S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 18, №2, стр. 1596–1614 (2021) УДК 517.9
DOI 10.33048/semi.2021.18.118 MSC 34A08, 26A33, 34B18

POSITIVE SOLUTIONS OF p-LAPLACIAN FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVE BOUNDARY CONDITION

F. HADDOUCHI

Abstract. In this paper, we show some results about the existence and uniqueness of the positive solution for a p-Laplacian fractional differential equations with fractional derivative boundary condition. Our results are based on Krasnosel'skii's fixed point theorem, the nonlinear alternative of Leray-Schauder type and contraction mapping principle. Three examples are given to illustrate the applicability of our main results.

Keywords: Caputo fractional differential equations, p-Laplacian operator, positive solutions, fixed-point theorem, existence, cone.

1. Introduction

Fractional derivatives and integrals are proved to be more useful in the modeling of different physical and natural phenomena. The p-Laplacian fractional boundary value problems related to nonlocal conditions have many applications in various fields, such as in the theory of heat conduction in materials with memory, nonlinear elasticity and glaciology, non-Newtonian mechanics, combustion theory, population biology, nonlinear flow laws and so on. For instance, when studying the steady-state turbulent flow with reaction, Bobisud (cf.[1]) introduced the differential equation

$$\big(\varphi_p\big(u'(t)\big)\big)' = f(t, u(t), u'(t))$$

with an operator $\varphi_p(x) = |x|^{p-1}x$. This problem appears in the study of non-Newtonian fluids. It yields the usual problem for diffusion in a porous medium when $p = 1$, i.e., $\varphi_p(x) = x$. The current analysis of these problems has a great interest and many methods are used to solve such problems. Recently, the study

Haddouchi, F., Positive solutions of p-Laplacian fractional differential equations with fractional derivative boundary condition.

Received October, 19, 2019, published December, 9, 2021.

of existence of positive solution to p-Laplacian fractional boundary value problems has gained much attention and is rapidly growing field. There are many papers concerning fractional differential equations with the p-Laplacian operator, see [8, 10, 16, 32, 17, 31, 33, 6, 28, 23].

In 2013, S. Ying et al. [29] considered the existence criteria for positive solutions of the nonlinear p-Laplacian fractional differential equation

$$\begin{cases} \Big(\varphi_p\big(D^{\alpha}u(t)\big)\Big)' = \varphi_p(\lambda)f(t,u(t),u'(t)),\ t\in(0,1),\\ k_0u(0)=k_1u(1),\ m_0u(0)=m_1u(1),\ u^{(r)}(0)=0,\ r=2,3,...,[\alpha], \end{cases}$$

where φ_p is the p-Laplacian operator, i.e., $\varphi_p(s)=|s|^{p-2}s$, $p>1$, and $\varphi_q=\varphi_p^{-1}$, $\frac{1}{p}+\frac{1}{q}=1$. D^{α} is the standard Caputo derivative and $f:[0,1]\times[0,\infty)\times(-\infty,\infty)\to[0,\infty)$ satisfies the Carathéodory type condition, $\alpha>2$ is real and $[\alpha]$ denotes the integer part of the real number α, $\lambda>0$, k_i,m_i $(i=0,1)$ are constants satisfying $0<k_1<k_0$ and $0<m_1<m_0$. They used the nonlinear alternative of Leray-Schauder type and the fixed-point theorems in Banach space to investigate the existence of at least single, twin, triple, n or $2n-1$ positive solutions.

In 2019, T. Xiaosong et al. [26] considered the following mixed fractional resonant boundary value problem with $p(t)$-Laplacian operator

$$\begin{cases} {}^{c}D^{\beta}\Big(\varphi_{p(t)}\big(D^{\alpha}u(t)\big)\Big) = f(t,u(t),D^{\alpha}u(t)),\ t\in[0,T],\\ t^{1-\alpha}u(t)\big|_{t=0}=0,\ D^{\alpha}u(0)=D^{\alpha}u(T), \end{cases}$$

where $0<\alpha,\beta\le 1$, $1<\alpha+\beta\le 2$, ${}^{c}D^{\beta}$ is Caputo fractional derivative and D^{α} is Riemann-Liouville fractional derivative, $\varphi_{p(t)}(.)$ is $p(t)$-Laplacian operator, $p(t)>1$, $p(t)\in C^1[0,T]$ with $p(0)=p(T)$, $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$. Under the appropriate conditions of the nonlinear term, the existence of solutions for the above mixed fractional resonant boundary value problem is obtained by using the continuation theorem of coincidence degree theory.

In 2019, Z. Li et al. [14] considered the existence of nontrivial solutions for a certain p-Laplacian fractional differential equation

$$\begin{cases} D^{\beta}\varphi_p\Big(\big(D^{\alpha}\big(p(t)u'(t)\big)\big)\Big) + f(t,u(t)) = 0,\ t\in(0,1),\\ au(0)-bp(0)u'(0)=0,\ cu(1)+dp(1)u'(1)=0,\ D^{\alpha}\big(p(t)u'(t)\big)\big|_{t=0}=0, \end{cases}$$

where a, b, c, d are constants with $0<ad+bc+ac\int_0^1\frac{1}{p(s)}ds<\infty$, $p(.):[0,1]\to(0,\infty)$ is continuous, and φ_p is the p-Laplacian operator, $0<\alpha,\beta<1$, D^{α} is the standard Caputo derivative. Under the assumption that $f(t,u)$ is a continuous function, and by the use of some fixed point theorems in cones, they studied the existence and uniqueness results.

In 2017, T. Yuansheng et al. [30] considered a class of four-point boundary value problem of fractional differential equations with p-Laplacian operator

$$\begin{cases} D^{\gamma}\Big(\varphi_p\big(D^{\alpha}u(t)\big)\Big) = f(t,u(t)),\ t\in(0,1),\\ u(0)=D^{\alpha}u(0)=0,\ D^{\beta}u(1)=\lambda u(\xi),\ D^{\alpha}u(1)=\mu D^{\alpha}u(\eta), \end{cases}$$

where $\alpha,\beta,\gamma\in\mathbb{R}$, $1<\alpha,\gamma\le 2$, $\beta>0$, $1+\beta\le\alpha$, and $\xi,\eta\in(0,1)$, $\lambda,\mu\in[0,\infty)$, $(1-\beta)\Gamma(\alpha)\ge\lambda\Gamma(\alpha-\beta)\xi^{\alpha-2}$, $1-\mu^{p-1}\eta^{\gamma-2}\ge 0$ and φ_p is the p-Laplacian operator, D^{ν} $(\nu\in\{\alpha,\beta,\gamma\})$ is the standard Riemann-Liouville differentiation and

$f \in C([0,1]\times[0,\infty)\to[0,\infty))$. By the use of the Leggett-Williams fixed-point theorem, the multiplicity results of positive solution are obtained.

And in the same year, L. Xiping et al. [27], studied the following four-point boundary value problem of fractional differential equation with mixed fractional derivatives and p-Laplacian operator

$$\begin{cases} D^{\alpha}\Big(\varphi_p\big({}^{c}D^{\beta}u(t)\big)\Big) = f\big(t,u(t),{}^{c}D^{\beta}u(t)\big),\ t\in(0,1],\\ {}^{c}D^{\beta}u(0) = u'(0) = 0,\\ u(1) = r_1u(\eta),\ {}^{c}D^{\beta}u(1) = r_2\ {}^{c}D^{\beta}u(\xi), \end{cases}$$

where $1<\alpha,\beta\le 2$, $r_1,r_2\ge 0$, D^{α} is the Riemann-Liouville fractional derivative operator, and ${}^{c}D^{\beta}$ is the Caputo fractional derivative operator, $p>1$, φ_p is the p-Laplacian operator and $f\in C([0,1]\times[0,+\infty)\times(-\infty,0],[0,+\infty))$. By means a method of lower and upper solutions, the authors established some new results on the existence of positive solutions.

In 2018, W. Han et al. [5] investigated the multiple positive solutions for the following nonlinear fractional differential equations coupled with the p-Laplacian operator and infinite-point boundary value conditions

$$\begin{cases} -D^{\beta}\Big(\varphi_p\big(D^{\alpha}u(t)-\lambda u(t)\big)\Big) = f(t,u(t)),\ t\in(0,1],\\ \lim_{t\to0^+} t^{1-\alpha}u(t) = \sum_{i=1}^{\infty}\mu_i u(\xi_i),\\ \lim_{t\to0^+} t^{2-\beta}\Big(\varphi_p\big(D^{\alpha}u(t)-\lambda u(t)\big)\Big) = \varphi_p\big(D^{\alpha}u(1)-\lambda u(1)\big) = 0, \end{cases}$$

where $0<\alpha\le 1$, $1<\beta\le 2$, $\lambda<0$, $\mu_i\ge 0$, $0<\xi_i<1$, $i\in\mathbb{N}^+$, $\sum_{i=1}^{\infty}\mu_i\xi_i^{\alpha-1}<1$, $f\in C([0,1]\times[0,\infty)\to[0,\infty))$, φ_p is the p-Laplacian operator and D^{α}, D^{β} are the Riemann-Liouville fractional derivatives. By means of the properties of Green's function and fixed point theorems, they established the suitable criteria to guarantee the existence of positive solutions.

In 2018, J. Tan, M. Li [24], studied the solutions for the following nonlinear fractional differential equations with p-Laplacian operator nonlocal boundary value problem in a Banach space E

$$\begin{cases} -D^{\beta}\Big(\varphi_p\big(D^{\alpha}u\big)\Big)(t) = f(t,u(t)),\ t\in(0,1),\\ u(0)=\theta,\ D^{\alpha}u(0)=\theta,\\ D^{\gamma}u(1) = \sum_{i=1}^{m-2}\alpha_i D^{\gamma}u(\xi_i), \end{cases}$$

where $D^{\alpha},D^{\beta},D^{\gamma}$ are the standard Riemann-Liouville fractional derivatives, θ is the zero element of E, $1<\alpha\le 2$, $0<\beta,\gamma\le 1$, $\alpha-\gamma\ge 1$, $I=[0,1]$, $f: I\times E\to E$ is continuous, $\alpha_i\ge 0$ $(i=1,2,...,m-2)$, $0<\xi_1<\xi_2<...<\xi_{m-2}<1$, $\sum_{i=1}^{m-2}\alpha_i\xi_i^{\alpha-\gamma-1}<1$, φ_p is the p-Laplacian operator. By means of the technique of the properties of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem, they established some new existence criteria.

For some other results on p-Laplacian fractional boundary value problems, we refer the reader to the papers [25, 18, 19, 22, 2, 7, 15, 11, 3, 21].

Motivated by these works, in this paper, we are concerned with the following three-point boundary value problem of fractional differential equations with p-Laplacian operator

$$\Big(\varphi_p\big(D^{\alpha}u(t)\big)\Big)' + a(t)f(t,u(t)) = 0,\ t\in(0,1), \tag{1.1}$$

$$D^{\alpha}u(0) = u'(0) = u''(0) = 0,\ u(1) + u'(1) = u'(\eta), \tag{1.2}$$

where $2 < \alpha \leq 3$, $\eta \in (0,1)$, φ_p is the p-Laplacian operator, D^{α} is the standard Caputo derivative.
Throughout this paper, we assume the following conditions:

(H1) $f \in C([0,1] \times [0,\infty), [0,\infty))$, and $f(t,.) \not\equiv 0$ on $[0,1]$;
(H2) $a \in C([0,1], [0,\infty))$ and $a(t) \not\equiv 0$ on any subinterval of $[0,1]$.

This paper is organized as follows. In section 2, we recall some special definitions, theorems and lemmas that will be used to prove our main results. In section 3, we discuss the existence and uniqueness of positive solution for (1.1)-(1.2). Finally, we give some examples to illustrate our results in section 4.

2. Preliminaries

In this section, we introduce first the following preliminary facts that will be used throughout this article.
At first, let $\mathcal{B}$ be the Banach space $C([0,1], \mathbb{R})$ when equipped with the usual supremum norm,

$$\|u\| = sup_{t\in[0,1]}|u(t)|.$$

Definition 1. *Let E be a real Banach space. A nonempty, closed, convex set $K \subset E$ is a cone if it satisfies the following two conditions:*

(i) *$x \in K$, $\lambda \geq 0$ imply $\lambda x \in K$;*
(ii) *$x \in K$, $-x \in K$ imply $x = 0$.*

Definition 2. *An operator $T : E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

Definition 3. *A function $u(t)$ is called a positive solution of (1.1) and (1.2) if $u \in C([0,1])$ and $u(t) > 0$ for all $t \in (0,1)$.*

Definition 4. *The Riemann-Liouville fractional integral of order α for a continuous function f is defined as*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds,\ \alpha > 0,$$

provided the integral exists, where $\Gamma(.)$ is the gamma function, which is defined by $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$.

Definition 5. *For at least n-times continuously differentiable function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as*

$${}^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds,\ n-1 < \alpha < n,\ n = [\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 1 ([12])**.** *For $\alpha > 0$, the general solution of the fractional differential equation ${}^{c}D^{\alpha}x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + ... + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$ $(n = [\alpha]+1)$.

According to Lemma 1, it follows that

$$I^{\alpha}\ {}^{c}D^{\alpha}x(t) = x(t) + c_0 + c_1 t + ... + c_{n-1}t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$ $(n = [\alpha] + 1)$.

Lemma 2 ([20], [12]). *If $\beta > \alpha > 0$ and $x \in L_1[0,1]$, then*
(i) ${}^{c}D^{\alpha}\ I^{\beta}x(t) = I^{\beta-\alpha}x(t)$, *holds almost everywhere on $[0,1]$ and it is valid at any point $t \in [0,1]$ if $x \in C[0,1]$;* ${}^{c}D^{\alpha}\ I^{\alpha}x(t) = x(t)$, *for all* $t \in [0,1]$.
(ii) ${}^{c}D^{\alpha}t^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)}t^{\lambda-\alpha-1}$, $\lambda > [\alpha]$ *and* ${}^{c}D^{\alpha}t^{\lambda-1} = 0$, $\lambda < [\alpha]$.

Definition 6. *Let $p > 0$, $q > 0$, the Euler beta function is defined by*

$$\mathfrak{B}(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt.$$

The basic properties of the Γ and $\mathfrak{B}$ functions which will be used in the following studies are listed below.

Proposition 1. *Let $\alpha > 0$, $p > 0$, $q > 0$ and n a positive integer. Then*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha),\ \Gamma\Big(n+\frac{1}{2}\Big) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}, \Gamma(n+1) = n!,$$

$$\mathfrak{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\ \mathfrak{B}(p,q) = \mathfrak{B}(p,q+1) + \mathfrak{B}(p+1,q),$$

$$\mathfrak{B}(p,q) = \mathfrak{B}(q,p),\ \mathfrak{B}(p+1,q) = \mathfrak{B}(p,q)\frac{p}{p+q}.$$

In particular,

$$\Gamma(1) = 1,\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},\ \mathfrak{B}(1,q) = \frac{1}{q}.$$

To prove our results, we need the following three well-known fixed point theorems.

Theorem 1. [13] *Let E be a Banach space, and let $K \subset E$, be a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$A : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$$

be a completely continuous operator such that

(a) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(b) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 2. [4] *Let E be a Banach space, E_1 a closed, convex subset of E, U an open subset of E_1 and $0 \in U$. Suppose that $A : \overline{U} \to E_1$ is a continuous, compact (that is $A(\overline{U})$ is a relatively compact subset of E_1) map. Then either*

(i) *A has a fixed point in $\overline{U}$, or*
(ii) *There is a $u \in \partial U$ (the boundary of U in E_1) and $\lambda \in (0,1)$ with $u = \lambda A(u)$.*

Theorem 3. *(Banach's fixed point theorem) Let (X,d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point u^* in X (i.e. $T(u^*) = u^*$).*

Consider the fractional boundary value problem

$$\Big(\varphi_p\big(D^{\alpha}u(t)\big)\Big)' + h(t) = 0,\ t \in (0,1), \tag{2.1}$$

$$D^{\alpha}u(0) = u'(0) = u''(0) = 0,\ u(1) + u'(1) = u'(\eta), \tag{2.2}$$

where $\alpha \in (2,3]$ and $h \in C[0,1]$.

Lemma 3. *Suppose that $h \in C([0,1],[0,\infty))$, then the boundary value problem (2.1)-(2.2) has a unique solution which can be expressed by*

$$u(t) = \int_0^1 \mathcal{K}(t,s)\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds,$$

where

$$\mathcal{K}(t,s) = \mathcal{G}(t,s) + \mathcal{H}(\eta,s), \tag{2.3}$$

$$\begin{aligned}
\mathcal{G}(t,s) &= \begin{cases} \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \\
\mathcal{H}(t,s) &= \begin{cases} \frac{(1-s)^{\alpha-2}-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le t \le s \le 1. \end{cases}
\end{aligned} \tag{2.4}$$

Proof. By integrating the equation (2.1), it follows that

$$\varphi_p\big(D^{\alpha}u(t)\big) - \varphi_p\big(D^{\alpha}u(0)\big) = -\int_0^t h(s)ds,$$

and so,

$$D^{\alpha}u(t) = -\varphi_q\bigg(\int_0^t h(s)ds\bigg).$$

From Lemma 1 , we get

$$u(t) = -I^{\alpha}\varphi_q\bigg(\int_0^t h(s)ds\bigg) + C_0 + C_1t + C_2t^2.$$

where $C_0, C_1, C_2 \in \mathbb{R}$ are constants. Using the boundary conditions (2.2), we have

$$C_1 = C_2 = 0.$$

So,

$$\begin{aligned}
u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds + C_0, \\
u'(t) &= -\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds.
\end{aligned} \tag{2.5}$$

By the boundary condition $u(1) + u'(1) = u'(\eta)$, we have

$$\begin{aligned}
C_0 = &\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds + \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds \\
&- \int_0^{\eta} \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds.
\end{aligned}$$

Inserting C_0 into (2.5), we get

$$
\begin{aligned}
u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds - \int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds + \int_0^t \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&\quad + \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds + \int_0^\eta \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&\quad + \int_\eta^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds - \int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&= \int_0^t \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&\quad + \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&\quad + \int_0^\eta \frac{(1-s)^{\alpha-2}-(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&\quad + \int_\eta^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&= \int_0^1 \Big(\mathcal{G}(t,s)+\mathcal{H}(\eta,s)\Big)\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds \\
&= \int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^s h(\tau)d\tau\Big)ds.
\end{aligned}
$$

□

Lemma 4. *The functions $\mathcal{G}(t,s)$ and $\mathcal{H}(t,s)$ defined by (2.4) are continuous on $[0,1]\times[0,1]$ and satisfy the following properties*

(i) $\mathcal{G}(t,s) \geq 0$, $\mathcal{H}(t,s) \geq 0$, *for all* $t,s \in [0,1]$;
(ii) $\big(1-t^{\alpha-1}\big)\mathcal{G}(s,s) \leq \mathcal{G}(t,s) \leq \mathcal{G}(s,s)$, *for all* $(t,s)\in[0,1]\times[0,1]$;
(iii) $\big(1-t^{\alpha-2}\big)\mathcal{H}(s,s) \leq \mathcal{H}(t,s) \leq \mathcal{H}(s,s)$, *for all* $(t,s)\in[0,1]\times[0,1]$,

where $\mathcal{G}(s,s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$, $\mathcal{H}(s,s) = \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$.

Proof. $\mathcal{G}$ and $\mathcal{H}$ are continuous by definition.
(i) If $0 \leq s \leq t$, then

$$
\begin{aligned}
\mathcal{G}(t,s) &= \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\geq \frac{(1-s)^{\alpha-1}-(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&= 0.
\end{aligned}
$$

For $s \geq t$, we have $\mathcal{G}(t,s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \geq 0$.
So,

$$\mathcal{G}(t,s) \geq 0, \text{ for all } t,s \in [0,1].$$

By a similar argument, we show that $\mathcal{H}(t,s) \geq 0,$ for all $t,s \in [0,1]$.
(ii) If $0 < s \leq t$, then

$$\begin{aligned}\mathcal{G}(t,s) &= \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &\geq \frac{(1-s)^{\alpha-1} - t^{\alpha-1}(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)} \\ &\geq \frac{(1-s)^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \big(1-t^{\alpha-1}\big)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}.\end{aligned}$$

On the other hand, we have $\mathcal{G}(t,s) \leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$.
Notice that

$$\mathcal{G}(s,s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}.$$

Thus,

$$\big(1-t^{\alpha-1}\big)\mathcal{G}(s,s) \leq \mathcal{G}(t,s) \leq \mathcal{G}(s,s), \text{ for all } t,s \in [0,1].$$

Now, if $s = 0$, then

$$\begin{aligned}\mathcal{G}(t,0) &= \frac{1-t^{\alpha-1}}{\Gamma(\alpha)} \\ &\leq \frac{1}{\Gamma(\alpha)} \\ &= \mathcal{G}(0,0).\end{aligned}$$

On the other hand,

$$\mathcal{G}(t,0) = \frac{1-t^{\alpha-1}}{\Gamma(\alpha)} \geq \big(1-t^{\alpha-1}\big)\mathcal{G}(0,0).$$

Therefore, item (ii) holds.
(iii) It follows directly from (ii). □

Lemma 5. *Let $\rho \in (0,1)$ be fixed. $\mathcal{K}(t,s)$ defined by (2.3) satisfies the following properties*

(i) $\mathcal{K}(t,s) \geq 0$, *for all* $t,s \in [0,1]$.
(ii) $\big(1-\eta^{\alpha-2}\big)\big(1-t^{\alpha-1}\big)\Phi(s) \leq \mathcal{K}(t,s) \leq \Phi(s)$, *for all* $(t,s) \in [0,1]\times[0,1]$.
(iii) $\big(1-\eta^{\alpha-2}\big)\big(1-\rho^{\alpha-1}\big)\Phi(s) \leq \mathcal{K}(t,s) \leq \Phi(s)$, *for all* $(t,s) \in [0,\rho]\times[0,1]$,

where $\Phi(s) = \frac{(\alpha-s)(1-s)^{\alpha-2}}{\Gamma(\alpha)}$.

Proof. Notice that (i) holds trivially. Next, we show (ii) holds. First, notice that

$$\begin{aligned}\mathcal{G}(s,s) + \mathcal{H}(s,s) &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &= \Phi(s).\end{aligned}$$

From Lemma 4 and (2.3), we have

$$\begin{aligned}\mathcal{K}(t,s) &= \mathcal{G}(t,s) + \mathcal{H}(\eta,s) \\ &\leq \mathcal{G}(s,s) + \mathcal{H}(s,s) \\ &= \Phi(s).\end{aligned}$$

On the other hand, from Lemma 4, we get

$$\begin{aligned}\mathcal{K}(t,s) &= \mathcal{G}(t,s)+\mathcal{H}(\eta,s)\\ &\geq \big(1-t^{\alpha-1}\big)\mathcal{G}(s,s)+\big(1-\eta^{\alpha-2}\big)\mathcal{H}(s,s)\\ &\geq \big(1-t^{\alpha-1}\big)\big(1-\eta^{\alpha-2}\big)\big[\mathcal{G}(s,s)+\mathcal{H}(s,s)\big]\\ &= \big(1-t^{\alpha-1}\big)\big(1-\eta^{\alpha-2}\big)\Phi(s).\end{aligned}$$

Therefore, (ii) holds.
(iii) It follows directly from (ii). □

Lemma 6. *Let $\rho\in(0,1)$ be fixed. If $h\in C([0,1],[0,\infty))$, then the unique solution of the fractional boundary value problem (2.1)-(2.2) is nonnegative and satisfies*

$$\min_{t\in[0,\rho]} u(t)\geq \gamma\|u\|,$$

where $\gamma=\big(1-\eta^{\alpha-2}\big)\big(1-\rho^{\alpha-1}\big)$.

Proof. The positiveness of $u(t)$ follows immediately from Lemma 3 and Lemma 5. For all $t\in[0,1]$, we have

$$\begin{aligned}u(t) &= \int_0^1 \mathcal{K}(t,s)\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds\\ &\leq \int_0^1 \Phi(s)\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds\end{aligned}$$

Then

$$\|u\|\leq \int_0^1 \Phi(s)\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds. \tag{2.6}$$

On the other hand, Lemma 3, Lemma 5 and (2.6) imply that, for any $t\in[0,\rho]$, we have

$$\begin{aligned}u(t) &= \int_0^1 \mathcal{K}(t,s)\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds\\ &\geq \big(1-\eta^{\alpha-2}\big)\big(1-\rho^{\alpha-1}\big)\int_0^1 \Phi(s)\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds\\ &= \gamma\int_0^1 \Phi(s)\varphi_q\bigg(\int_0^s h(\tau)d\tau\bigg)ds\\ &\geq \gamma\|u\|.\end{aligned} \tag{2.7}$$

Therefore,

$$\min_{t\in[0,\rho]} u(t)\geq \gamma\|u\|.$$

□

Let $\rho\in(0,1)$ be fixed. Introduce the cone that we shall use in the sequel.

$$\mathcal{P}=\bigg\{u\in\mathcal{B}:\ u(t)\geq 0,\ t\in[0,1],\ \min_{t\in[0,\rho]} u(t)\geq\gamma\|u\|\bigg\},$$

and define the operator $\mathcal{A}:\mathcal{P}\to\mathcal{B}$ by

$$\mathcal{A}u(t)=\int_0^1 \mathcal{K}(t,s)\varphi_q\bigg(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\bigg)ds, \tag{2.8}$$

where $\mathcal{K}(t,s)$ is defined by (2.3).

Remark 4. *By Lemma 3, the fixed points of the operator $\mathcal{A}$ in $\mathcal{P}$ are the nonnegative solutions of the boundary value problem* (1.1)-(1.2).

The following properties of the p-Laplacian operator will play an important role in the rest of the paper.

Lemma 7 ([9], Lemma 1.3)**.** *Let φ_p be a p-Laplacian operator. Then*

(i) *If $1 < p \leq 2$, $xy > 0$, and $|x|, |y| \geq m > 0$, then*

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)m^{p-2}|x-y|.$$

(ii) *If $p > 2$, $|x|, |y| \leq M$, then*

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)M^{p-2}|x-y|.$$

Lemma 8. *The operator $\mathcal{A}$ defined in* (2.8) *is completely continuous and satisfies $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$.*

Proof. From Lemma 6 and under assumption (H1), it follows that $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$. In view of the assumption of nonnegativeness and continuity of $f(t,u(t))$, $\mathcal{K}(t,s)$ and Lebesgue's dominated convergence theorem, we conclude that $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is continuous. Let D be an arbitrary bounded set in $\mathcal{P}$. Then, there exists $M > 0$ such that $D \subset \{u \in \mathcal{P} : \|u\| < M\}$. Set

$$L = \max\{f(t,u)/\ t \in [0,1], u \in D\}.$$

From Lemmas 3 and 5, for any $u \in D$, we have

$$\begin{aligned} \mathcal{A}u(t) &= \int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &\leq \int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^1 a(\tau)Ld\tau\Big)ds \\ &\leq L^{q-1}\varphi_q\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1 \Phi(s)ds. \end{aligned}$$

Thus,

$$\|\mathcal{A}u\| \leq L^{q-1}\varphi_q\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1 \Phi(s)ds.$$

Hence, $\mathcal{A}(D)$ is uniformly bounded. On the other hand, let $u \in D$, $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then, from Lemmas 3 and 4, we have

$$\begin{aligned}
|\mathcal{A}u(t_1) - \mathcal{A}u(t_2)| &= \left| \int_0^1 \mathcal{K}(t_1,s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau \Big)ds \right. \\
&\quad \left. - \int_0^1 \mathcal{K}(t_2,s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau \Big)ds \right| \\
&= \left| \int_0^1 \Big[\mathcal{K}(t_1,s) - \mathcal{K}(t_2,s)\Big]\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau \Big)ds \right| \\
&= \left| \int_0^1 \Big[\mathcal{G}(t_1,s) - \mathcal{G}(t_2,s)\Big]\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau \Big)ds \right| \\
&\leq L^{q-1} \int_0^1 \big|\mathcal{G}(t_1,s) - \mathcal{G}(t_2,s)\big|\varphi_q\Big(\int_0^s a(\tau)d\tau \Big)ds.
\end{aligned}$$

The continuity of $\mathcal{G}$ implies that the right-side of the above inequality tends to zero if $t_2 \to t_1$. That is to say, $\mathcal{A}(D)$ is equicontinuous. Thus, the Arzela-Ascoli theorem implies that $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is completely continuous. □

3. Existence of positive solutions

Set

$$\Lambda_1 = \left(\varphi_q\Big(\int_0^1 a(\tau)d\tau \Big) \int_0^1 \Phi(s)ds \right)^{-1}, \ \Lambda_2 = \left(\gamma \int_0^\rho \Phi(s)\varphi_q\Big(\int_0^s a(\tau)d\tau \Big)ds \right)^{-1}.$$

Then we find that $0 < \Lambda_1 < \Lambda_2$. In fact,

$$\begin{aligned}
\Lambda_2^{-1} &= \gamma \int_0^\rho \Phi(s)\varphi_q\Big(\int_0^s a(\tau)d\tau \Big)ds \\
&< \int_0^\rho \Phi(s)\varphi_q\Big(\int_0^s a(\tau)d\tau \Big)ds \\
&\leq \varphi_q\Big(\int_0^1 a(\tau)d\tau \Big) \int_0^1 \Phi(s)ds \\
&= \Lambda_1^{-1}.
\end{aligned}$$

Theorem 5. *Assume that* (H1)-(H2) hold. If there exist constants $\rho_1 > 0$, $\rho_2 > 0$, $M_1 \in (0, \Lambda_1]$, and $M_2 \in [\Lambda_2, \infty)$, where $\rho_1 < \rho_2$ and $M_2\rho_1 < M_1\rho_2$, such that f satisfies

(i) $f(t,u) \leq \varphi_p(M_1\rho_2)$ for all $u \in [0,\rho_2]$ and $t \in [0,1]$, and
(ii) $f(t,u) \geq \varphi_p(M_2\rho_1)$ for all $u \in [\gamma\rho_1, \rho_1]$ and $t \in [0,\rho]$,

then the problem (1.1)-(1.2) has at least one positive solution $u \in \mathcal{P}$ satisfying $\rho_1 < \|u\| < \rho_2$.

Proof. Define the open set

$$\Omega_{\rho_2} = \{u \in \mathcal{B} : \|u\| < \rho_2\}.$$

Let $u \in \mathcal{P} \cap \partial\Omega_{\rho_2}$. Then, from assumption (i) and Lemma 5, we have

$$\begin{aligned}
\mathcal{A}u(t) &= \int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\leq \int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^s a(\tau)\varphi_p(M_1\rho_2)d\tau\Big)ds \\
&\leq M_1\rho_2\int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^1 a(\tau)d\tau\Big)ds \\
&\leq M_1\rho_2\varphi_q\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1 \Phi(s)ds \\
&\leq \Lambda_1\Lambda_1^{-1}\rho_2 \\
&= \rho_2,
\end{aligned}$$

which implies that

$$\|\mathcal{A}u\| \leq \|u\| \text{ for all } u \in \mathcal{P} \cap \partial\Omega_{\rho_2}. \tag{3.1}$$

Next, define the open set $\Omega_{\rho_1} = \{u \in \mathcal{B} : \|u\| < \rho_1\}$.
For any $u \in \mathcal{P} \cap \partial\Omega_{\rho_1}$, by using (H1)-(H2), assumption (ii) and Lemma 5, for $t \in [0,\rho]$, we then get

$$\begin{aligned}
\mathcal{A}u(t) &= \int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\geq \int_0^\rho \mathcal{K}(t,s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\geq \gamma\int_0^\rho \Phi(s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\geq M_2\rho_1\gamma\int_0^\rho \Phi(s)\varphi_q\Big(\int_0^s a(\tau)d\tau\Big)ds \\
&\geq \rho_1\Lambda_2\gamma\int_0^\rho \Phi(s)\varphi_q\Big(\int_0^s a(\tau)d\tau\Big)ds \\
&= \rho_1\Lambda_2\Lambda_2^{-1} \\
&= \rho_1,
\end{aligned}$$

which implies that

$$\|\mathcal{A}u\| \geq \|u\| \text{ for all } u \in \mathcal{P} \cap \partial\Omega_{\rho_1}. \tag{3.2}$$

Therefore by (b) in Theorem 1, $\mathcal{A}$ has at least one fixed point in $\mathcal{P} \cap (\overline{\Omega}_{\rho_2} \setminus \Omega_{\rho_1})$. So there exists at least one solution of (1.1)-(1.2) with $\rho_1 < \|u\| < \rho_2$. □

By a closely similar way, we can obtain the following result.

Theorem 6. *Assume that* (H1)-(H2) hold. If there exist constants $\rho_1 > 0$, $\rho_2 > 0$, $M_1 \in (0,\Lambda_1]$, and $M_2 \in [\Lambda_2,\infty)$, where $\gamma\rho_2 < \rho_1 < \rho_2$, and $M_1\rho_1 > M_2\rho_2$, such that f satisfies

(i) $f(t,u) \geq \varphi_p(M_2\rho_2)$ for all $u \in [\gamma\rho_2,\rho_2]$ and $t \in [0,\rho]$, and
(ii) $f(t,u) \leq \varphi_p(M_1\rho_1)$ for all $u \in [0,\rho_1]$ and $t \in [0,1]$,

then the problem (1.1)-(1.2) has at least one positive solution $u \in \mathcal{P}$ satisfying $\rho_1 < \|u\| < \rho_2$.

Theorem 7. *Assume that* (H1)-(H2) hold. Further, assume that there exists a constant $\nu > 0$ such that

$$\nu > L^{q-1}\varphi_q\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1 \Phi(s)ds, \tag{3.3}$$

where $L = \max\{f(t,u)/\ (t,u) \in [0,1]\times[0,\nu]\}$.
Then the fractional boundary value problem (1.1)-(1.2) has at least one positive solution.

Proof. Let

$$\mathcal{U} = \{u \in \mathcal{P} : \|u\| < \nu\}.$$

By virtue of Lemma 8, the operator $\mathcal{A} : \overline{\mathcal{U}} \to \mathcal{P}$ is completely continuous. Assume that there exist $u \in \overline{\mathcal{U}}$ and $\lambda \in (0,1)$ such that $u = \lambda\mathcal{A}u$. Then we have

$$\begin{aligned} |u(t)| = |\lambda(\mathcal{A}u)(t)| &= \Big|\lambda\int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds\Big| \\ &\le \int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^s a(\tau)Ld\tau\Big)ds \\ &\le L^{q-1}\varphi_q\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1 \Phi(s)ds. \end{aligned}$$

So,

$$\|u\| \le L^{q-1}\varphi_q\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1 \Phi(s)ds.$$

Thus from (3.3), we have that $\|u\| < \nu$, which means that $u \notin \partial\mathcal{U}$. Hence, it follows that there is no $u \in \partial\mathcal{U}$ such that $u = \lambda\mathcal{A}u$ for some $\lambda \in (0,1)$. Therefore by Theorem 2, we conclude that the fractional boundary value problem (1.1)-(1.2) has at least one positive solution. □

Theorem 8. *Assume that* (H1)-(H2) hold, and $1 < p < 2$. In addition, we assume that the following assumptions hold:

(C1) There exists a nonnegative function $k \in C[0,1]$ such that

$$f(t,u) \le k(t), \text{ for any } (t,u) \in [0,1]\times[0,\infty). \tag{3.4}$$

(C2) There exists a constant L with

$$0 < L < \frac{\Gamma(\alpha+1)}{(\alpha+1)(q-1)}\Big(\int_0^1 a(t)dt\Big)^{-1}\Big(\int_0^1 a(t)k(t)dt\Big)^{2-q}$$

such that

$$|f(t,u) - f(t,v)| \le L|u-v|, \text{ for any } t \in [0,1] \text{ and } u,v \in [0,\infty). \tag{3.5}$$

Then the boundary value problem (1.1) and (1.2) has a unique solution.

Proof. By (3.4), for $t \in [0,1]$, we get

$$\begin{aligned} \int_0^t a(s)f(s,u(s))ds &\le \int_0^1 a(s)f(s,u(s))ds \\ &\le \int_0^1 a(s)k(s)ds = M. \end{aligned}$$

From (ii) in Lemma 7 and (3.5), for any $u, v \in \mathcal{B}$, we have

$$\begin{aligned}
|\mathcal{A}u(t) - \mathcal{A}v(t)| &= \left| \int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau \Big)ds \right. \\
&\quad \left. - \int_0^1 \mathcal{K}(t,s)\varphi_q\Big(\int_0^s a(\tau)f(\tau,v(\tau))d\tau \Big)ds \right| \\
&= \left| \int_0^1 \mathcal{K}(t,s)\Big(\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau \Big) \right. \\
&\quad \left. -\varphi_q\Big(\int_0^s a(\tau)f(\tau,v(\tau))d\tau \Big)\Big)ds \right| \\
&\leq \int_0^1 \mathcal{K}(t,s)\left| \varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau \Big) \right. \\
&\quad \left. -\varphi_q\Big(\int_0^s a(\tau)f(\tau,v(\tau))d\tau \Big)\right| ds \\
&\leq \int_0^1 \mathcal{K}(t,s)(q-1)M^{q-2}\Big(\int_0^s a(\tau)\Big| f(\tau,u(\tau)) \\
&\quad -f(\tau,v(\tau))\Big| d\tau \Big)ds \\
&\leq L\|u-v\|(q-1)M^{q-2}\Big(\int_0^1 a(\tau)d\tau \Big)\Big(\int_0^1 \Phi(s)ds \Big) \\
&= L_1\|u-v\|,
\end{aligned}$$

where

$$\begin{aligned}
L_1 &= L(q-1)M^{q-2}\Big(\int_0^1 a(\tau)d\tau \Big)\Big(\int_0^1 \Phi(s)ds \Big) \\
&= \frac{L(q-1)M^{q-2}(\alpha+1)}{\Gamma(\alpha+1)} \int_0^1 a(\tau)d\tau.
\end{aligned}$$

By definition of L, we have $0 < L_1 < 1$. Then

$$\|\mathcal{A}u - \mathcal{A}v\| \leq L_1\|u-v\|.$$

By virtue of Theorem 3, it follows that there exists a unique fixed point for the operator $\mathcal{A}$, which corresponds to the unique solution for problem (1.1) and (1.2). □

Theorem 9. *Assume that* (H1)-(H2) hold, and $p > 2$. In addition, we assume that there exist constants $\mu > 0$, $0 < \sigma < \frac{2}{2-q}$ such that

$$a(t)f(t,u) \geq \mu\sigma t^{\sigma-1}, \text{ for any } (t,u) \in (0,1] \times [0,\infty), \tag{3.6}$$

and

$$|f(t,u) - f(t,v)| \leq k|u-v|, \text{ for any } t \in [0,1] \text{ and } u, v \in [0,\infty), \tag{3.7}$$

where

$$0 < k < \frac{\big(\sigma(q-2)+\alpha\big)\Gamma(\alpha-1)}{(q-1)\mu^{q-2}\big(\sigma(q-2)+\alpha+1\big)\mathfrak{B}\big(\alpha-1,\sigma(q-2)+1\big)}\left(\int_0^1 a(\tau)d\tau\right)^{-1}.$$

Then the boundary value problem (1.1) and (1.2) has a unique solution.

Proof. By (3.6), we get

$$\int_0^t a(s)f(s,u(s))ds \geq \mu t^{\sigma}, \text{ for any } (t,u)\in[0,1]\times[0,\infty).$$

By (i) in Lemma 7 and (3.7), for any $u, v \in \mathcal{B}$, we have

$$\begin{aligned}
|\mathcal{A}u(t)-\mathcal{A}v(t)| &= \left|\int_0^1 \mathcal{K}(t,s)\bigg(\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)\right.\\
&\quad \left.-\varphi_q\Big(\int_0^s a(\tau)f(\tau,v(\tau))d\tau\Big)\bigg)ds\right|\\
&\leq \int_0^1 \mathcal{K}(t,s)\bigg|\varphi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)\\
&\quad -\varphi_q\Big(\int_0^s a(\tau)f(\tau,v(\tau))d\tau\Big)\bigg|ds\\
&\leq \int_0^1 \mathcal{K}(t,s)(q-1)(\mu s^{\sigma})^{q-2}\bigg(\int_0^s a(\tau)\Big|f(\tau,u(\tau))\\
&\quad -f(\tau,v(\tau))\Big|d\tau\bigg)ds\\
&\leq (q-1)\mu^{q-2}k\|u-v\|\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1 \mathcal{K}(t,s)s^{\sigma(q-2)}ds\\
&\leq (q-1)\mu^{q-2}k\|u-v\|\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1 \big[\mathcal{G}(s,s)\\
&\quad +\mathcal{H}(s,s)\big]s^{\sigma(q-2)}ds\\
&= (q-1)\mu^{q-2}k\|u-v\|\Big(\int_0^1 a(\tau)d\tau\Big)\\
&\quad \times\left(\frac{\mathfrak{B}\big(\alpha,\sigma(q-2)+1\big)}{\Gamma(\alpha)}+\frac{\mathfrak{B}\big(\alpha-1,\sigma(q-2)+1\big)}{\Gamma(\alpha-1)}\right)\\
&= \frac{(q-1)\mu^{q-2}k\big(\sigma(q-2)+\alpha+1\big)}{\big(\sigma(q-2)+\alpha\big)\Gamma(\alpha-1)}\Big(\int_0^1 a(\tau)d\tau\Big)\\
&\quad \times\mathfrak{B}\big(\alpha-1,\sigma(q-2)+1\big)\|u-v\|\\
&= L\|u-v\|,
\end{aligned}$$

where

$$L=\frac{(q-1)\mu^{q-2}k\big(\sigma(q-2)+\alpha+1\big)}{\big(\sigma(q-2)+\alpha\big)\Gamma(\alpha-1)}\Big(\int_0^1 a(\tau)d\tau\Big)\mathfrak{B}\big(\alpha-1,\sigma(q-2)+1\big).$$

By definition of k, we have $0 < L < 1$. Then

$$\|\mathcal{A}u - \mathcal{A}v\| \leq L\|u - v\|.$$

This implies that $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is a contraction mapping. By Theorem 3, we get that $\mathcal{A}$ has a unique fixed point in $\mathcal{B}$, which is a solution of the problem (1.1) and (1.2). □

4. Examples

Example 4.1. *Consider the nonlinear boundary value problem*

$$\begin{cases} \left(\varphi_{\frac{3}{2}}\left(D^{\frac{5}{2}}u(t)\right)\right)' + \frac{1}{2}te^t \ln(u+1) = 0, \ t \in (0,1), \\ D^{\frac{5}{2}}u(0) = u'(0) = u''(0) = 0, \ u(1) + u'(1) = u'(\eta), \end{cases} \tag{4.1}$$

where $f(t,u) = \frac{1}{2}t\ln(u+1)$, $a(t) = e^t$, $\alpha = \frac{5}{2}$, $p = \frac{3}{2}$, $\eta \in (0,1)$, *and then* $q = 3$, $f \in C([0,1] \times [0,\infty), [0,\infty))$. *By taking* $\nu = 1$, *we obtain*

$$L = \max\{f(t,u)/\ t \in [0,1], u \in [0,1]\} = \frac{1}{2}\ln(2),$$

and

$$L^{q-1}\varphi_q\left(\int_0^1 a(\tau)d\tau\right)\int_0^1 \Phi(s)ds \approx 0.372 < \nu = 1.$$

By means of Theorem 7, the boundary value problem (4.1) *has at least one positive solution.*

Example 4.2. *As a second example we consider the following boundary value problem*

$$\begin{cases} \left(\varphi_{\frac{3}{2}}\left(D^{\frac{13}{5}}u(t)\right)\right)' + te^{-t}\sin^2 u = 0, \ t \in (0,1), \\ D^{\frac{13}{5}}u(0) = u'(0) = u''(0) = 0, \ u(1) + u'(1) = u'(\eta), \end{cases} \tag{4.2}$$

where $f(t,u) = e^{-t}\sin^2 u$, $a(t) = t$, $\alpha = \frac{13}{5}$, $p = \frac{3}{2}, \eta \in (0,1)$, *and then* $q = 3$, $f \in C([0,1] \times [0,\infty), [0,\infty))$.
Taking the nonnegative function $k(t) = e^{-t}$, *then* $k \in C[0,1]$ *and* $f(t,u) \leq k(t)$. *Choosing* $L = 2$, *for any* $t \in [0,1]$ *and* $u, v \in [0,\infty)$, *we have*

$$\begin{aligned} |f(t,u) - f(t,v)| &= e^{-t}|\sin^2 u - \sin^2 v| \\ &\leq 2|u - v| \\ &= L|u - v|, \end{aligned}$$

and

$$\begin{aligned} \frac{\Gamma(\alpha+1)}{(\alpha+1)(q-1)}\left(\int_0^1 a(t)dt\right)^{-1}\left(\int_0^1 a(t)k(t)dt\right)^{2-q} &= \frac{13}{18}\Gamma\left(\frac{13}{5}\right)\left(1 - 2e^{-1}\right)^{-1} \\ &\approx 3.90744 \\ &> 2 = L. \end{aligned}$$

From Theorem 8, the boundary value problem (4.2) *has a unique solution.*

Example 4.3. *Let the following boundary value problem*

$$\begin{cases} \Big(\varphi_{\frac{7}{2}}\big(D^{\frac{5}{2}}u(t)\big)\Big)' + \frac{1}{160}t\sqrt{t}(348+\sqrt{u}+t) = 0,\ t\in(0,1), \\ D^{\frac{5}{2}}u(0) = u'(0) = u''(0) = 0,\ u(1)+u'(1) = u'(\frac{1}{2}), \end{cases} \tag{4.3}$$

where $f(t,u) = \frac{1}{400}(348+\sqrt{u}+t), a(t) = \frac{5}{2}t\sqrt{t}, \alpha = \frac{5}{2}, p = \frac{7}{2}, \eta = \frac{1}{2}$, *and then* $q = \frac{7}{5},\ f \in C([0,1]\times[0,\infty),[0,\infty))$. *By a simple computation, we obtain*

$$\begin{aligned}
\Lambda_1 &= \left(\varphi_q\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1 \Phi(s)ds\right)^{-1} \\
&= \left(\varphi_{\frac{7}{5}}\Big(\int_0^1 \frac{5}{2}\tau\sqrt{\tau}d\tau\Big)\int_0^1 \frac{(\frac{5}{2}-s)(1-s)^{\frac{1}{2}}}{\Gamma(\frac{5}{2})}ds\right)^{-1} \\
&= \frac{15\sqrt{\pi}}{28} \\
&\approx 0.94952 \\
\Lambda_2 &= \left(\gamma\int_0^\rho \Phi(s)\varphi_q\Big(\int_0^s a(\tau)d\tau\Big)ds\right)^{-1} \\
&= \left(\gamma\int_0^\rho \Phi(s)\varphi_{\frac{7}{5}}\Big(\int_0^s \frac{5}{2}\tau\sqrt{\tau}d\tau\Big)ds\right)^{-1} \\
&= \left(\gamma\int_0^\rho s\Phi(s)ds\right)^{-1} \\
&= \left(\frac{\big(1-\eta^{\alpha-2}\big)\big(1-\rho^{\alpha-1}\big)}{(1-\alpha)\Gamma(\alpha)}\left((1-\rho)^{\alpha-1}\left(\rho(\alpha-\rho)+\frac{(\alpha-2\rho)(1-\rho)}{\alpha}\right.\right.\right. \\
&\quad \left.\left.\left. -\frac{2(1-\rho)^2}{\alpha(\alpha+1)}\right)+\frac{2}{\alpha(\alpha+1)}-1\right)\right)^{-1} \\
&= \left(\frac{8}{9\sqrt{\pi}}\Big(1-\frac{1}{\sqrt{2}}\Big)\big(1-\rho^{\frac{3}{2}}\big)\left(\frac{27}{35}-(1-\rho)^{\frac{3}{2}}\left(\rho\Big(\frac{5}{2}-\rho\Big)\right.\right.\right. \\
&\quad \left.\left.\left. +\frac{2}{5}\Big(\frac{5}{2}-2\rho\Big)(1-\rho)-\frac{8}{35}(1-\rho)^2\right)\right)\right)^{-1}
\end{aligned}$$

Choosing $M_1 = \Lambda_1$, $M_2 = \Lambda_2$, $\rho_1 = \frac{1}{120}$ *and* $\rho_2 = 1$. *With the use of the Mathematica software, we easy to check that* $M_2\rho_1 = \frac{\Lambda_2}{120} < M_1\rho_2 = \frac{15\sqrt{\pi}}{28}$ *for all* $\rho\in\big[\frac{1}{5},\frac{4}{5}\big]$. *By a simple computation, we obtain* $M_1^{\frac{5}{2}} = \frac{225\sqrt{\frac{15}{7}}\pi^{\frac{5}{4}}}{1568} \approx 0.87855$, *and* $\big(\frac{M_2}{120}\big)^{\frac{5}{2}} \le 0.86233$ *for all* $\rho\in\big[\frac{1}{5},\frac{4}{5}\big]$. *Again, we see that* f *satisfies the following*

relations:

$$f(t,u) = \frac{348+\sqrt{u}+t}{400} \leq 0.875 < \varphi_{\frac{7}{2}}\big(M_1\rho_2\big) = M_1^{\frac{5}{2}} \approx 0.87855, t \in [0,1], u \in [0,1],$$

$$f(t,u) = \frac{348+\sqrt{u}+t}{400} \geq 0.87 > \varphi_{\frac{7}{2}}\big(M_2\rho_1\big) = \Big(\frac{M_2}{120}\Big)^{\frac{5}{2}}, t \in [0,\rho], u \in \left[\frac{\gamma}{120}, \frac{1}{120}\right].$$

So, all the assumptions of Theorem 5 are satisfied. With the use of Theorem 5, the fractional boundary value problem (4.3) *has at least one positive solution* u *such that* $\frac{1}{120} < \|u\| < 1$.

References

[1] L.E. Bobisud, *Steady-state turbulent flow with reaction*, Rocky Mt. J. Math., **21**:3 (1991), 993–1007. Zbl 0760.34023

[2] T. Chen, W. Liu, *An anti-periodic boundary value problem for the fractional differential equation with a p-Laplacian operator*, Appl. Math. Lett., **25**:11 (2012), 1671–1675. Zbl 1248.35219

[3] L. Cheng, W. Liu, Q. Ye, *Boundary value problem for a coupled system of fractional differential equations with p-Laplacian operator at resonance*, Electron. J. Differ. Equ., **2014** (2014), Paper No. 60. Zbl 1292.34002

[4] A. Granas, J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer, New York, 2003. Zbl 1025.47002

[5] H. Wang, S. Liu, H. Li, *Positive solutions to p-Laplacian fractional differential equations with infinite-point boundary value conditions*, Adv. Difference Equ., **2018** (2018), Paper 425. Zbl 1448.34061

[6] Hongling Lu, Zhenlai Han Zhenlai, Shurong Sun, *Multiplicity of positive solutions for Sturm-Liouville boundary value problems of fractional differential equations with p-Laplacian*, Bound. Value Probl., **2014** (2014), Paper No 26. Zbl 1304.34010

[7] Z. Hu, W. Liu, J. Liu, *Existence of solutions of fractional differential equation with p-Laplacian operator at resonance*, Abstr. Appl. Anal., **2014** (2014), Art. ID 809637. Zbl 7023121

[8] L. Hu, S. Zhang, *Existence results for a coupled system of fractional differential equations with p-Laplacian operator and infinite-point boundary conditions*, Bound. Value Probl., **2017** (2017), Paper No. 88. Zbl 1367.26018

[9] H. Khan, W. Chen, H. Sun, *Analysis of positive solution and Hyers-Ulam stability for a class of singular fractional differential equations with p-Laplacian in Banach space*, Math. Methods Appl. Sci., **41**:9 (2018), 3430–3440. Zbl 1394.34017

[10] A. Khan, Y. Li, K. Shah, T.S. Khan, *On coupled p-Laplacian fractional differential equations with nonlinear boundary conditions*, Complexity, **2017** (2017), Art. ID 8197610. Zbl 1373.93157

[11] H. Khan, Y. Li, H. Sun, A. Khan, *Existence of solution and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator*, J. Nonlinear Sci. Appl., **10**:10 (2017), 5219–5229. Zbl 1412.47140

[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006. Zbl 1092.45003

[13] M.A. Krasnosel'skii, *Positive solutions of operator equations*, P. Noordhoff, Groningen, 1964. Zbl 0121.10604

[14] Li Zhang, Fanglei Wang, Yuanfang Ru, *Existence of nontrivial solutions for fractional differential equations with p-Laplacian*, J. Funct. Spaces, **2019** (2019), Art. ID 3486410. Zbl 1409.34018

[15] Z. Liu, L. Lu, *A class of BVPs for nonlinear fractional differential equations with p-Laplacian operator*, Electron. J. Qual. Theory Differ. Equ., **2012** (2012), Paper No. 70. Zbl 1340.34299

[16] R. Luca, *Positive solutions for a system of fractional differential equations with p-Laplacian operator and multi-point boundary conditions*, Nonlinear Anal., Model. Control., **23**:5 (2018), 771–801. Zbl 1420.34020

[17] N.I Mahmudov, S. Unul, *Existence of solutions of fractional boundary value problems with p-Laplacian operator*, Bound. Value Probl., **2015** (2015) Paper No. 99. Zbl 6580555

[18] K. Perera, M. Squassina, Y. Yang, *A note on the Dancer-Fučík spectra of the fractional p-Laplacian and Laplacian operators*, Adv. Nonlinear Anal., **4**:1 (2015), 13–23. Zbl 1432.35228
[19] K. Perera, M. Squassina, Y. Yang, *Bifurcation and multiplicity results for critical fractional p-Laplacian problems*, Math. Nachr., **289**:2-3 (2016), 332–342. Zbl 1336.35050
[20] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, **198**, Academic Press, San Diego, 1999. Zbl 0924.34008
[21] K.R. Prasad, B.M.B. Krushna, *Multiple positive solutions for a coupled system of p-Laplacian fractional order two-point boundary value problems*, Int. J. Differ. Equ., **2014** (2014), Art. ID 485647. Zbl 1294.34008
[22] P. Pucci, M. Xiang, B. Zhang, *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations*, Adv. Nonlinear Anal., **5**:1 (2016), 27–55. Zbl 1334.35395
[23] T. Shen, W. Liu, X. Shen, *Existence and uniqueness of solutions for several BVPs of fractional differential equations with p-Laplacian operator*, Mediterr. J. Math., **13**:6 (2016), 4623–4637. Zbl 1349.34018
[24] J. Tan, M. Li, *Solutions of fractional differential equations with p-Laplacian operator in Banach spaces*, Bound. Value Probl., **2018** (2018), Paper No. 15. Zbl 6996323
[25] J. Wang, H. Xiang, *Upper and lower solutions method for a class of singular fractional boundary value problems with p-Laplacian operator*, Abstr. Appl. Anal., **2010** (2010), Art. ID 971824. Zbl 1209.34005
[26] X. Tang, X. Wang, Z. Wang, P. Ouyang, *The existence of solutions for mixed fractional resonant boundary value problem with $p(t)$-Laplacian operator*, J. Appl. Math. Comput., **61**:1-2 (2019), 559—572. Zbl 1431.34012
[27] X. Liu, M. Jia, W. Ge, *The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator*, Appl. Math. Lett., **65** (2017), 56–62. Zbl 1357.34018
[28] C. Yang, J. Yan, *Positive solutions for third-order Sturm-Liouville boundary value problems with p-Laplacian*, Comput. Math. Appl., **59**:6 (2010), 2059–2066. Zbl 1189.34056
[29] Y. Su, Q. Li, Xi-Lan Liu, *Existence criteria for positive solutions of p-Laplacian fractional differential equations with derivative terms*, Adv. Difference Equ., **2013** (2013), Paper No 119. Zbl 1380.34022
[30] Y. Tian, S. Sun, Z. Bai, *Positive solutions of fractional differential equations with p-Laplacian*, J. Funct. Spaces, **2017** (2017), Art. ID 3187492. Zbl 1462.34019
[31] Y. Li, *Existence of positive solutions for fractional differential equation involving integral boundary conditions with p-Laplacian operator*, Adv. Difference Equ., **2017** (2017), Paper No. 135. Zbl 1444.34013
[32] Y. Wang, S. Liu, Z. Han, *Eigenvalue problems for fractional differential equations with mixed derivatives and generalized p-Laplacian*, Nonlinear Anal. Model. Control., **23**:6 (2018), 830–850. Zbl 1420.34027
[33] Z. Han, H. Lu, C. Zhang, *Positive solutions for eigenvalue problems of fractional differential equation with generalized p-Laplacian*, Appl. Math. Comput., **257** (2015), 526–536. Zbl 1338.34015

Faouzi Haddouchi
Faculty of Physics,
University of Sciences and Technology of Oran-MB,
Oran, 31000, Algeria
Laboratory of Fundamental and Applied Mathematics of Oran,
1, University of Oran,
Oran, 31000, Algeria
Email address: fhaddouchi@gmail.com