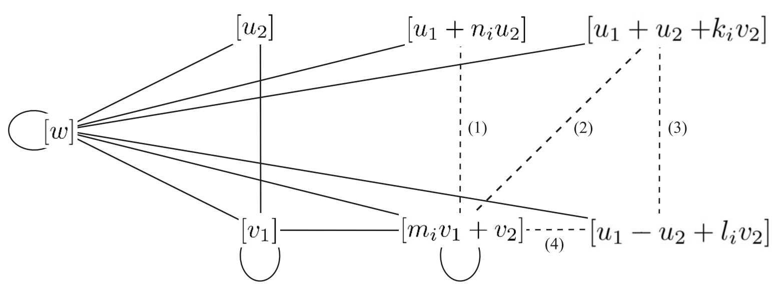
(3)



(1) *if* $k_i + k_j = 0$;
(2) *if* $m_i + n_j + 1 = 0$.

pic. 16

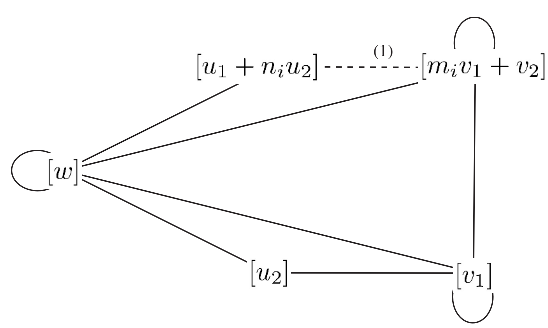
(4)



(1) *if* $n_i + m_j = 0$;
(2) *if* $m_i = -1$;
(3) *if* $k_i = l_j$;
(4) *if* $m_i = 1$.

pic. 17

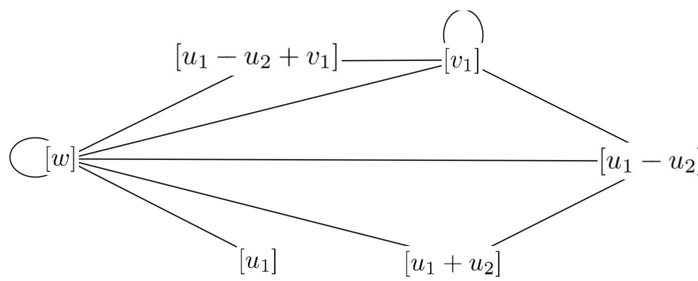
(5)



(1) *if* $m_i + \delta n_j = 0$.

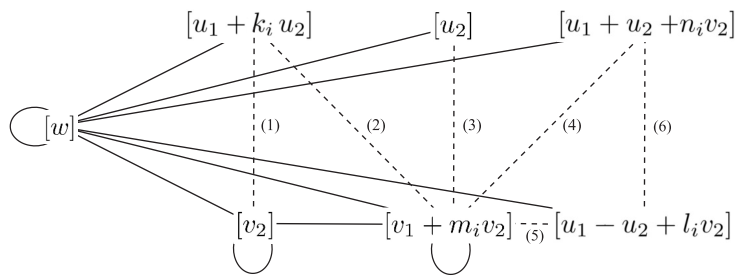
pic. 18

(6)



pic. 19

(7)

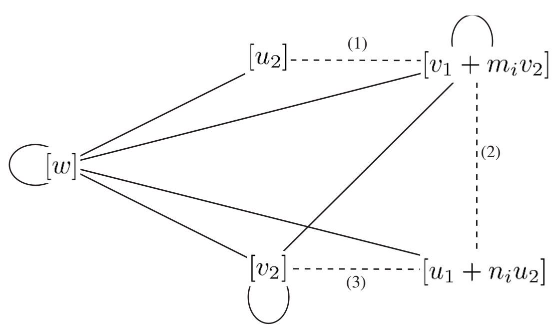


$k_i \neq \pm 1$

(1) *if* $k_i = -\zeta$;
(2) *if* $k_i m_j + k_i \zeta + m_j \zeta + 1 = 0$;
(3) *if* $m_i = -\zeta$;
(4) *if* $m_i = -1$;
(5) *if* $m_i = 1$.
(6) *if* $(1+\zeta) l_i = (1-\zeta) n_j$.

pic. 20

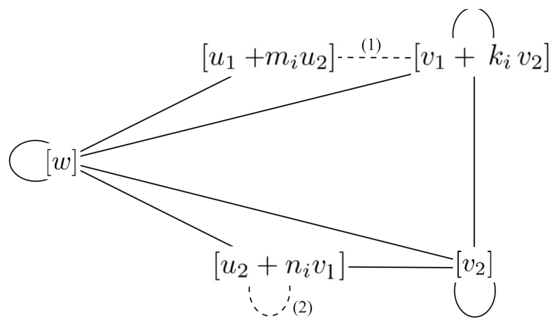
(8)



(1) *if* $m_i = -\frac{\xi}{\delta}$;
(2) *if* $\xi m_i + \xi n_j + \delta m_i n_j + 1 = 0$, $n_j \neq -\frac{\xi}{\delta}$, $m_i \neq -\frac{\xi}{\delta}$;
(3) *if* $n_i = -\frac{\xi}{\delta}$.

pic. 21

(9)



(1) *if* $k_i + m_j = 0$*;*
(2) *if* $n_i + n_j = 0$.

pic. 22

Proof. The results for the rings of characteristic $p = 2$ were proved in [24] and were stated here for the sake of completeness.

Consider the case $p \neq 2$. The hardest part is to find the equivalence classes, listed in the statement of the theorem. After obtaining these classes and the geometric representation of the graph the rest of the proof for every case consists mainly of checking the statement of the theorem via direct computation. As an example, we will only consider case 7 as it is the hardest one we encountered.

Case 7. Let
$\begin{pmatrix} u_1u_1 & u_1u_2 \\ u_2u_1 & u_2u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v_2 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \omega, \begin{pmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{pmatrix} = \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix} \omega,$
where $\zeta \neq \pm 1$ and $\dfrac{1-\zeta}{1+\zeta} \notin (F^*)^3$.

We want to show that

$$R = [1] \bigcup_{k_i \in F\setminus\{\pm 1\}} [u_1 + k_i u_2] \bigcup_{n_i \in F} [u_1 + u_2 + n_i v_2] \bigcup_{l_i \in F} [u_1 - u_2 + l_i v_2] \cup [u_2]$$
$$\bigcup_{m_i \in F} [v_1 + m_i v_2] \cup [v_2] \cup [w] \cup [0],$$

$$[u_1 + k_i u_2] = F^*(u_1 + k_i u_2) + Fv_1 + Fv_2 + Fw, k_i \neq \pm 1,$$
$$[u_1 + u_2 + n_i v_2] = F^*(u_1 + u_2 + n_i v_2) + F(v_1 + v_2) + Fw,$$
$$[u_1 - u_2 + l_i v_2] = F^*(u_1 - u_2 + l_i v_2) + F(v_1 - v_2) + Fw,$$
$$[u_2] = F^* u_2 + Fv_1 + Fv_2 + Fw,$$
$$[v_1 + m_i v_2] = F^*(v_1 + m_i v_2) + Fw,$$
$$[v_2] = F^* v_2 + Fw,$$
$$[w] = F^* w.$$

In that case

$$xx' = 0 \Leftrightarrow \alpha_1\alpha_1' + \alpha_2\alpha_2' = 0,\ \alpha_1\alpha_2' + \alpha_2\alpha_1' = 0,$$
$$\alpha_1(\beta_1' + \beta_2'\zeta) + \alpha_2(\beta_1'\zeta + \beta_2') + \beta_1(\alpha_1' + \alpha_2'\zeta) + \beta_2(\alpha_1'\zeta + \alpha_2') = 0.$$

If $x \in u_1 + k_i u_2 + Fv_1 + Fv_2 + Fw$, then $\alpha_1 = 1, \alpha_2 = k_i$ for some $k_i \in F$, $k_i \neq \pm 1$, $i \in \{1, \dots, q\}$, and

$$xx' = 0 \Leftrightarrow \alpha_1' + k_i\alpha_2' = 0,\ \alpha_2' + k_i\alpha_1' = 0,$$
$$\beta_1' + \beta_2'\zeta + k_i(\beta_1'\zeta + \beta_2') + \beta_1(\alpha_1' + \alpha_2'\zeta) + \beta_2(\alpha_1'\zeta + \alpha_2') = 0.$$

Since $\alpha_1' + k_i\alpha_2' = 0$ and $\alpha_2' + k_i\alpha_1' = 0$, then $\alpha_1'(1 - k_i^2) = 0$, which means that $\alpha_1' = \alpha_2' = 0$. Therefore

$$xx' = 0 \Leftrightarrow \alpha_1' = \alpha_2' = 0, \beta_1'(1 + k_i\zeta) + \beta_2'(k_i + \zeta) = 0.$$

If $k_i \neq -\zeta$, then

$$\mathrm{Ann}(r) = F\left(v_1 - \frac{1 + k_i\zeta}{k_i + \zeta} v_2\right) + Fw.$$

If $k_i = -\zeta$, then $\beta_1' = 0$ and

$$\mathrm{Ann}(x) = Fv_2 + Fw.$$

If $x \in (u_1 + u_2 + n_i v_2) + F(v_1 + v_2) + Fw$, then $\alpha_1 = \alpha_2 = 1$, $\beta_2 = \beta_1 + n_i$ and

$$xx' = 0 \Leftrightarrow \alpha_2' = -\alpha_1', \beta_2' = \frac{1 - \zeta}{1 + \zeta}\, n_i\alpha_1' - \beta_1'.$$

Therefore

$$\mathrm{Ann}(x) = F\left(u_1 - u_2 + \frac{1 - \zeta}{1 + \zeta}\, n_i v_2\right) + F(v_1 - v_2) + Fw.$$

If $x \in (u_1 - u_2 + l_i v_2) + F(v_1 - v_2) + Fw$, then $\alpha_1 = 1$, $\alpha_2 = -1$, $\beta_2 = -\beta_1 + l_i$ and

$$xx' = 0 \Leftrightarrow \alpha_2' = \alpha_1', \beta_2' = \frac{1 + \zeta}{1 - \zeta}\, l_i\alpha_1' + \beta_1'.$$

Hence

$$\mathrm{Ann}(x) = F\left(u_1 + u_2 + \frac{1 + \zeta}{1 - \zeta}\, l_i v_2\right) + F(v_1 + v_2) + Fw.$$

If $x \in u_2 + Fv_1 + Fv_2 + Fw$, then $\alpha_1 = 0, \alpha_2 = 1$ and

$$xx' = 0 \Leftrightarrow \alpha_2' = \alpha_1' = 0, \beta_2' = -\beta_1'\zeta.$$

Thus

$$\mathrm{Ann}(x) = F(v_1 - \zeta v_2) + Fw.$$

If $x \in v_1 + m_i v_2 + Fw$, then $\alpha_1 = \alpha_2 = 0, \beta_1 = 1, \beta_2 = m_i$ for some $m_i \in F$, $i \in \{1, \dots, q\}$ and

$$xx' = 0 \Leftrightarrow \alpha_1' + \alpha_2' m_i + \alpha_2'\zeta + \alpha_1' m_i\zeta = 0.$$

If $m_i \neq -\zeta$, then $\alpha_2' = -\frac{1 + m_i\zeta}{m_i + \zeta}\alpha_1'$ and

$$\mathrm{Ann}(x) = F\left(u_1 - \frac{1 + m_i\zeta}{m_i + \zeta}\, u_2\right) + Fv_1 + Fv_2 + Fw.$$

If $m_i = -\zeta$, then $\alpha_1' = 0$ and

$$\mathrm{Ann}(x) = Fu_2 + Fv_1 + Fv_2 + Fw.$$

If $x \in v_2 + F\omega$, then $\alpha_1 = \alpha_2 = 0, \beta_1 = 0, \beta_2 = 1$ and

$$xx' = 0 \Leftrightarrow \alpha_2' = -\alpha_1'\zeta.$$

So we obtain that

$$\mathrm{Ann}(x) = F(u_1 - \zeta u_2) + Fv_1 + Fv_2 + Fw.$$

If $x = w$, then $xx' = 0$ for all $x' \in J$ and thus $\mathrm{Ann}(w) = J$.

Moving on, we have that

$$\begin{aligned}
|R^*| + \left|\bigcup_{k_i\in F\setminus\{\pm1\}} (F^*(u_1+k_iu_2)+Fv_1+Fv_2+Fw)\right| + \\
+ |F^*(u_1+u_2+n_iv_2)+F(v_1+v_2)+Fw| + \\
+ |F^*(u_1-u_2+l_iv_2)+F(v_1-v_2)+Fw| + |F^*u_2+Fv_1+Fv_2+Fw| + \\
+ \left|\bigcup_{m_i\in F} (F^*(v_1+m_iv_2)+Fw)\right| + |F^*v_2+Fw| + |F^*w| + |\{0\}| = \\
= (q-1)q^5 + (q-1)(q-2)q^3 + (q-1)q^3 + (q-1)q^3 + \\
(q-1)q^3 + (q-1)q^2 + (q-1)q + (q-1) + 1 = q^6.
\end{aligned}$$

Thus

$$\begin{aligned}
R = [1] \bigcup_{k_i\in F\setminus\{\pm1\}} [u_1+k_iu_2] \bigcup_{n_i\in F} [u_1+u_2+n_iv_2] \bigcup_{l_i\in F} [u_1-u_2+l_iv_2] \cup [u_2] \\
\bigcup_{m_i\in F} [v_1+m_iv_2] \cup [v_2] \cup [w] \cup [0]
\end{aligned}$$

and for all $k_i, l_i, m_j, n_i \in F$, $i, j \in \{1, \dots, q\}$,

$$\mathrm{Ann}[u_1+k_iu_2] = \left[v_1 - \frac{1+k_i\zeta}{k_i+\zeta}v_2\right] \cup [w] \cup [0],\ k_i \neq -\zeta,$$

$$\mathrm{Ann}[u_1-\zeta u_2] = [v_2] \cup [w] \cup [0],$$

$$\mathrm{Ann}[u_1+u_2+n_iv_2] = \left[u_1-u_2+\frac{1-\zeta}{1+\zeta}\,n_iv_2\right] \cup [v_1-v_2] \cup [w] \cup [0],$$

$$\mathrm{Ann}[u_1-u_2+l_iv_2] = \left[u_1+u_2+\frac{1+\zeta}{1-\zeta}\,n_iv_2\right] \cup [v_1+v_2] \cup [w] \cup [0],$$

$$\mathrm{Ann}[u_2] = [v_1-\zeta v_2] \cup [w] \cup [0],$$

$$\mathrm{Ann}[v_1+m_jv_2] = \left[u_1 - \frac{1+m_j\zeta}{m_j+\zeta}\,u_2\right] \bigcup_{m_i\in F} [v_1+m_iv_2] \cup [v_2] \cup [w] \cup [0],\ m_j \neq \pm1, -\zeta,$$

$$\mathrm{Ann}[v_1-\zeta v_2] = [u_2] \bigcup_{m_i\in F} [v_1+m_iv_2] \cup [v_2] \cup [w] \cup [0],$$

$$\mathrm{Ann}[v_1-v_2] = \bigcup_{n_i\in F} [u_1+u_2+n_iv_2] \bigcup_{m_i\in F} [v_1+m_iv_2] \cup [v_2] \cup [w] \cup [0],$$

$$\mathrm{Ann}[v_1+v_2] = \bigcup_{l_i\in F} [u_1-u_2+l_iv_2] \bigcup_{m_i\in F} [v_1+m_iv_2] \cup [v_2] \cup [w] \cup [0],$$

$$\mathrm{Ann}[v_2] = [u_1-\zeta u_2] \bigcup_{m_i\in F} [v_1+m_iv_2] \cup [v_2] \cup [w] \cup [0],$$

$$\mathrm{Ann}[w] = J.$$

The geometric representation of the graph can be seen on picture 20.

□

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Evgeniy Vladimirovich Zhuravlev
Altai State University,
61, Lenina ave.,
Barnaul, 656049, Russia
Email address: evzhuravlev@mail.ru

Olga Alexandrovna Filina
Altai State University,
61, Lenina ave.,
Barnaul, 656049, Russia
Email address: olya-filina@mail.ru