S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 18, №2, стр. 1572–1595 (2021) УДК 517.98, 519.677
DOI 10.33048/semi.2021.18.117 MSC 44A30

SINGULAR VALUE DECOMPOSITION OF A NORMAL RADON TRANSFORM OPERATOR ACTING ON 3D SYMMETRIC 2-TENSOR FIELDS

A.P. POLYAKOVA

Abstract. A problem of 3D 2-tensor field potential part reconstruction by the known value of its normal Radon transform is considered. A singular value decomposition of the operator is constructed for solving the problem. Basic fields are constructed with the use of Jacobi polynomials, Gegenbauer polynomials, and spherical harmonics.

Keywords: symmetric tensor field, potential field, potential, normal Radon transform, singular value decomposition of an operator, system of orthogonal polynomials.

1. Introduction

A classic operator of integral geometry, acting on vector and 2-tensor fields, is a ray transform [1]. In a two-dimensional case, for complete reconstruction of the vector field it is necessary to know the values of two ray transforms, longitudinal and transverse ones, since each of them has a nonzero kernel, while for complete reconstruction of a symmetric, that is, invariant with respect to all index transpositions, 2-tensor field in $\mathbb{R}^2$, the values of three ray transforms have to be known (for details, see [2]). In a three-dimensional case, by longitudinal ray transform only the solenoidal part of a vector or symmetric 2-tensor field can be reconstructed. To reconstruct the potential part of the field, it is necessary to have other data. One of the operators that allows to reconstruct the potential part of a 3D vector and

Polyakova, A.P., Singular value decomposition of a normal Radon transform operator acting on 3D symmetric 2-tensor fields.

The research was carried out within the state assignment for Sobolev Institute of Mathematics SB RAS (project №-0314-2019-0011) and with a partial support of RFBR and DFG according to the project 19-51-12008.

Received May, 17, 2021, published December, 7, 2021.

symmetric 2-tensor field is the normal Radon transform. In work [3], the issue of inverting the Radon transform of a symmetric 2-tensor field in $\mathbb{R}^3$ and, in particular, the normal Radon transform, is studied. Note also the paper [4], in which two approaches to reconstruction of a three-dimensional potential vector and symmetric 2-tensor fields by known values of the normal Radon transform based on the method of approximate inverse are proposed.

In this paper, a singular value decomposition of a normal Radon transform operator acting on three-dimensional symmetric 2-tensor fields is constructed. Singular value decompositions of Radon transform operators [5]–[10] and longitudinal ray transform [11], acting on scalar fields in $\mathbb{R}^3$, are well known. In the mentioned works, basic fields are constructed based on different variations of orthogonal polynomials and spherical harmonics. In particular, to construct the basic fields in work [9], Jacobi polynomials and Gegenbauer polynomials were used. Based on this result along with the relation of ray transforms and the normal Radon transform with the Radon transform, singular value decompositions of operators of ray transforms of vector [12] and symmetric 2-tensor [13] fields in $\mathbb{R}^2$, and decomposition of the normal Radon transform operator of vector fields in $\mathbb{R}^3$ [14],[15], and also the result of this article were obtained. Note the papers dedicated to the development, realisation, and studying of algorithms of numerical solution to problems of vector and tensor tomography in $\mathbb{R}^2$ and $\mathbb{R}^3$ with the use of the SV-decompositions [16]–[19], mentioned above. We should mention work [20], in which a singular value decomposition of the operator of the fan Radon transform, acting on 2-dimensional tensor fields of arbitrary valency, is constructed.

In paper [19], an algorithm of numerical solution to a problem of 2-tensor tomography on reconstruction of a three-dimensional potential symmetric 2-tensor field by its known normal Radon transform, based on the method of truncated singular value decomposition, is proposed. In particular, the operator of the normal Radon transform is represented in the form of a series with respect to singular values and basic elements in the image space, then the inverse operator represents a set of a similar structure, where the preimages of these basic elements and the same singular values are involved (for details, see [7],[21],[10]). It is necessary to note that [19] is only dedicated to numerical solution of the three-dimensional 2-tensor tomography, the orthogonality of basic fields of degree $N \leqslant 50$ in the main space was verified using Wolfram Mathematica 9 programm package. In this work, this statement is theoretically justified for fields of arbitrary degree, thus, a singular value decomposition of the normal Radon transform operator acting on a three-dimensional symmetric 2-tensor field is constructed.

2. Main definitions

We introduce the notations $B = \{x \in \mathbb{R}^3 \mid |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\}$ for a unit ball, $\partial B = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ for a unit sphere, $Z = \{(s, \xi) \mid \xi \in \mathbb{R}^3, |\xi| = 1, s \in (-1, 1)\}$ for a cylinder.

We will denote functions by $f(x), g(x), \dots$ For potentials, we will use the notation $\phi(x), \psi(x), \dots$ A set of symmetric m-tensor fields $\mathbf{w}(x) = (w_{i_1 \dots i_m}(x))$, $\mathbf{u}(x) = (u_{i_1 \dots i_m}(x))$, $\mathbf{v}(x) = (v_{i_1 \dots i_m}(x))$, $i_1, \dots, i_m = 1, 2, 3$, defined in B, is denoted by

$S^m(B)$. An inner product in $S^m(B)$ is introduced by the formula

$$\langle \mathbf{u}(x), \mathbf{v}(x)\rangle = \sum_{i_1,..,i_m=1}^{3} u_{i_1..i_m}(x) v_{i_1..i_m}(x).$$

A functional space $L_2(S^m(B))$ consists of square integrable symmetric m-tensor fields, defined in B. An inner product of two tensor fields $\mathbf{u}$ and $\mathbf{v}$ from the space $L_2(S^m(B))$ is given by the formula:

$$(\mathbf{u}, \mathbf{v})_{L_2(S^m(B))} = \int_B \langle \mathbf{u}(x), \mathbf{v}(x)\rangle dx.$$

We denote the Sobolev spaces for symmetric m-tensor fields by $H^k(S^m(B))$. By $H_0^k(S^m(B))$ we denote the space of symmetric 2-tensor fields from $H^k(S^m(B))$, vanishing on the boundary of the region along with all their derivatives up to the $(k-1)$-the order. Moreover, we will use the weight space $L_2(Z, \rho)$, where $\rho(s) > 0$ is given on Z. The innerr product of the functions f and g from $L_2(Z, \rho)$ is defined by the formula:

$$(f, g)_{L_2(Z,\rho)} = \int_Z f(x) g(x) \rho(x) dx.$$

Differential operators. We will use the following operators:

1) *Inner differential operator*

$$\mathrm{d} : H^k(S^m(B)) \to H^{k-1}(S^{m+1}(B)),$$

acting on a potential ψ and a vector field $\mathbf{v}$ in the following way:

$$(\mathrm{d}\psi)_i = \frac{\partial \psi}{\partial x_i}, \qquad (\mathrm{d}\mathbf{v})_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

2) *Curl operator*

$$\mathrm{curl} : H^k(S^1(B)) \to H^{k-1}(S^1(B)),$$

acting on a vector field $\mathbf{w}$ by the formula

$$\mathrm{curl}\, \mathbf{w} = \left(\frac{\partial w_3}{\partial x_2} - \frac{\partial w_2}{\partial x_3}, \frac{\partial w_1}{\partial x_3} - \frac{\partial w_3}{\partial x_1}, \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}\right).$$

3) *Divergence operator*

$$\mathrm{div} : H^k(S^{m+1}(B)) \to H^{k-1}(S^m(B)),$$

acting on a tensor field $\mathbf{w}$ according to the rule:

$$(\mathrm{div}\, \mathbf{w})_{i_1..i_m} = \sum_{j=1}^{3} \frac{\partial w_{i_1..i_m j}}{\partial x_j}.$$

Recall that a m-tensor field $\mathbf{u} \in H^k(S^m(B))$ is called *potential*, if there exists a $(m-1)$-tensor field $\mathbf{v} \in H^{k+1}(S^{m-1}(B))$ (the potential), such that $\mathbf{u} = \mathrm{d}\mathbf{v}$. A field $\mathbf{w} \in H^k(S^m(B))$ is called *solenoidal*, if $\mathrm{div}\, \mathbf{w} = 0 \in H^{k-1}(S^{m-1}(B))$. Obviously, the vector field $\mathbf{w} = \mathrm{curl}\, \mathbf{u}$ is solenoidal. Similarly, a symmetric 2-tensor field $\mathbf{w}$ is solenoidal, if $(w_{i1}, w_{i2}, w_{i3}) = \mathrm{curl}\, \mathbf{v}^i,\ i = 1, 2, 3$ for some vector fields $\mathbf{v}^i$.

It is known [1] that there exists a unique decomposition of every symmetric m-tensor field $\mathbf{v} \in L_2(S^m(B))$:

$$\mathbf{v} = \mathbf{w} + \mathrm{d}\mathbf{u},$$
$$\mathbf{w} \in H^1(S^m(B)), \qquad \mathrm{div}\mathbf{w} = 0, \qquad \mathbf{u} \in H_0^1(S^{m-1}(B)).$$

In work [19], it was shown that for a symmetric 2-tensor field $\mathbf{v} \in L_2(S^2(B))$ there exists a unique decomposition

$$\mathbf{v} = \mathbf{w} + \mathrm{d}(\mathrm{curl}\,\mathbf{u}) + \mathrm{d}^2\phi,$$

where

$$\mathbf{w} \in H^1(S^2(B)), \qquad \mathrm{div}\mathbf{w} = 0,$$
$$\mathbf{u} \in H^2(S^1(B)), \qquad \mathrm{curl}\,\mathbf{u} \in H_0^1(S^1(B)),$$
$$\phi \in H_0^2(B).$$

Integral operators. The plane $P_{\xi,s}$ in $\mathbb{R}^3$ is given by the normal equation $\langle \xi, x\rangle - s = 0$ for $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$, $|\xi| = 1$. Here $|s|$ is the distance from the plane to the origin, and ξ is the normal vector of the plane.

The Radon transform $\mathcal{R}f : L_2(B) \to L_2(Z, \rho)$ of the scalar function $f(x)$ is given by the formula

$$[\mathcal{R}f](s, \xi) = \int\limits_{P_{\xi,s}\cap B} f(x)\, dx.$$

The normal Radon transform $\mathcal{R}_2^{\perp} : L_2(S^2(B)) \to L_2(Z, \rho)$ of a symmetric 2-tensor field $\mathbf{u}(x)$ is defined by the formula

$$[\mathcal{R}_2^{\perp}\mathbf{u}](s, \xi) = \int\limits_{P_{\xi,s}\cap B} \langle \mathbf{u}(x), \xi^2\rangle\, dx.$$

We will formulate the properties of the normal Radon transform, obtained in work [19], in the form of a statement.

Statement 1 *1) For the function $\psi \in H_0^2(B)$, the following equality holds:*

$$[\mathcal{R}_2^{\perp}(\mathrm{d}^2\psi)](s, \xi) = \frac{\partial^2}{\partial s^2}[\mathcal{R}\psi](s, \xi).$$

2) The kernel of the normal Radon transform operator, acting on symmetric 2-tensor fields, consists of any linear combinations of two following types of fields:

— solenoidal symmetric 2-tensor fields $\mathbf{w}$, for which $(w_{i1}, w_{i2}, w_{i3}) = \mathrm{curl}\,\mathbf{u}^i$, $i = 1, 2, 3$ is fulfilled, where $\mathbf{u}^i \in H^2(S^1(B)) \cup H_0^1(S^1(B))$, $i = 1, 2, 3$;

— potential symmetric 2-tensor fields of the form $\mathbf{w} = \mathrm{d}(\mathrm{curl}\,\mathbf{v})$, such that $\mathbf{v} \in H^2(S^1(B)) \cup H_0^1(S^1(B))$.

Statement 1 implies that by the known normal Radon transform of a symmetric 2-tensor field only its potential part of the form $\mathrm{d}^2\psi$, $\psi \in H^2(B)$, can be reconstructed.

Orthogonal polynomials. Recall the definitions and some properties of orthogonal polynomials necessary for construction of a singular value decomposition of the normal Radon transform operator.

Jacobi polynomials $P_n^{(p,q)}(t)$ of degree n with indices (p,q), given on the segment $[0,1]$, are defined by the explicit formula

$$P_n^{(p,q)}(t) = 1 + \sum_{k=1}^{n} (-1)^k C_n^k \frac{(p+n)(p+n+1)\dots(p+n+k-1)}{q(q+1)\dots(q+k-1)} t^k,$$

where C_n^k are binomial coefficients. On the segment $[0,1]$, these polynomials are orthogonal with weight $t^{q-1}(1-t)^{p-q}$, that is, for $n \neq m$, the following equality holds:

$$\int_0^1 t^{q-1}(1-t)^{p-q}\, P_n^{(p,q)}(t)\, P_m^{(p,q)}(t)\, dt = 0.$$

The norm of Jacobi polynomials can be calculated by the following formula:

$$(1) \qquad \|P_n^{(p,q)}\|^2 = \int_0^1 t^{q-1}(1-t)^{p-q}\, P_n^{(p,q)}(t)\, P_n^{(p,q)}(t)\, dt = \frac{n!\Gamma(q)\Gamma(q)\Gamma(p-q+n+1)}{\Gamma(q+n)\Gamma(p+n)(p+2n)}.$$

The first and second derivatives of a Jacobi polynomial are calculated by the formulas

$$(2) \qquad \begin{aligned} \left(P_n^{(p,q)}\right)'(t) &= -\frac{n(n+p)}{q} P_{n-1}^{(p+2,q+1)}(t), \\ \left(P_n^{(p,q)}\right)''(t) &= \frac{n(n-1)(n+p)(n+p+1)}{q(q+1)} P_{n-2}^{(p+4,q+2)}(t). \end{aligned}$$

Gegenbauer polynomials $C_n^{(\mu)}(t)$ of degree n with index μ are defined by the explicit formula

$$C_n^{(\mu)}(t) = \sum_{k=0}^{[n/2]} (-1)^k \frac{\Gamma(n-k+\mu)}{\Gamma(\mu)\, k!\, (n-2k)!} (2t)^{n-2k},$$

where $\Gamma(\alpha)$ is the Gamma function, and $[\,\cdot\,]$ is the integer part of the number. On the segment $[-1,1]$, Gegenbauer polynomials are orthogonal with weight $(1-t^2)^{\mu-1/2}$. The norm of Gegenbauer polynomials can be calculated by the formula

$$\|C_n^{(\mu)}\|^2 = \int_{-1}^{1} C_n^{(\mu)}(t) C_n^{(\mu)}(t)(1-t^2)^{\mu-1/2} dt = \frac{\pi 2^{1-2\mu}\Gamma(n+2\mu)}{n!(n+\mu)\Gamma^2(\mu)}.$$

Legendre polynomials $L_k(t)$ of degree k represent a special case of Gegenbauer polynomials: $L_k(t) = C_k^{(0.5)}(t)$. The derivative of a Legendre polynomial is calculated by the formula

$$\frac{d}{dt} L_k(t) = \frac{k}{1-t^2}(L_{k-1}(t) - tL_k(t)).$$

Legendre polynomials are orthogonal on the segment $[-1,1]$. The norm of Legendre polynomials can be calculated by the formula

$$\|L_k\|^2 = \int_{-1}^{1} L_k^2(t) dt = \frac{2}{2k+1}.$$

An associated Legendre polynomial $L_{kl}(t)$ of degree k with an integer index $l = 0, \ldots, k$ satisfies the Legendre equation, that is, the following equality holds

$$\frac{d}{dt}\left((1-t^2)\frac{d}{dt}L_{kl}(t)\right) + \left(k(k+1) - \frac{l^2}{1-t^2}\right)L_{kl}(t) = 0, \tag{3}$$

and is defined in terms of Legendre polynomial:

$$L_{kl}(t) = (1-t^2)^{l/2}\frac{d^l}{dt^l}L_k(t). \tag{4}$$

On the segment $[-1, 1]$, associated Legendre polynomials are orthogonal.

A spherical function Y_{kl} of order k with an integer index $l = -k, \ldots, k$ is defined by the formula

$$Y_{kl}(\theta, \varphi) = L_{k|l|}(\cos\theta) \cdot \begin{cases} \cos l\varphi, & l \geqslant 0, \\ \sin |l|\varphi, & l < 0. \end{cases}$$

Spherical functions are orthogonal on the unit sphere. The norm of a spherical function is calculated by the formula

$$\|Y_{kl}\| = \begin{cases} \sqrt{\dfrac{4\pi}{2k+1}}, & l = 0, \\ \sqrt{\dfrac{2\pi}{2k+1}\dfrac{(k+|l|)!}{(k-|l|)!}}, & l \neq 0. \end{cases}$$

Harmonic polynomials $H_{kl}(x)$ of degree k with an integer index $l = -k, \ldots, k$ in the spherical coordinate system have the form

$$H_{kl}(r, \theta, \varphi) = r^k Y_{kl}(\theta, \varphi).$$

3. Singular value decomposition of the normal Radon transform operator acting on symmetric 2-tensor fields

We will construct basic potential symmetric 2-tensor fields in the original space $L_2(S^2(B))$ by the potential method, that is, we choose a basic system of functions in the potential space $H_0^2(B)$, and then, applying the differential operator d^2, we form it into a 2-tensor potential basis in the original space $L_2(S^2(B))$. For the basis of the original system, we choose the polynomials of the following form:

$$\Phi_{kln}(x) = (1-|x|^2)^2\, H_{kl}(x)\, P_n^{(k+3.5,k+1.5)}(|x|^2), \quad k, n = 0, 1, 2, \ldots, \quad l = -k, \ldots k,$$

in the spherical coordinate system ($x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$) they have the form

$$\Phi_{kln}(r, \theta, \varphi) = (1-r^2)^2\, r^k\, P_n^{(k+3.5,k+1.5)}(r^2)\, Y_{kl}(\theta, \varphi).$$

Applying to the above potentials the operator d^2, we obtain a family of basic potential symmetric 2-tensor fields

$$\mathbf{T}_{kln}(x) = \mathrm{d}^2\Phi_{kln}(x), \qquad k, n = 0, 1, 2, \ldots, \quad l = -k, \ldots, k.$$

The basic system of potentials is defined by the equalities $\widetilde{\Phi}_{kln}(x) = \lambda_{kln}\Phi_{kln}(x)$, $k, n = 0, 1, 2, \ldots$, $l = -k, \ldots k$, where

$$\lambda_{kln} = \frac{\Gamma(n+k+1.5)}{(n+2)!\Gamma(k+1.5)\|Y_{kl}\|}\sqrt{\frac{2n+k+3.5}{8}}.$$

The main result of this paper is

Theorem 1 *The system of potential symmetric 2-tensor fields* $\widetilde{\mathbf{T}}_{kln} = \mathrm{d}^2\widetilde{\Phi}_{kln}$ *is orthonormal in the space* $L_2(S^2(B))$.

Now we formulate the results obtained in work [19].

Statement 2 *The images of potential symmetric 2-tensor fields* $\widetilde{\mathbf{T}}_{kln}$, $k, n = 0, 1, 2, \ldots$, $l = -k, \ldots, k$, *under the action of the normal Radon transform operator have the form*

$$[\mathcal{R}_2^{\perp}\widetilde{\mathbf{T}}_{kln}](s,\,\theta,\,\varphi) = b_{kln}(1-s^2)C_{2n+k+2}^{(1.5)}(s)Y_{kl}(\theta,\varphi),$$

where

$$b_{kln} = \frac{(-1)^n 4\pi}{(2n+k+3)(2n+k+4)\|Y_{kl}\|}\sqrt{\frac{2n+k+3.5}{2}}.$$

Statement 3 *The system of functions*

$$G_{kln}(s,\,\theta,\,\varphi) = \frac{(-1)^n\sqrt{2n+k+3.5}}{\sqrt{(2n+k+3)(2n+k+4)}\|Y_{kl}\|}(1-s^2)C_{2n+k+2}^{(1.5)}(s)Y_{kl}(\theta,\varphi),$$
$$k, n = 0, 1, 2, \ldots, \quad l = -k, \ldots, k,$$

is orthonormal in the space $L_2(Z, (1-s^2)^{-1})$.

Statement 2 and the definitions of the functions G_{kln} yield the equalities:

$$[\mathcal{R}_2^{\perp}\widetilde{\mathbf{T}}_{kln}](s,\,\theta,\,\varphi) = \sigma_{kn}G_{kln}(s,\,\theta,\,\varphi), \qquad k, n = 0, 1, 2, \ldots, \quad l = -k, \ldots, k,$$

where the numbers

$$\sigma_{kn} = \frac{2\sqrt{2}\pi}{\sqrt{(2n+k+3)(2n+k+4)}},$$

taking into account Theorem 1 and Statement 3, are singular values of the normal Radon transform operator. Therefore, the following theorem holds.

Theorem 2 *The singular value decomposition of the normal Radon transform operator*

$$\mathcal{R}_2^{\perp} \,:\, L_2(S^2(B)) \to L_2(Z, (1-s^2)^{-1})$$

has the form

$$g(s,\theta,\varphi) := [\mathcal{R}_2^{\perp}\mathbf{w}](s,\theta,\varphi) = \sum_{k,n=0}^{\infty}\sum_{l=-k}^{k}\sigma_{kn}\left(\mathbf{w}, \widetilde{\mathbf{T}}_{kln}\right)_{L_2(S^2(B))} G_{kln}(s,\theta,\varphi),$$

and the inverse operator is calculated by the formula

$$\mathbf{w}(x) = \left((\mathcal{R}_2^{\perp})^{-1}g\right)(x) = \sum_{k,n=0}^{\infty}\sum_{l=-k}^{k}\sigma_{kn}^{-1}(g, G_{kln})_{L_2(Z,(1-s^2)^{-1})}\widetilde{\mathbf{T}}_{kln}(x).$$

4. Proof of Theorem 1

For brevity, we introduce the following notation:

$$P_n := P_n^{(k+3.5,k+1.5)}(r^2), \quad P_n' := \frac{\partial P_n^{(k+3.5,k+1.5)}}{\partial(r^2)}(r^2), \quad P_n'' := \frac{\partial^2 P_n^{(k+3.5,k+1.5)}}{\partial(r^2)^2}(r^2),$$
$$Y_{kl} := Y_{kl}(\varphi,\theta), \qquad Y_{kl,\varphi}' := \frac{\partial Y_{kl}}{\partial\varphi}(\varphi,\theta), \qquad Y_{kl,\theta}' := \frac{\partial Y_{kl}}{\partial\theta}(\varphi,\theta),$$
$$Y_{kl,\varphi\varphi}'' := \frac{\partial^2 Y_{kl}}{\partial\varphi^2}(\varphi,\theta), \qquad Y_{kl,\varphi\theta}'' := \frac{\partial^2 Y_{kl}}{\partial\varphi\partial\theta}(\varphi,\theta), \qquad Y_{kl,\theta\theta}'' := \frac{\partial^2 Y_{kl}}{\partial\theta^2}(\varphi,\theta).$$

Lemma 1 *The values of components of basic symmetric 2-tensfor fields can be calculated by the formulas (in the spherical coordinate system)*

$$\begin{aligned}\left(\mathbf{T}_{kln}\right)_{ij}(r,\theta,\varphi) &= \left(2A_{ij}r^{k+2} - 2B_{ij}(1-r^2)r^k + C_{ij}(1-r^2)^2r^{k-2}\right)P_n \\ &+ \left(-4A_{ij}(1-r^2)r^{k+2} + B_{ij}(1-r^2)^2r^k\right)P_n' + A_{ij}(1-r^2)^2r^{k+2}P_n'', \\ &i,j = 1,2,3,\end{aligned}$$

where

$$\begin{aligned}
A_{11} &= Y_{kl}\left(4\cos^2\varphi\sin^2\theta\right),\\
B_{11} &= Y_{kl}\left(4k\cos^2\varphi\sin^2\theta+2\right) + Y'_{kl,\varphi}\left(-2\sin 2\varphi\right) + Y'_{kl,\theta}\left(2\cos^2\varphi\sin 2\theta\right),\\
C_{11} &= Y_{kl}\left(k(k-2)\cos^2\varphi\sin^2\theta + k\right) + Y'_{kl,\varphi}\left(\sin 2\varphi(\operatorname{ctg}^2\theta - k + 1)\right)\\
&+ Y''_{kl,\varphi\varphi}\left(\sin^2\varphi(1+\operatorname{ctg}^2\theta)\right) + Y'_{kl,\theta}\left((k-1)\cos^2\varphi\sin 2\theta + \sin^2\varphi\operatorname{ctg}\theta\right)\\
&+ Y''_{kl,\varphi\theta}\left(-\sin 2\varphi\operatorname{ctg}\theta\right) + Y''_{kl,\theta\theta}\left(\cos^2\varphi\cos^2\theta\right);\\
A_{12} = A_{21} &= Y_{kl}\left(2\sin 2\varphi\sin^2\theta\right),\\
B_{12} = B_{21} &= Y_{kl}\left(2k\sin 2\varphi\sin^2\theta\right) + Y'_{kl,\varphi}\left(2\cos 2\varphi\right) + Y'_{kl,\theta}\left(\sin 2\varphi\sin 2\theta\right),\\
C_{12} = C_{21} &= Y_{kl}\left(k(k-2)\sin 2\varphi\sin^2\theta/2\right) + Y'_{kl,\varphi}\left(\cos 2\varphi(k-1-\operatorname{ctg}^2\theta)\right)\\
&+ Y''_{kl,\varphi\varphi}\left(-\sin 2\varphi(1+\operatorname{ctg}^2\theta)/2\right) + Y'_{kl,\theta}\left(\sin 2\varphi((k-1)\sin 2\theta - \operatorname{ctg}\theta)/2\right)\\
&+ Y''_{kl,\varphi\theta}\left(\cos 2\varphi\operatorname{ctg}\theta\right) + Y''_{kl,\theta\theta}\left(\sin 2\varphi\cos^2\theta/2\right);\\
A_{22} &= Y_{kl}\left(4\sin^2\varphi\sin^2\theta\right),\\
B_{22} &= Y_{kl}\left(4k\sin^2\varphi\sin^2\theta+2\right) + Y'_{kl,\varphi}\left(2\sin 2\varphi\right) + Y'_{kl,\theta}\left(2\sin^2\varphi\sin 2\theta\right),\\
C_{22} &= Y_{kl}\left(k(k-2)\sin^2\varphi\sin^2\theta + k\right) + Y'_{kl,\varphi}\left(\sin 2\varphi(k-1-\operatorname{ctg}^2\theta)\right)\\
&+ Y''_{kl,\varphi\varphi}\left(\cos^2\varphi(1+\operatorname{ctg}^2\theta)\right) + Y'_{kl,\theta}\left((k-1)\sin^2\varphi\sin 2\theta + \cos^2\varphi\operatorname{ctg}\theta\right)\\
&+ Y''_{kl,\varphi\theta}\left(\sin 2\varphi\operatorname{ctg}\theta\right) + Y''_{kl,\theta\theta}\left(\sin^2\varphi\cos^2\theta\right);\\
A_{13} = A_{31} &= Y_{kl}\left(2\cos\varphi\sin 2\theta\right),\\
B_{13} = B_{31} &= Y_{kl}\left(2k\cos\varphi\sin 2\theta\right) + Y'_{kl,\varphi}\left(-2\sin\varphi\operatorname{ctg}\theta\right) + Y'_{kl,\theta}\left(2\cos\varphi\cos 2\theta\right),
\end{aligned}$$

$$\begin{aligned}
C_{13} = C_{31} &= Y_{kl}\left(k(k-2)\cos\varphi\sin 2\theta/2\right) + Y'_{kl,\varphi}\left(-k\sin\varphi\operatorname{ctg}\theta\right)\\
&+ Y'_{kl,\theta}\left((k-1)\cos\varphi\cos 2\theta\right) + Y''_{kl,\varphi\theta}\sin\varphi + Y''_{kl,\theta\theta}\left(-\cos\varphi\sin 2\theta/2\right);\\
A_{23} = A_{32} &= Y_{kl}\left(2\sin\varphi\sin 2\theta\right),\\
B_{23} = B_{32} &= Y_{kl}\left(2k\sin\varphi\sin 2\theta\right) + Y'_{kl,\varphi}\left(2\cos\varphi\operatorname{ctg}\theta\right) + Y'_{kl,\theta}\left(2\sin\varphi\cos 2\theta\right),\\
C_{23} = C_{32} &= Y_{kl}\left(k(k-2)\sin\varphi\sin 2\theta/2\right) + Y'_{kl,\varphi}\left(k\cos\varphi\operatorname{ctg}\theta\right)\\
&+ Y'_{kl,\theta}\left((k-1)\sin\varphi\cos 2\theta\right) + Y''_{kl,\varphi\theta}(-\cos\varphi) + Y''_{kl,\theta\theta}\left(-\sin\varphi\sin 2\theta/2\right);\\
A_{33} &= Y_{kl}\left(4\cos^2\theta\right),\\
B_{33} &= Y_{kl}\left(4k\cos^2\theta + 2\right) + Y'_{kl,\theta}\left(-2\sin 2\theta\right),\\
C_{33} &= Y_{kl}\left(k(k-2)\cos^2\theta + k\right) + Y'_{kl,\theta}\left(-(k-1)\sin 2\theta\right) + Y''_{kl,\theta\theta}\left(\sin^2\theta\right).
\end{aligned}$$

Proof. Transforming the Cartesian coordinate system into the spherical one, we have that

$$\begin{aligned}\frac{\partial^2\Phi_{kln}}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial\Phi_{kln}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\Phi_{kln}}{\partial r}\cos\varphi\sin\theta + \frac{\partial\Phi_{kln}}{\partial\theta}\frac{\cos\varphi\cos\theta}{r} - \frac{\partial\Phi_{kln}}{\partial\varphi}\frac{\sin\varphi}{r\sin\theta}\right)\\
&= \frac{\partial^2\Phi_{kln}}{\partial r^2}\cos^2\varphi\sin^2\theta + \frac{\partial^2\Phi_{kln}}{\partial\theta^2}\frac{\cos^2\varphi\cos^2\theta}{r^2} + \frac{\partial^2\Phi_{kln}}{\partial\varphi^2}\frac{\sin^2\varphi}{r^2\sin^2\theta}\\
&+ 2\frac{\partial^2\Phi_{kln}}{\partial r\partial\theta}\frac{\cos^2\varphi\sin\theta\cos\theta}{r} - 2\frac{\partial^2\Phi_{kln}}{\partial r\partial\varphi}\frac{\sin\varphi\cos\varphi}{r} - 2\frac{\partial^2\Phi_{kln}}{\partial\theta\partial\varphi}\frac{\sin\varphi\cos\varphi\cos\theta}{r^2\sin\theta}\\
&+ \frac{\partial\Phi_{kln}}{\partial r}\frac{\cos^2\varphi\cos^2\theta + \sin^2\varphi}{r} + \frac{\partial\Phi_{kln}}{\partial\theta}\frac{\cos\theta(\sin^2\varphi - 2\cos^2\varphi\sin^2\theta)}{r^2\sin\theta}\\
&+ 2\frac{\partial\Phi_{kln}}{\partial\varphi}\frac{\sin\varphi\cos\varphi}{r^2\sin^2\theta}.\end{aligned}$$

Substituting the expression for Φ_{kln} and grouping the summands by functions of r^2, we obtain the statement of Lemma for $(T_{kln})_{11}(r,\theta,\varphi)$.

In a similar way, using the formulas

$$\begin{aligned}\frac{\partial\Phi_{kln}}{\partial y} &= \frac{\partial\Phi_{kln}}{\partial r}\sin\varphi\sin\theta + \frac{\partial\Phi_{kln}}{\partial\theta}\frac{\sin\varphi\cos\theta}{r} + \frac{\partial\Phi_{kln}}{\partial\varphi}\frac{\cos\varphi}{r\sin\theta},\\
\frac{\partial\Phi_{kln}}{\partial z} &= \frac{\partial\Phi_{kln}}{\partial r}\cos\theta - \frac{\partial\Phi_{kln}}{\partial\theta}\frac{\sin\theta}{r},\end{aligned}$$

we obtain the remaining statements of Lemma. □

Orthogonality in the case $\mathbf{n_1 \neq n_2}$, $\mathbf{k_1 = k_2}$, $\mathbf{l_1 = l_2}$. In what follows, we assume that $l \geqslant 0$, the case when $l < 0$ is proved in a similar way. Applying Lemma 1, we calculate the scalar product

$$\begin{aligned}(\mathbf{T}_{kln_1},\mathbf{T}_{kln_2})_{L_2(S^2(B))} &= \iiint\limits_B \sum_{i,j=1}^{3}((T_{kln_1})_{ij}(x,y,z)(T_{kln_2})_{ij}(x,y,z))\,dx\,dy\,dz\\
&= \int_0^1\int_0^{2\pi}\int_0^{\pi}\left(\sum_{i,j=1}^{3}(T_{kln_1})_{ij}(r,\theta,\varphi)(T_{kln_2})_{ij}(r,\theta,\varphi)\right)r^2\sin\theta d\theta\,d\varphi\,dr.\end{aligned}$$

We substitute one r into dr, multiply by the second r the whole integrand and consider separately the integral over r^2. Using the orthogonality properties of Jacobi polynomials

$$\int_0^1 r^{2k+1}(1-r^2)^2P_{n_1}P_{n_2}dr^2 = 0, \qquad \int_0^1 r^{2k+3}(1-r^2)^3P'_{n_1}P'_{n_2}dr^2 = 0,$$

$$\int_0^1 r^{2k+5}(1-r^2)^4P''_{n_1}P''_{n_2}dr^2 = 0,$$

we obtain

$$(\mathbf{T}_{kln_1}, \mathbf{T}_{kln_2})_{L_2(S^2(B))} = \frac{1}{2}\int_0^1\int_0^{2\pi}\int_0^{\pi}\sum_{i,j=1}^{3}\left(B_{ij}^2(2k+1) - A_{ij}B_{ij}\frac{(2k+3)(2k+1)}{2}\right.$$

$$\left. - A_{ij}C_{ij}\frac{(2k+3)(2k+1)}{2} - 2C_{ij}^2 + B_{ij}C_{ij}(2k-1)\right)(1-r^2)^2 r^{2k-1} P_{n_1}P_{n_2}\sin\theta d\theta\, d\varphi\, dr^2$$

$$+ \frac{1}{2}\int_0^1\int_0^{2\pi}\int_0^{\pi}\sum_{i,j=1}^{3}\left(C_{ij}^2 + A_{ij}C_{ij}(k^2-1/4) - B_{ij}C_{ij}(k-1/2)\right)$$

$$\cdot (1-r^2)^2 r^{2k-3} P_{n_1}P_{n_2}\sin\theta d\theta\, d\varphi\, dr^2$$

$$+ \frac{1}{2}\int_0^1\int_0^{2\pi}\int_0^{\pi}\sum_{i,j=1}^{3}\left(B_{ij}^2 - 2A_{ij}C_{ij} - A_{ij}B_{ij}\frac{2k+3}{2}\right)(1-r^2)^3 r^{2k+1} P'_{n_1}P'_{n_2}\sin\theta d\theta\, d\varphi\, dr^2.$$

Consider separately the last integral. Substituting the expressions for A_{ij}, B_{ij}, C_{ij} and taking the sum, we obtain the following expression

$$\frac{1}{2}\int_0^1 (1-r^2)^3 r^{2k+1} P'_{n_1}P'_{n_2} dr^2 \cdot \int_0^{2\pi}\int_0^{\pi}\sum_{i,j=1}^{3}\left(B_{ij}^2 - 2A_{ij}C_{ij} - A_{ij}B_{ij}\frac{2k+3}{2}\right)\sin\theta d\theta\, d\varphi$$

$$= \frac{1}{2}\int_0^1 (1-r^2)^3 r^{2k+1} P'_{n_1}P'_{n_2} dr^2$$

$$\cdot \int_0^{2\pi}\int_0^{\pi}\left(-8k(k+1)Y_{kl}^2 + 8Y'_{kl,\theta}Y'_{kl,\theta} + 8\frac{1}{\sin^2\theta}Y'_{kl,\varphi}Y'_{kl,\varphi}\right)\sin\theta d\theta\, d\varphi.$$

In work [15], it is proved that

$$\int_0^{2\pi}\int_0^{\pi}\left(-k(k+1)Y_{kl}^2 + Y'_{kl,\theta}Y'_{kl,\theta} + \frac{1}{\sin^2\theta}Y'_{kl,\varphi}Y'_{kl,\varphi}\right)\sin\theta d\theta\, d\varphi = 0,$$

hence, the last integral turns to zero, and it remains to prove that the sum of the first two integrals equals zero.

Note that the integrands in the considered integrals are linearly dependent. In particular, for every $i, j = 1, 2, 3$, we have

$$B_{ij}^2(2k+1) - (A_{ij}B_{ij} + A_{ij}C_{ij})\frac{(2k+3)(2k+1)}{2} - 2C_{ij}^2 + B_{ij}C_{ij}(2k-1)$$

$$= (2k+1)\left(B_{ij}^2 - 2A_{ij}C_{ij} - A_{ij}B_{ij}\frac{2k+3}{2}\right) - 2C_{ij}\left(C_{ij} + A_{ij}(k^2-1/4) - B_{ij}(k-1/2)\right),$$

therefore, to prove the orthogonality, it suffices to show that the second integral equals zero. In particular, we prove that

$$I := \int_0^{2\pi}\int_0^{\pi}\sum_{i,j=1}^{3}\left(C_{ij}^2 + A_{ij}C_{ij}(k^2-1/4) - B_{ij}C_{ij}(k-1/2)\right)\sin\theta d\theta\, d\varphi = 0.$$

Substituting the expressions for A_{ij}, B_{ij}, C_{ij} and taking the sum, we obtain

$$I = \int_0^{2\pi}\int_0^{\pi} \Big(k(k-2)(k^2-1)Y_{kl}^2 + \operatorname{ctg}\theta\, Y_{kl}Y'_{kl,\theta} + (\operatorname{ctg}^2\theta+1)Y_{kl}Y''_{kl,\varphi\varphi} + Y_{kl}Y''_{kl,\theta\theta}$$
$$+ 2(\operatorname{ctg}^2\theta+1)(\operatorname{ctg}^2\theta - k^2 + k)Y'_{kl,\varphi}Y'_{kl,\varphi} - 4\operatorname{ctg}\theta(\operatorname{ctg}^2\theta+1)Y'_{kl,\varphi}Y''_{kl,\varphi\theta}$$
$$+ (\operatorname{ctg}^2\theta - 2k(k-1))Y'_{kl,\theta}Y'_{kl,\theta} + 2\operatorname{ctg}\theta(\operatorname{ctg}^2\theta+1)Y'_{kl,\theta}Y''_{kl,\varphi\varphi}$$
$$+ (\operatorname{ctg}^2\theta+1)^2 Y''_{kl,\varphi\varphi}Y''_{kl,\varphi\varphi} + 2(\operatorname{ctg}^2\theta+1)Y''_{kl,\varphi\theta}Y''_{kl,\varphi\theta} + Y''_{kl,\theta\theta}Y''_{kl,\theta\theta}\Big)\sin\theta d\theta\, d\varphi\,.$$

Representing Y_{kl} in terms of associated Legendre polynomials, substituting $t = \cos\theta$ and integrating over φ, we obtain

$$I = -\pi\int_{-1}^{1}\Big(k(k-2)(k^2-1)\,L_{kl}^2 - tL_{kl}L'_{kl} - \frac{l^2}{1-t^2}L_{kl}^2 + L_{kl}(-tL'_{kl} + (1-t^2)L''_{kl})$$
$$+ \frac{2l^2}{(1-t^2)}L_{kl}^2\Big(\frac{t^2}{1-t^2} - k(k-1)\Big) + \frac{4l^2t}{1-t^2}L_{kl}L'_{kl}$$
$$+ L'_{kl}L'_{kl}(1-t^2)\Big(\frac{t^2}{1-t^2} - 2k(k-1)\Big) + \frac{2l^2t}{1-t^2}L_{kl}L'_{kl} + \frac{l^4}{(1-t^2)^2}L_{kl}^2$$
$$+ 2l^2L'_{kl}L'_{kl} + \Big(-tL'_{kl} + (1-t^2)L''_{kl}\Big)^2\Big)\,dt$$
$$= -\pi\int_{-1}^{1}\Big(k(k-2)(k^2-1)\,L_{kl}^2 - 2tL_{kl}L'_{kl} - \frac{l^2}{1-t^2}L_{kl}^2 + (1-t^2)L_{kl}L''_{kl} + \frac{2l^2t^2}{(1-t^2)^2}L_{kl}^2$$
$$- \frac{2l^2k(k-1)}{1-t^2}L_{kl}^2 + \frac{6l^2t}{1-t^2}L_{kl}L'_{kl} + 2t^2L'_{kl}L'_{kl} - 2k(k-1)(1-t^2)L'_{kl}L'_{kl}$$
$$+ \frac{l^4}{(1-t^2)^2}L_{kl}^2 + 2l^2L'_{kl}L'_{kl} + (1-t^2)^2L''_{kl}L''_{kl} - 2t(1-t^2)L'_{kl}L''_{kl}\Big)\,dt.$$

Squaring the Legendre equation (3) and subtracting from the integrand, we obtain

$$I = -\pi\int_{-1}^{1}\Big(k(k+1)(-4k+2)L_{kl}^2 - \frac{l^2}{1-t^2}L_{kl}^2 + (1-2k-2k^2)(1-t^2)L_{kl}L''_{kl}$$
$$+ \frac{2l^2t^2}{(1-t^2)^2}L_{kl}^2 + \frac{4kl^2}{1-t^2}L_{kl}^2 + \frac{2l^2t}{1-t^2}L_{kl}L'_{kl} - 2t^2L'_{kl}L'_{kl} - 2k(k-1)(1-t^2)L'_{kl}L'_{kl}$$
$$+ 2l^2L'_{kl}L'_{kl} + 2t(1-t^2)L'_{kl}L''_{kl} + 2l^2L_{kl}L''_{kl} + 2t(2k^2+2k-1)L_{kl}L'_{kl}\Big)\,dt.$$

Lemma 2 *For the derivatives k_1, k_2, the following equality holds:*

$$(1-t^2)L_{k_1l}L'_{k_2l}\Big|_{-1}^{1} = 0. \tag{5}$$

Proof. For $l = 0$, we have

$$(1-t^2)L_{k_1l}L'_{k_2l}\Big|_{-1}^{1} = (1-t^2)L_{k_1}L'_{k_2}\Big|_{-1}^{1} = 0,$$

since Legendre polynomials and their derivatives do not have singularities when $t = \pm 1$. If $l = 1$, then

$$
\begin{aligned}
(1-t^2)L_{k_1 l}L'_{k_2 l}\Big|_{-1}^{1} &= (1-t^2)L_{k_1 1}L'_{k_2 1}\Big|_{-1}^{1} \\
&\overset{(4)}{=} ((1-t^2)\left((1-t^2)^{1/2}L'_{k_1}\right)\left((1-t^2)^{1/2}L'_{k_2}\right)'\Big|_{-1}^{1} \\
&= (1-t^2)\left((1-t^2)^{1/2}L'_{k_1}\right)\left(\frac{-t}{(1-t^2)^{1/2}}L'_{k_2} + (1-t^2)^{1/2}L''_{k_2}\right)\Big|_{-1}^{1} \\
&= (1-t^2)\left(-tL'_{k_1}L'_{k_2} + (1-t^2)L'_{k_1}L''_{k_2}\right)\Big|_{-1}^{1} = 0.
\end{aligned}
$$

For $l \geqslant 2$, the equality is proved in a similar way. □

Further, note that

$$
\begin{aligned}
&\int_{-1}^{1}\left(\frac{(4k-1)l^2}{1-t^2}L_{kl}^2 - k(k+1)(4k-1)L_{kl}^2\right)dt \\
=&\int_{-1}^{1}(4k-1)\left(\frac{l^2}{1-t^2}L_{kl} - k(k+1)L_{kl}\right)L_{kl}\,dt \overset{(3)}{=} \int_{-1}^{1}(4k-1)\left((1-t^2)L'_{kl}\right)'L_{kl}\,dt \\
=&(4k-1)(1-t^2)L_{kl}L'_{kl}\Big|_{-1}^{1} - \int_{-1}^{1}(4k-1)(1-t^2)L'_{kl}L'_{kl}\,dt \\
\overset{(5)}{=}& -\int_{-1}^{1}(4k-1)(1-t^2)L'_{kl}L'_{kl}\,dt,
\end{aligned}
$$

hence,

$$
\begin{aligned}
I = -\pi\int_{-1}^{1}\Big(&k(k+1)L_{kl}^2 + \underline{(1-2k-2k^2)(1-t^2)L_{kl}L''_{kl}} + \frac{2l^2t^2}{(1-t^2)^2}L_{kl}^2 \\
&+ \frac{2l^2t}{1-t^2}L_{kl}L'_{kl} - 2t^2L'_{kl}L'_{kl}\underline{-(2k^2+2k-1)(1-t^2)L'_{kl}L'_{kl}} + 2l^2L'_{kl}L'_{kl} \\
&+ 2t(1-t^2)L'_{kl}L''_{kl} + 2l^2L_{kl}L''_{kl}\underline{+2t(2k^2+2k-1)L_{kl}L'_{kl}}\Big)\,dt.
\end{aligned}
$$

Grouping the underlined summands, we obtain

$$\int_{-1}^{1}\Big((1-2k-2k^2)(1-t^2)L_{kl}L''_{kl}-(2k^2+2k-1)(1-t^2)L'_{kl}L'_{kl}$$
$$+2t(2k^2+2k-1)L_{kl}L'_{kl}\Big)\,dt$$
$$=\int_{-1}^{1}(1-2k-2k^2)\left((1-t^2)L_{kl}L''_{kl}+(1-t^2)L'_{kl}L'_{kl}-2tL_{kl}L'_{kl}\right)\,dt$$
$$=\int_{-1}^{1}(1-2k-2k^2)\left((1-t^2)L_{kl}L'_{kl}\right)'\,dt$$
$$=(1-2k-2k^2)(1-t^2)L_{kl}L'_{kl}\Big|_{-1}^{1}\overset{(5)}{=}0.$$

Therefore, we have

$$I=-\pi\int_{-1}^{1}\left(k(k+1)L^2_{kl}+\frac{2l^2t^2}{(1-t^2)^2}L^2_{kl}+\frac{2l^2t}{1-t^2}L_{kl}L'_{kl}-2t^2L'_{kl}L'_{kl}+2l^2L'_{kl}L'_{kl}\right.$$
$$\left.+2t(1-t^2)L'_{kl}L''_{kl}+2l^2L_{kl}L''_{kl}\right)\,dt.$$

From Legendre equation (3) follows the equality

$$\int_{-1}^{1}k(k+1)L^2_{kl}\,dt=\int_{-1}^{1}\left(\frac{l^2}{1-t^2}L^2_{kl}-\left((1-t^2)L'_{kl}\right)'L_{kl}\right)\,dt$$
$$=-(1-t^2)L_{kl}L'_{kl}\Big|_{-1}^{1}+\int_{-1}^{1}\left(\frac{l^2}{1-t^2}L^2_{kl}+(1-t^2)L'_{kl}L'_{kl}\right)\,dt$$
$$\overset{(5)}{=}\int_{-1}^{1}\left(\frac{l^2}{1-t^2}L^2_{kl}+(1-t^2)L'_{kl}L'_{kl}\right)\,dt,$$

hence,

$$I=-\pi\int_{-1}^{1}\left(\frac{l^2(1+t^2)}{(1-t^2)^2}L^2_{kl}+\frac{2l^2t}{1-t^2}L_{kl}L'_{kl}-2t^2L'_{kl}L'_{kl}+2l^2L'_{kl}L'_{kl}\right.$$
$$\left.+2t(1-t^2)L'_{kl}L''_{kl}+2l^2L_{kl}L''_{kl}+(1-t^2)L'_{kl}L'_{kl}\right)\,dt$$
$$\overset{(3)}{=}-\pi\int_{-1}^{1}\left(\frac{l^2(1+t^2)}{(1-t^2)^2}L^2_{kl}+\frac{2l^2t}{1-t^2}L_{kl}L'_{kl}-2t^2L'_{kl}L'_{kl}+2l^2L'_{kl}L'_{kl}\right.$$
$$\left.+2tL'_{kl}\left[2tL'_{kl}-k(k+1)L_{kl}+\frac{l^2}{1-t^2}L_{kl}\right]+2l^2L_{kl}L''_{kl}+(1-t^2)L'_{kl}L'_{kl}\right)\,dt$$

$$= -\pi \int_{-1}^{1} \left(\frac{l^2(1+t^2)}{(1-t^2)^2} L_{kl}^2 + \frac{4l^2 t}{1-t^2} L_{kl} L'_{kl} + (1+t^2) L'_{kl} L'_{kl} + 2l^2 L'_{kl} L'_{kl} \right.$$
$$\left. -2tk(k+1) L_{kl} L'_{kl} + 2l^2 L_{kl} L''_{kl} \right) dt.$$

Since

$$\int_{-1}^{1} \frac{2l^2 t}{1-t^2} L_{kl} L'_{kl}\, dt = \int_{-1}^{1} \frac{l^2 t}{1-t^2} (L_{kl} L_{kl})'\, dt = \frac{l^2 t}{1-t^2} L_{kl}^2 \Big|_{-1}^{1} - \int_{-1}^{1} \left(\frac{l^2 t}{1-t^2} \right)' L_{kl}^2\, dt$$
$$= \frac{l^2 t}{1-t^2} L_{kl}^2 \Big|_{-1}^{1} - \int_{-1}^{1} \frac{l^2(1+t^2)}{(1-t^2)^2} L_{kl}^2\, dt$$

and

$$\int_{-1}^{1} \left(2l^2 L'_{kl} L'_{kl} + 2l^2 L_{kl} L''_{kl} \right) dt = 2l^2 \int_{-1}^{1} \left(L_{kl} L'_{kl} \right)' dt = 2l^2 L_{kl} L'_{kl} \Big|_{-1}^{1},$$

we have that

$$I = -\pi \left(\frac{l^2 t}{1-t^2} L_{kl}^2 + 2l^2 L_{kl} L'_{kl} \right) \Big|_{-1}^{1}$$
$$-\pi \int_{-1}^{1} \left(2t L'_{kl} \left(\frac{l^2}{1-t^2} L_{kl} - k(k+1) L_{kl} \right) + (1+t^2) L'_{kl} L'_{kl} \right) dt$$
$$\overset{(3)}{=} -\pi \left(\frac{l^2 t}{1-t^2} L_{kl}^2 + 2l^2 L_{kl} L'_{kl} \right) \Big|_{-1}^{1} - \pi \int_{-1}^{1} \left(2t L'_{kl} \left((1-t^2) L'_{kl} \right)' + (1+t^2) L'_{kl} L'_{kl} \right) dt$$
$$= -\pi \left(\frac{l^2 t}{1-t^2} L_{kl}^2 + 2l^2 L_{kl} L'_{kl} + 2t(1-t^2) L'_{kl} L'_{kl} \right) \Big|_{-1}^{1}$$
$$-\pi \int_{-1}^{1} \left(-(2t L'_{kl})'(1-t^2) L'_{kl} + (1+t^2) L'_{kl} L'_{kl} \right) dt$$
$$= -\pi \left(\frac{l^2 t}{1-t^2} L_{kl}^2 + 2l^2 L_{kl} L'_{kl} + 2t(1-t^2) L'_{kl} L'_{kl} \right) \Big|_{-1}^{1}$$
$$-\pi \int_{-1}^{1} \left((-1+3t^2) L'_{kl} L'_{kl} - 2t(1-t^2) L'_{kl} L''_{kl} \right) dt$$
$$= -\pi \left(\frac{l^2 t}{1-t^2} L_{kl}^2 + 2l^2 L_{kl} L'_{kl} + 2t(1-t^2) L'_{kl} L'_{kl} \right) \Big|_{-1}^{1} - \pi \int_{-1}^{1} \left(-t(1-t^2) L'_{kl} L'_{kl} \right)' dt$$
$$= -\pi \left(\frac{l^2 t}{1-t^2} L_{kl}^2 + 2l^2 L_{kl} L'_{kl} + t(1-t^2) L'_{kl} L'_{kl} \right) \Big|_{-1}^{1}.$$

If $l = 0$, then $L_{kl} = L_{k0} = L_k$ and $I = -\pi t(1-t^2) L'_k L'_k \Big|_{-1}^{1} = 0$.

If $l=1$, then $L_{k1} \overset{(4)}{=} (1-t^2)^{1/2}\dfrac{d}{dt}L_k = (1-t^2)^{1/2}L_k'$, therefore,

$$\begin{aligned}
I =& -\pi \left(\frac{t}{1-t^2}(1-t^2)L_k'L_k' + 2(1-t^2)^{1/2}L_k' \left((1-t^2)^{1/2}L_k'\right)' \right.\\
&\left. +t(1-t^2)\left((1-t^2)^{1/2}L_k'\right)'\left((1-t^2)^{1/2}L_k'\right)'\right)\Bigg|_{-1}^{1}\\
=& -\pi \left(tL_k'L_k' + 2(1-t^2)^{1/2}L_k'\left(\frac{-t}{(1-t^2)^{1/2}}L_k' + (1-t^2)^{1/2}L_k''\right)\right.\\
&\left. +t(1-t^2)\left(\frac{t^2}{1-t^2}L_k'L_k' - 2tL_k'L_k'' + (1-t^2)L_k''L_k''\right)\right)\Bigg|_{-1}^{1}\\
=& -\pi\Big(tL_k'L_k' - 2tL_k'L_k' + 2(1-t^2)L_k'L_k'' + t^3L_k'L_k' - 2t^2(1-t^2)L_k'L_k''\\
&+ t(1-t^2)^2L_k''L_k''\Big)\Bigg|_{-1}^{1}\\
=& -\pi\left(-tL_k'L_k' + t^3L_k'L_k'\right)\Bigg|_{-1}^{1} = 0,
\end{aligned}$$

since the derivatives of Legendre polynomials do not have singularities when $t=\pm 1$. Using the similar reasoning, it is easy to show that $I=0$ for $l \geqslant 2$.

Therefore, the orthogonality in the case $k_1=k_2$, $l_1=l_2$ and $n_1 \neq n_2$ is proved.

Orthogonality in the case when $\mathbf{k_1 \neq k_2}$ or $\mathbf{l_1 \neq l_2}$, $\mathbf{n_1}$ and $\mathbf{n_2}$ are arbitrary.

Lemma 3 *The values of components of basic symmetric 2-tensor fields can be calculated by formulas (in the spherical coordinate system)*

$$\left(\mathbf{T}_{kln}\right)_{ij}(r,\theta,\varphi) = A_{ij}r^{k+2}F'' + B_{ij}r^kF' + C_{ij}r^{k-2}F,\;\; i,j=1,2,3,$$

where A_{ij}, B_{ij}, C_{ij} are from Lemma 1 and $F=(1-r^2)^2P_n^{(k+3.5,\,k+1.5)}(r^2)$.

This fact is proved by regrouping the summands in the representation from Lemma 1.

Using Lemma 3, we will calculate the scalar product

$$\begin{aligned}
J :=& \left(\mathbf{T}_{k_1l_1n_1}, \mathbf{T}_{k_2l_2n_2}\right)_{L_2(S^2(B))}\\
=& \int_0^1\int_0^{2\pi}\int_0^{\pi}\left(\sum_{i,j=1}^{3}(T_{k_1l_1n_1})_{ij}(r,\theta,\varphi)(T_{k_2l_2n_2})_{ij}(r,\theta,\varphi)\right)r^2\sin\theta d\theta\, d\varphi\, dr.
\end{aligned}$$

For brevity, we denote

$$\begin{aligned}
&A_\alpha := A_{ij}(k_\alpha,l_\alpha), && B_\alpha := B_{ij}(k_\alpha,l_\alpha),\\
&C_\alpha := C_{ij}(k_\alpha,l_\alpha), && F_\alpha := (1-r^2)^2P_{n_\alpha}^{(k_\alpha+3.5,k_\alpha+1.5)}(r^2), \qquad \alpha=1,2.
\end{aligned}$$

Further we omit the summation indices (ij), but the summation is implied. Also we introduce the notation $k = k_1 + k_2$. Then

$$
\begin{aligned}
J = &\int_0^1 \int_0^{2\pi} \int_0^{\pi} \left(A_1 r^{k_1+2} F_1'' + B_1 r^{k_1} F_1' + C_1 r^{k_1-2} F_1\right) \\
&\qquad \cdot \left(A_2 r^{k_2+2} F_2'' + B_2 r^{k_2} F_2' + C_2 r^{k_2-2} F_2\right) r^2 \sin\theta d\theta\, d\varphi\, dr \\
= &\frac{1}{2} \int_0^1 \int_0^{2\pi} \int_0^{\pi} \left(A_1 A_2 r^{k+5} F_1'' F_2'' + B_1 A_2 r^{k+3} F_1' F_2'' + C_1 A_2 r^{k+1} F_1 F_2''\right. \\
&\qquad + A_1 B_2 r^{k+3} F_1'' F_2' + B_1 B_2 r^{k+1} F_1' F_2' + C_1 B_2 r^{k-1} F_1 F_2' \\
&\qquad \left. + A_1 C_2 r^{k+1} F_1'' F_2 + B_1 C_2 r^{k-1} F_1' F_2 + C_1 C_2 r^{k-3} F_1 F_2\right) \sin\theta d\theta\, d\varphi\, dr^2.
\end{aligned}
$$

Consider separately the integrals over r^2

$$
\begin{aligned}
&\int_0^1 B_1 A_2 r^{k+3} F_1' F_2''\, dr^2 = \int_0^1 B_1 A_2 r^{k+3} F_1' (F_2')'\, dr^2 \\
=& B_1 A_2 r^{k+3} F_1' F_2' \Big|_0^1 - \int_0^1 \left(B_1 A_2 \frac{k+3}{2} r^{k+1} F_1 F_2' + B_1 A_2 r^{k+3} F_1'' F_2'\right) dr^2 \\
=& \int_0^1 \left(-B_1 A_2 \frac{k+3}{2} r^{k+1} F_1' F_2' - B_1 A_2 r^{k+3} F_1'' F_2'\right) dr^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_0^1 C_1 A_2 r^{k+1} F_1 F_2''\, dr^2 &= \int_0^1 \left(-C_1 A_2 \frac{k+1}{2} r^{k-1} F_1 F_2' - C_1 A_2 r^{k+1} F_1' F_2'\right) dr^2, \\
\int_0^1 A_1 C_2 r^{k+1} F_1'' F_2\, dr^2 &= \int_0^1 \left(-A_1 C_2 \frac{k+1}{2} r^{k-1} F_1' F_2 - A_1 C_2 r^{k+1} F_1' F_2'\right) dr^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
J = \int_0^1 \int_0^{2\pi} \int_0^{\pi} \Bigg(&A_1 A_2 r^{k+5} F_1'' F_2'' + (A_1 B_2 - B_1 A_2) r^{k+3} F_1'' F_2' \\
&+ \left(C_1 B_2 - C_1 A_2 \frac{k+1}{2}\right) r^{k-1} F_1 F_2' + \left(B_1 C_2 - A_1 C_2 \frac{k+1}{2}\right) r^{k-1} F_1' F_2 \\
&+ \left(B_1 B_2 - C_1 A_2 - A_1 C_2 - B_1 A_2 \frac{k+3}{2}\right) r^{k+1} F_1' F_2' \\
&+ C_1 C_2 r^{k-3} F_1 F_2 \Bigg) \sin\theta d\theta\, d\varphi\, dr^2.
\end{aligned}
$$

Since

$$\int_0^1 r^{k-1}F_1F_2'\,dr^2 = r^{k-1}F_1F_2\Big|_0^1 - \int_0^1 \left(\frac{k-1}{2}r^{k-3}F_1F_2 + r^{k-1}F_1'F_2\right)\,dr^2$$
$$= \int_0^1 \left(-\frac{k-1}{2}r^{k-3}F_1F_2 - r^{k-1}F_1'F_2\right)\,dr^2,$$

we obtain

$$J = \int_0^1\int_0^{2\pi}\int_0^{\pi}\Bigg(A_1A_2r^{k+5}F_1''F_2'' + (A_1B_2 - B_1A_2)r^{k+3}F_1''F_2'$$
$$+ \left(B_1B_2 - C_1A_2 - A_1C_2 - B_1A_2\frac{k+3}{2}\right)r^{k+1}F_1'F_2'$$
$$+ \left(B_1C_2 - A_1C_2\frac{k+1}{2} - C_1B_2 + C_1A_2\frac{k+1}{2}\right)r^{k-1}F_1'F_2$$
$$+ \left(C_1C_2 - C_1B_2\frac{k-1}{2} + C_1A_2\frac{k^2-1}{4}\right)r^{k-3}F_1F_2\Bigg)\sin\theta d\theta\,d\varphi\,dr^2.$$

Directly calculating the value of A_1A_2, we obtain $A_1A_2 = 16Y_{k_1l_1}Y_{k_2l_2}$. Then

$$\int_0^1\int_0^{2\pi}\int_0^{\pi} A_1A_2r^{k+5}F_1''F_2''\sin\theta d\theta\,d\varphi dr^2 = 16\int_0^1 r^{k+5}F_1''F_2''dr^2\int_0^{2\pi}\int_0^{\pi} Y_{k_1l_1}Y_{k_2l_2}\sin\theta d\theta d\varphi = 0$$

due to orthogonality of the spherical functions Y_{kl} when $k_1 \neq k_2$ or $l_1 \neq l_2$. Similarly, calculating the value A_1B_2, A_2B_1, A_2C_1 and A_1C_2, we obtain

$$A_1B_2 = 16k_2Y_{k_1l_1}Y_{k_2l_2}, \qquad A_1C_2 = 4k_2(k_2-1)Y_{k_1l_1}Y_{k_2l_2},$$
$$A_2B_1 = 16k_1Y_{k_1l_1}Y_{k_2l_2}, \qquad A_2C_1 = 4k_1(k_1-1)Y_{k_1l_1}Y_{k_2l_2}.$$

Hence,

$$\int_0^{2\pi}\int_0^{\pi} A_1B_2\sin\theta d\theta\,d\varphi = \int_0^{2\pi}\int_0^{\pi} A_2B_1\sin\theta d\theta\,d\varphi = \int_0^{2\pi}\int_0^{\pi} A_1C_2\sin\theta d\theta\,d\varphi$$
$$= \int_0^{2\pi}\int_0^{\pi} A_2C_1\sin\theta d\theta\,d\varphi = 0$$

and

$$J = \int_0^1\int_0^{2\pi}\int_0^{\pi}\Big(B_1B_2r^{k+1}F_1'F_2' + (B_1C_2 - C_1B_2)\,r^{k-1}F_1'F_2$$
$$+ \left(C_1C_2 - C_1B_2\frac{k-1}{2}\right)r^{k-3}F_1F_2\Big)\sin\theta d\theta\,d\varphi\,dr^2.$$

Calculating C_1B_2, we obtain

$$\begin{aligned}C_1B_2 &= (4k_1k_2(k_2-1)+2k_2(k_2+1))Y_{k_1l_1}Y_{k_2l_2} + 2(1+\operatorname{ctg}^2\theta)Y''_{k_1l_1,\varphi\varphi}Y_{k_2l_2}\\ &+ 2\operatorname{ctg}\theta Y'_{k_1l_1,\theta}Y_{k_2l_2} + 2Y''_{k_1l_1,\theta\theta}Y_{k_2l_2} + 4(k_1-1)(1+\operatorname{ctg}^2\theta)Y'_{k_1l_1,\varphi}Y'_{k_2l_2,\varphi}\\ &+ 4(k_1-1)Y'_{k_1l_1,\theta}Y'_{k_2l_2,\theta}.\end{aligned}$$

Integrating the first summand $(4k_1k_2(k_2-1)+2k_2(k_2+1))Y_{k_1l_1}Y_{k_2l_2}$ over θ and φ, we obtain 0 due to orthogonality of spherical functions. We have $Y_{kl} = L_{kl}(\cos\theta)\cos l\varphi$, and for $l_1 \neq l_2$ the following equality holds

$$\int_0^{2\pi} \cos l_1\varphi \cos l_2\varphi\, d\varphi = \int_0^{2\pi} \sin l_1\varphi \sin l_2\varphi\, d\varphi = \int_0^{2\pi} \sin l_1\varphi \cos l_2\varphi\, d\varphi = 0, \tag{6}$$

hence, for the case $l_1 \neq l_2$, we have that $C_1B_2 = 0$. Further, considering C_1B_2, we assume that $l_1 = l_2 = l$.

Lemma 4 *The following equality holds:*

$$\int_0^{2\pi}\int_0^{\pi} (Y'_{k_1l,\theta}Y'_{k_2l,\theta} + (1+\operatorname{ctg}^2\theta)Y'_{k_1l,\varphi}Y'_{k_2l,\varphi})\sin\theta d\theta\, d\varphi = 0.$$

Proof. Representing the spherical functions in terms of associated Legendre polynomials, integrating over φ, and substituting $t = \cos\theta$, we obtain

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\pi} (Y'_{k_1l,\theta}Y'_{k_2l,\theta} + (1+\operatorname{ctg}^2\theta)Y'_{k_1l,\varphi}Y'_{k_2l,\varphi})\sin\theta d\theta\, d\varphi\\ &= -\pi\int_{-1}^{1}\left(L'_{k_1l}(t)L'_{k_2l}(t)(1-t^2) + L_{k_1l}(t)L_{k_2l}(t)\frac{l^2}{1-t^2}\right)dt\\ &= -\pi L_{k_1l}(t)L'_{k_2l}(t)(1-t^2)\Big|_{-1}^{1}\\ &\quad -\pi\int_{-1}^{1}\left(-L_{k_1l}(t)\left(L'_{k_2l}(t)(1-t^2)\right)' + L_{k_1l}(t)L_{k_2l}(t)\frac{l^2}{1-t^2}\right)dt\\ &\overset{(5)}{=} -\pi\int_{-1}^{1} L_{k_1l}(t)\left(-\left(L'_{k_2l}(t)(1-t^2)\right)' + L_{k_2l}(t)\frac{l^2}{1-t^2}\right)dt\\ &\overset{(3)}{=} -\pi\int_{-1}^{1} L_{k_1l}(t)\left(k_2(k_2+1)L_{k_2l}(t)\right)dt = 0\end{aligned}$$

due to orthogonality of associated Legendre polynomials. □

Using the result of Lemma 4, we obtain

$$C_1B_2 = 2(1+\operatorname{ctg}^2\theta)Y''_{k_1l_1,\varphi\varphi}Y_{k_2l_2} + 2\operatorname{ctg}\theta Y'_{k_1l_1,\theta}Y_{k_2l_2} + 2Y''_{k_1l_1,\theta\theta}Y_{k_2l_2}.$$

We represent the spherical functions in terms of associated Legendre polynomials, integrate over φ, and substitute $t = \cos\theta$:

$$\int_0^{2\pi}\int_0^{\pi} C_1 B_2 \sin\theta d\theta\, d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(2(1+\operatorname{ctg}^2\theta) Y''_{k_1 l_1,\varphi\varphi} Y_{k_2 l_2} + 2\operatorname{ctg}\theta Y'_{k_1 l_1,\theta} Y_{k_2 l_2} + 2 Y''_{k_1 l_1,\theta\theta} Y_{k_2 l_2}\right) \sin\theta d\theta\, d\varphi$$

$$= -\pi \int_{-1}^{1} \left(-\frac{2l^2}{1-t^2} L_{k_1 l} L_{k_2 l} - 2t L'_{k_1 l} L_{k_2 l} + 2L''_{k_1 l} L_{k_2 l}(1-t^2) - 2t L'_{k_1 l} L_{k_2 l} \right) dt$$

$$= -\pi \int_{-1}^{1} 2L_{k_2 l} \left(-\frac{l^2}{1-t^2} L_{k_1 l} - 2t L'_{k_1 l} + L''_{k_1 l}(1-t^2) \right) dt$$

$$= -\pi \int_{-1}^{1} 2L_{k_2 l} \left(-\frac{l^2}{1-t^2} L_{k_1 l} + (L'_{k_1 l}(1-t^2))' \right) dt$$

$$\overset{(3)}{=} -\pi \int_{-1}^{1} 2L_{k_2 l} \left(-k_1(k_1+1) L_{k_1 l}\right) dt = 0$$

due to orthogonality of associated Legendre polynomials. Similarly, for $l_1 \neq l_2$ or $k_1 \neq k_2$

$$\int_0^{2\pi}\int_0^{\pi} B_1 C_2 \sin\theta d\theta\, d\varphi = 0,$$

and therefore, we have

$$J = \int_0^{1}\int_0^{2\pi}\int_0^{\pi} \left(B_1 B_2 r^{k+1} F'_1 F'_2 + C_1 C_2 r^{k-3} F_1 F_2\right) \sin\theta d\theta\, d\varphi\, dr^2.$$

It is easy to prove that

$$B_1 B_2 = (16k_1k_2 + 8(k_1+k_2) + 12) Y_{k_1 l_1} Y_{k_2 l_2} + 8(1+\operatorname{ctg}^2\theta) Y'_{k_1 l_1,\varphi} Y'_{k_2 l_2,\varphi} + 8 Y'_{k_1 l_1,\theta} Y'_{k_2 l_2,\theta}.$$

Due to orthogonality of spherical functions and Lemma 4,

$$\int_0^{2\pi}\int_0^{\pi} B_1 B_2 \sin\theta d\theta\, d\varphi = 0.$$

It remains to prove that

$$\int_0^{2\pi}\int_0^{\pi} C_1 C_2 \sin\theta d\theta\, d\varphi = 0.$$

Calculating the value of C_1C_2, we obtain

$$\begin{aligned}
C_1C_2 &= k_1k_2(k_1k_2 - k_1 - k_2 + 3)Y_{k_1l_1}Y_{k_2l_2} \\
&+ 2(\operatorname{ctg}^2\theta + 1)(\operatorname{ctg}^2\theta + k_1k_2 - k_1 - k_2 + 1)Y'_{k_1l_1,\varphi}Y'_{k_2l_2,\varphi} \\
&+ (2(k_1k_2 - k_1 - k_2 + 1) + \operatorname{ctg}^2\theta)Y'_{k_1l_1,\theta}Y'_{k_2l_2,\theta} + (\operatorname{ctg}^2\theta + 1)^2 Y''_{k_1l_1,\varphi\varphi}Y''_{k_2l_2,\varphi\varphi} \\
&+ 2(\operatorname{ctg}^2\theta + 1)Y''_{k_1l_1,\varphi\theta}Y''_{k_2l_2,\varphi\theta} + Y''_{k_1l_1,\theta\theta}Y''_{k_2l_2,\theta\theta} + k_1\operatorname{ctg}\theta\, Y_{k_1l_1}Y'_{k_2l_2,\theta} \\
&+ k_2\operatorname{ctg}\theta\, Y'_{k_1l_1,\theta}Y_{k_2l_2} + k_1(\operatorname{ctg}^2\theta + 1)Y_{k_1l_1}Y''_{k_2l_2,\varphi\varphi} \\
&+ k_2(\operatorname{ctg}^2\theta + 1)Y''_{k_1l_1,\varphi\varphi}Y_{k_2l_2} + k_1Y_{k_1l_1}Y''_{k_2l_2,\theta\theta} + k_2Y''_{k_1l_1,\theta\theta}Y_{k_2l_2} \\
&+ (-2\operatorname{ctg}\theta(\operatorname{ctg}^2\theta + 1))Y'_{k_1l_1,\varphi}Y''_{k_2l_2,\varphi\theta} + (-2\operatorname{ctg}\theta(\operatorname{ctg}^2\theta + 1))Y''_{k_1l_1,\varphi\theta}Y'_{k_2l_2,\varphi} \\
&+ \operatorname{ctg}\theta(\operatorname{ctg}^2\theta + 1)Y''_{k_1l_1,\varphi\varphi}Y'_{k_2l_2,\theta} + \operatorname{ctg}\theta(\operatorname{ctg}^2\theta + 1)Y'_{k_1l_1,\theta}Y''_{k_2l_2,\varphi\varphi}.
\end{aligned}$$

Obviously, when $l_1 \neq l_2$, due to (6),

$$\int_0^{2\pi}\int_0^{\pi} C_1C_2 \sin\theta d\theta\, d\varphi = 0.$$

Next, we put $l_1 = l_2 = l$ and $k_1 \neq k_2$. Due to orthogonality of spherical functions and using Lemma 4, we obtain

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi}\Big(k_1k_2(k_1k_2 - k_1 - k_2 + 3)Y_{k_1l}Y_{k_2l} \\
&\qquad + 2(\operatorname{ctg}^2\theta + 1)(\operatorname{ctg}^2\theta + k_1k_2 - k_1 - k_2 + 1)Y'_{k_1l,\varphi}Y'_{k_2l,\varphi} \\
&\qquad + (2(k_1k_2 - k_1 - k_2 + 1) + \operatorname{ctg}^2\theta)Y'_{k_1l,\theta}Y'_{k_2l,\theta}\Big)\sin\theta d\theta\, d\varphi \\
&= \int_0^{2\pi}\int_0^{\pi}\big(2\operatorname{ctg}^2\theta(\operatorname{ctg}^2\theta + 1)Y'_{k_1l,\varphi}Y'_{k_2l,\varphi} + \operatorname{ctg}^2\theta\, Y'_{k_1l,\theta}Y'_{k_2l,\theta}\big)\sin\theta d\theta\, d\varphi.
\end{aligned}$$

Representing the spherical functions in terms of associated Legendre polynomials, we substitute $t = \cos\theta$ and integrate over φ:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi} C_1C_2 \sin\theta d\theta\, d\varphi \\
&= -\pi\int_{-1}^{1}\Big(\frac{2l^2t^2}{(1-t^2)^2}L_{k_1l}L_{k_2l} + t^2L'_{k_1l}L'_{k_2l} + \frac{l^4}{(1-t^2)^2}L_{k_1l}L_{k_2l} + 2l^2L'_{k_1l}L'_{k_2l} \\
&\qquad + (1-t^2)^2L''_{k_1l}L''_{k_2l} - t(1-t^2)L'_{k_1l}L''_{k_2l} - t(1-t^2)L''_{k_1l}L'_{k_2l} + t^2L'_{k_1l}L'_{k_2l} \\
&\qquad - k_1tL_{k_1l}L'_{k_2l} - k_2tL'_{k_1l}L_{k_2l} - \frac{k_1l^2}{1-t^2}L_{k_1l}L_{k_2l} - \frac{k_2l^2}{1-t^2}L_{k_1l}L_{k_2l} \\
&\qquad + k_1(1-t^2)L_{k_1l}L''_{k_2l} - k_1tL_{k_1l}L'_{k_2l} + k_2(1-t^2)L''_{k_1l}L_{k_2l} - k_2tL'_{k_1l}L_{k_2l} \\
&\qquad + \frac{2l^2t}{1-t^2}L_{k_1l}L'_{k_2l} + \frac{2l^2t}{1-t^2}L'_{k_1l}L_{k_2l} + \frac{l^2t}{1-t^2}L_{k_1l}L'_{k_2l} + \frac{l^2t}{1-t^2}L'_{k_1l}L_{k_2l}\Big)\, dt.
\end{aligned}$$

Grouping the summands with a multiplier k_1 and using the Legendre equation, we obtain

$$\int_{-1}^{1} k_1 L_{k_1 l} \left((1-t^2) L''_{k_2 l} - 2t L'_{k_2 l} - \frac{l^2}{1-t^2} L_{k_2 l} \right) dt$$
$$= \int_{-1}^{1} k_1 L_{k_1 l} \left(-k_2(k_2+1) L_{k_2 l} \right) dt = 0$$

due to orthogonality of associated Legendre polynomials. Similarly, the integral over the summands with a factor k_2 equals zero.

The following equality holds:

$$\int_{-1}^{1} \left((1-t^2) L''_{k_1 l} - 2t L'_{k_1 l} - \frac{l^2}{1-t^2} L_{k_1 l} \right) \left((1-t^2) L''_{k_2 l} - 2t L'_{k_2 l} - \frac{l^2}{1-t^2} L_{k_2 l} \right) dt$$
$$\overset{(3)}{=} \int_{-1}^{1} k_1(k_1+1) k_2(k_2+1) L_{k_1 l} L_{k_2 l} \, dt = 0.$$

Therefore, subtracting the integrand of the first integral from $C_1 C_2$, we obtain

$$\int_0^{2\pi} \int_0^{\pi} C_1 C_2 \sin\theta d\theta \, d\varphi$$
$$= -\pi \int_{-1}^{1} \Big(\frac{2l^2 t^2}{(1-t^2)^2} L_{k_1 l} L_{k_2 l} - 2t^2 L'_{k_1 l} L'_{k_2 l} + 2l^2 L'_{k_1 l} L'_{k_2 l} + t(1-t^2) L'_{k_1 l} L''_{k_2 l}$$
$$+ t(1-t^2) L''_{k_1 l} L'_{k_2 l} + \frac{l^2 t}{1-t^2} L_{k_1 l} L'_{k_2 l} + \frac{l^2 t}{1-t^2} L'_{k_1 l} L_{k_2 l}$$
$$+ l^2 L''_{k_1 l} L_{k_2 l} + l^2 L_{k_1 l} L''_{k_2 l} \Big) dt$$
$$= -\pi \int_{-1}^{1} \left(\frac{2l^2 t^2}{(1-t^2)^2} L_{k_1 l} L_{k_2 l} - 2t^2 L'_{k_1 l} L'_{k_2 l} + l^2 (L_{k_1 l} L_{k_2 l})'' \right.$$
$$\left. + t(1-t^2)(L'_{k_1 l} L'_{k_2 l})' + \frac{l^2 t}{1-t^2} (L_{k_1 l} L_{k_2 l})' \right) dt$$
$$= -\pi \left(l^2 (L_{k_1 l} L_{k_2 l})' + t(1-t^2) L'_{k_1 l} L'_{k_2 l} + \frac{l^2 t}{1-t^2} L_{k_1 l} L_{k_2 l} \right) \Big|_{-1}^{1}$$
$$-\pi \int_{-1}^{1} \left(\frac{2l^2 t^2}{(1-t^2)^2} L_{k_1 l} L_{k_2 l} - 2t^2 L'_{k_1 l} L'_{k_2 l} - (t(1-t^2))' L'_{k_1 l} L'_{k_2 l} \right.$$
$$\left. - \left(\frac{l^2 t}{1-t^2} \right)' L_{k_1 l} L_{k_2 l} \right) dt$$

$$= -\pi\left(l^2(L_{k_1 l}L_{k_2 l})' + t(1-t^2)L'_{k_1 l}L'_{k_2 l} + \frac{l^2 t}{1-t^2}L_{k_1 l}L_{k_2 l}\right)\Bigg|_{-1}^{1}$$
$$-\pi\int_{-1}^{1}\left(-\frac{l^2}{(1-t^2)}L_{k_1 l}L_{k_2 l} - (1-t^2)L'_{k_1 l}L'_{k_2 l}\right)dt.$$

We calculate separately the final integral

$$\int_{-1}^{1}\left(-\frac{l^2}{(1-t^2)}L_{k_1 l}L_{k_2 l} - (1-t^2)L'_{k_1 l}L'_{k_2 l}\right)dt$$
$$= -(1-t^2)L'_{k_1 l}L_{k_2 l}\Big|_{-1}^{1} + \int_{-1}^{1}\left(-\frac{l^2}{(1-t^2)}L_{k_1 l}L_{k_2 l} + \left((1-t^2)L'_{k_1 l}\right)' L_{k_2 l}\right)dt$$
$$\overset{(5),(3)}{=}\int_{-1}^{1}\left(-\frac{l^2}{(1-t^2)}L_{k_1 l}L_{k_2 l} + \left(-k_1(k_1+1)L_{k_1 l} + \frac{l^2}{1-t^2}L_{k_1 l}\right)L_{k_2 l}\right)dt$$
$$= \int_{-1}^{1} -k_1(k_1+1)L_{k_1 l}L_{k_2 l}\,dt = 0.$$

We have obtained that

$$\int_0^{2\pi}\int_0^{\pi} C_1 C_2 \sin\theta d\theta\, d\varphi = -\pi\left(l^2(L_{k_1 l}L_{k_2 l})' + t(1-t^2)L'_{k_1 l}L'_{k_2 l} + \frac{l^2 t}{1-t^2}L_{k_1 l}L_{k_2 l}\right)\Bigg|_{-1}^{1}.$$

If $l = 0$, then due to that fact that derivatives of Legendre polynomials do not have singularities given $t = \pm 1$, obviously, we have that

$$\int_0^{2\pi}\int_0^{\pi} C_1 C_2 \sin\theta d\theta\, d\varphi = 0.$$

If $l = 1$, then taking into account that

$$L_{k1} = \sqrt{1-t^2}L'_k, \qquad L'_{k1} = \frac{-t}{\sqrt{1-t^2}}L'_k + (1-t^2)^{1/2}L''_k,$$

we have

$$\int_0^{2\pi}\int_0^{\pi} C_1 C_2 \sin\theta d\theta\, d\varphi$$
$$= -\pi\Bigg(((1-t^2)L'_{k_1}L'_{k_2})' + t(1-t^2)\left(\frac{-t}{\sqrt{1-t^2}}L'_{k_1} + \sqrt{1-t^2}L''_{k_1}\right)$$
$$\cdot\left(\frac{-t}{\sqrt{1-t^2}}L'_{k_2} + (1-t^2)^{1/2}L''_{k_2}\right) + \frac{t}{1-t^2}(1-t^2)L'_{k_1}L'_{k_2}\Bigg)\Bigg|_{-1}^{1}$$
$$= -\pi\left(-2tL'_{k_1}L'_{k_2} + t^3 L'_{k_1}L'_{k_2} + tL'_{k_1}L'_{k_2}\right)\Big|_{-1}^{1} = 0.$$

If $l \geqslant 2$, then every summand has a factor $(1-t^2)$, therefore, the whole sum for $t = \pm 1$ equals zero.

Hence, the orthogonality in the case when $l_1 \neq l_2$ or $k_1 \neq k_2$ is proved.

Calculating the norm of the basic fields.

To construct an orthonormal basis in the original space, it remains to calculate the norm of the basic fields

$$\|\mathbf{T}_{kln}\|^2 = \int_0^1 \int_0^{2\pi} \int_0^{\pi} \left(\sum_{i,j=1}^{3} (T_{kln})^2_{ij}(r,\theta,\varphi) \right) r^2 \sin\theta d\theta\, d\varphi\, dr.$$

Using Lemma 1, the formulas

$$\|P_n\|^2 = \int_0^1 r^{2k+1}(1-r^2)^2 P_n P_n\, dr^2, \qquad \|P'_n\|^2 = \int_0^1 r^{2k+3}(1-r^2)^3 P'_n P'_n\, dr^2,$$

$$\|P''_n\|^2 = \int_0^1 r^{2k+5}(1-r^2)^4 P''_n P''_n\, dr^2, \qquad \|Y_{kl}\|^2 = \int_0^{2\pi} \int_0^{\pi} Y_{kl} Y_{kl} \sin\theta\, d\theta\, d\varphi,$$

and transforms similar to the ones used when proving orthogonality, we obtain

$$\|\mathbf{T}_{kln}\|^2 = 16(k+1.5)(k+2.5)\|P_n\|^2\|Y_{kl}\|^2 + 32(k+2.5)\|P'_n\|^2\|Y_{kl}\|^2 + 8\|P''_n\|^2\|Y_{kl}\|^2.$$

Using the formulas for the norm (1) and the derivatives (2) of Jacobi polynomials, it is easy to show that

$$\|P_n\|^2 = \frac{n!\,(\Gamma(k+1.5))^2\Gamma(n+3)}{\Gamma(k+n+1.5)\Gamma(k+n+3.5)(k+2n+3.5)},$$

$$\|P'_n\|^2 = \frac{n(n+k+3.5)^2 n!\,(\Gamma(k+2.5))^2\Gamma(n+3)}{(k+1.5)^2\Gamma(k+n+1.5)\Gamma(k+n+4.5)(k+2n+3.5)},$$

$$\|P''_n\|^2 = \frac{n(n-1)(n+k+3.5)^2(n+k+4.5)^2 n!\,(\Gamma(k+3.5))^2\Gamma(n+3)}{(k+1.5)^2(k+2.5)^2\Gamma(k+n+1.5)\Gamma(k+n+4.5)(k+2n+5.5)}.$$

Then

$$\|\mathbf{T}_{kln}\|^2 = \frac{8((n+2)!)^2(\Gamma(k+1.5))^2\|Y_{kl}\|^2}{(\Gamma(n+k+1.5))^2(k+2n+3.5)}.$$

Therefore, Theorem 1 is proved.

5. Conclusion

In this paper, a singular value decomposition of the normal Radon transform operator, acting on a three-dimensional symmetric 2-tensor field, is constructed.

References

[1] V.A. Sharafutdinov, *Integral geometry for tensor fields*, VSP, Utrecht, 1994. Zbl 0883.53004

[2] E.Yu. Derevtsov, I.E. Svetov, *Tomography of tensor fields in the plain*, Eurasian Journal of Mathematical and Computer Applications, **3**:2 (2015), 24–68.

[3] M. Defrise, G.T. Gullberg, *3D reconstruction of tensors and vectors*, LBNL, Berkeley, 2005. (Technical Report в„– LBNL-54936)

[4] I.E. Svetov, *The method of approximate inverse for the Radon transform operator acting on functions and for the normal Radon transform operators acting on vector and symmetric 2-tensor fields in* $\mathbb{R}^3$, Sib. Électron. Mat. Izv., **17** (2020), 1073–1087. Zbl 1461.44002

[5] A.M. Cormack, *Representation of a function by its line integrals, with some radiological applications. I*, J. Appl. Phys., **34** (1963), 2722–2727. Zbl 0117.32303

[6] A.M. Cormack, *Representation of a function by its line integrals, with some radiological applications. II*, J. Appl. Phys., **35** (1964), 2908–2913. Zbl 0122.18401

[7] M. Davison, *A singular value decomposition for the Radon transform in n-dimensional Euclidean space*, Numer. Funct. Anal. Optim., **3** (1981), 231–240. Zbl 0467.65069

[8] E.T. Quinto, *Singular value decomposition and inversion methods for the exterior Radon transform and a spherical transform*, J. Math. Anal. Appl., **95**:2 (1983), 437–448. Zbl 0569.44005

[9] A.K. Louis, *Orthogonal function series expansions and the null space of the Radon transform*, SIAM J. Math. Anal., **15**:3 (1984), 621–633. Zbl 0533.42018

[10] P. Maass, *Singular value decomposition for Radon transform*, in G.T. Herman, A.K. Louis, F. Natterer, (eds), *Mathematical Methods in Tomography*, Springer-Verlag, Berlin, Heidelberg, 1990, 6–14.

[11] P. Maass, *The x-ray transform: singular value decomposition and resolution*, Inverse Probl., **3**:4 (1987), 729–741. Zbl 0636.44003

[12] E.Yu. Derevtsov, A.V. Efimov, A.K. Louis, T. Schuster, *Singular value decomposition and its application to numerical inversion for ray transforms in 2D vector tomography*, J. Inverse Ill-Posed Problems, **19**:4-5 (2011), 689–715. Zbl 1279.33015

[13] E.Yu. Derevtsov, A.P. Polyakova, *An application of the SVD-method to a problem of integral geometry of 2-tensor fields*, J. Math. Sci., New York, **202**:1 (2014), 50–71. Zbl 1289.53028

[14] A. Polyakova, *Reconstruction of potential part of 3D vector field by using singular value decomposition*, J. Physics: Conference Series, **410** (2013), 012015.

[15] A.P. Polyakova, *Reconstruction of a vector field in a ball from its normal Radon transform*, J. Math. Sci., New York, **205**:3 (2015), 418–439. Zbl 1349.65731

[16] I.E. Svetov, A.P. Polyakova, *Comparison of two algorithms for the numerical solution of two-dimensional vector tomography*, Sib. Électron. Math. Izv., **10** (2013), 90–108. Zbl 1330.65208

[17] I.E. Svetov, A.P. Polyakova, *Approximate solution of two-dimensional 2-tensor tomography problem using truncated singular value decomposition*, Sib. Électron. Math. Izv., **12** (2015), 480–499. Zbl 1342.44004

[18] A.P. Polyakova, I.E. Svetov, *Numerical solution of the problem of reconstructing a potential vector field in the unit ball from its normal Radon transform*, J. Appl. Ind. Math., **9**:4 (2015), 547–558. Zbl 1349.65702

[19] A.P. Polyakova, I.E. Svetov, *Numerical solution of reconstruction problem of a potential symmetric 2-tensor field in a ball from its normal Radon transform*, Sib. Électron. Math. Izv., **13** (2016), 154–174. Zbl 1342.44003

[20] S.G. Kazantsev, A.A. Bukhgeim, *Singular value decomposition for 2D fan-beam Radon transform of tensor fields*, J. Inverse Ill-Posed Probl., **12**:3 (2004), 245–278. Zbl 1058.65143

[21] F. Natterer, *The mathematics of computerized tomography*, Teubner Verlag, Stuttgart, 1986. Zbl 0617.92001

Anna Petrovna Polyakova
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: anna.polyakova@ngs.ru