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SOME QUESTIONS ON THE RELATIONSHIP OF THE FACTORIZATION PROBLEM OF MATRIX FUNCTIONS AND THE TRUNCATED WIENER—HOPF EQUATION IN THE WIENER ALGEBRA

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Abstract. The paper studies the relationship between the equation in convolutions of the second kind on a finite interval (which is also called the truncated Wiener—Hopf equation) and the factorization problem (which is also called the Riemann—Hilbert boundary value problem or the Riemann boundary value problem). In our works published in 2020–2021 (Izv. vuz.), a new approach (method) was proposed to solve the Riemann boundary value problem in the Wiener algebra of order 2. The method is to reduce the Riemann problem to the truncated Wiener—Hopf equation. In this paper, the method is being developed. Here, in the factorization problem, a matrix-function of a sufficiently general form with an arbitrary total index is studied, more general formulas for the relationship between the solutions of the factorization problem and the corresponding truncated Wiener—Hopf equation are also found. In addition, new results are obtained in the theory of equations in convolutions based on the revealed relationship between the problems under consideration.

Keywords: Wiener algebra, factorization problem, partial indices, truncated Wiener—Hopf equation.

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Introduction

This paper studies the relationship between the equation in convolutions of the 2nd kind on a finite interval (which is also called the truncated Wiener—Hopf equation) and the factorization problem (which is also called the Riemann—Hilbert boundary value problem or the Riemann boundary value problem). In works [1]–[2], we proposed a new approach to solving the Riemann boundary problem in the Wiener algebra of order 2. This approach (method) м the Riemann problem to the truncated Wiener—Hopf equation. In this article, the method is further developed. Here in the factorization problem, we study the matrix function of a sufficiently general form with an arbitrary total index and find more general formulas of relationship between solutions of the factorization problem and the corresponding truncated Wiener—Hopf equation. Moreover, new results are obtained in the theory of equations in convolutions on the basis of the established relationship between the considered problems.

It is well known that to study the truncated Wiener—Hopf equation, boundary value problems for analytical functions of Riemann type (for the history of the question, see [3]) are applied, however, the results of such application are only obtained for some special classes of kernels of integral operator of the convolution (see [4]–[7]). On the other hand, it is well known that the factorization problem can be reduced to the Fredholm integral equation or singular integral equation with Cauchy kernel (for the historical information, see [8]–[11]). Along the way, classical results in the boundary value problem theory for analytical functions have been obtained. However, in the general case these integral equations did not turn out to be less complex than the original boundary value problems. Although the truncated Wiener—Hopf equation is one of the most studied among the Fredholm integral equations of second kind, the question of solving the factorization problem with the help of this equation was not studied prior to works [1]–[2].

Before turning to the exact formulations of the studied problems, recall the notion of a Wiener algebra. For integer $2 \geq n, l \geq 1$, we suppose that $L_{n\times l}$ is a space of $n \times l$ matrix functions with elements from $L_1(R)$, $\mathcal{F}f$ is a Fourier image of the matrix function $f \in L_{n\times l}$:

$$\mathcal{F}f(x) = \int\limits_{-\infty}^{\infty} f(t)e^{ixt}\,dt,\ \ x \in \mathbb{R},$$

where $\mathbb{R}$ is the extended real line (R is the real line); $\mathbb{W}^{n\times n}$ is a Wiener algebra of continuous matrix functions of the form $C + \mathcal{F}f$, where C is a constant matrix of order n and $f \in L_{n\times n}$; $\mathbb{W}^{n\times n}_{+}$ ($\mathbb{W}^{n\times n}_{-}$) is a subalgebra in $\mathbb{W}^{n\times n}$, consisting of matrix functions of the form $C + \mathcal{F}f$, such that $f(t) = 0$ when $t < 0$ (when $t > 0$). We denote by $\mathbb{W}^{n\times 1}$, $\mathbb{W}^{n\times 1}_{\pm}$ the groups consisting of column vectors of matrix functions from the algebras $\mathbb{W}^{n\times n}$, $\mathbb{W}^{n\times n}_{\pm}$ respectively.

If A is some algebra, then we denote by $\mathcal{G}A$ the group of invertible elements in A. When $n = 1$, we will drop the superscript $n \times n$ for $\mathbb{W}$.

Consider the factorization problem on the extended line $\mathbb{R}$, on which it is required to find the vector functions $\Phi^{\pm} \in \mathbb{W}^{2\times 1}_{\pm}$ (given the restriction $\Phi^{+}_{2}(\infty) = 0$) by the boundary condition:

$$\Phi^{+}(x) = G(x)\,\Phi^{-}(x),\ \ x \in \mathbb{R}, \tag{0.1}$$

where

$$G(x) = \begin{pmatrix} 1+g_{11}(x) & g_{12}(x) \\ g_{21}(x) & 1+g_{22}(x) \end{pmatrix} \in \mathcal{G}\mathbb{W}^{2\times 2},$$
$$g_{jl} \in \mathbb{W},\ g_{jl}(\infty) = 0,\ j,l = 1,2. \tag{0.2}$$

When speaking of a solution of factorization problem (0.1)–(0.2), we mean a nontrivial solution.

We will provide the fundamental result of the theory of matrix function factorization in the Wiener algebra of order 2 (see, for example, [8, Ch. 6], [9], [10, §7]).

Theorem 1. *Let $G \in \mathcal{G}\mathbb{W}^{2\times 2}$. Then the matrix $G(x)$ admits standard (left) factorization, that is, it can be represented in the form of the following matrix product:*

$$G(x) = G_+(x)D(x)G_-(x),\ x \in \mathbb{R}, \tag{0.3}$$

where $G_\pm \in \mathcal{G}\mathbb{W}_\pm^{2\times 2}$ ($G_\pm$ are multiplier factors), $D(x)$ is a diagonal matrix function,

$$D(x) = \Big\{\Big(\frac{x-i}{x+i}\Big)^{\kappa_1}, \Big(\frac{x-i}{x+i}\Big)^{\kappa_2}\Big\},$$

$\kappa_1 \geq \kappa_2$ are partial indices of the matrix G (integers),

$$\kappa := \operatorname{Ind}\det\, G(t) \equiv \frac{1}{2\pi}\Delta_{\mathbb{R}} \arg\det\, G(t) = \kappa_1 + \kappa_2$$

is the total index of the matrix G.

If $\kappa_j = 0,\ j = 1,2$, then the standard factorization (0.3) is said to be canonical.

The factorization problem is one of the mostly sought after among the problems of complex analysis due to its broad application in different areas of mathematics and natural science. To date, the factorization problem (of the Riemann problem) is far from over. The main unsolved results in the factorization problem theory include the problem of finding (calculating) the invariants of the problem — the private indices of the matrix $G(x)$ and the problem of constructing (calculating) the elements of the matrix functions $G_\pm(x)$ in the decomposition (0.3).

Now consider the equation in convolutions of second kind on the finite interval $(0,\tau)$, to which we will reduce factorization problem (0.1)–(0.2):

$$u(t) - \int_0^\tau k(t-s)u(s)\,ds = f(t), \quad t \in (0,\tau), \tag{0.4}$$

where

$$k \in L_1(R), \quad f \in L_1(0,\tau),\ \tau > 0. \tag{0.5}$$

It is easy to see that the values of the function $k(t)$ outside the interval $(-\tau,\tau)$ do not influence the solution of equation (0.4) (we will search for the latter in $L_1(0,\tau)$).

§1. Preliminary statements, additional constructions and assumptions

Given C=0, we will supply the corresponding algebras, subalgebras, and groups with the subscript 0 ($\mathbb{W}_0^{n\times l}$, $\mathbb{W}_{0\pm}^{n\times l}$, $l \in \{1,n\}$). On the algebra $\mathbb{W}_0$, we define the projectors P_0^+ and P_0^- that are complementary to each other by the formulas

$$P_0^\pm\ :\ W_0\ \to\ \mathbb{W}_{0\pm},\ P_0^\pm \mathcal{F}g(x) = \int_{-\infty}^{\infty} e^{ixt}g(t)\theta(\pm t)\,dt,\ x \in \mathbb{R},$$

where θ is a Heaviside step function.

Note the following properties of the linear operators $P_0^\pm$:

$$P_0^+ + P_0^- = I,\ \mathcal{F}^{-1}\{P_0^\pm \mathcal{F} g(x)\}(t) = g(t)\theta(\pm t),\ t \in R,$$

where I is a unit operator, $\mathcal{F}^{-1}$ is the inverse Fourier transform.

We will list the restrictions on the elements of the matrix G in (0.2). In the work, we will assume the existence of the parameter $\tau \in (0,\infty)$, such that

$$\begin{gathered} g_{11}(x) = \int_{-\tau}^{\tau} \mu_{11}(t) e^{ixt}\,dt,\ g_{12}(x) = \int_{-\infty}^{\tau} \mu_{12}(t) e^{ixt}\,dt, \\ g_{21}(x) = \int_{-\tau}^{\infty} \mu_{21}(t) e^{ixt}\,dt\ \ \Big(g_{22}(x) = \int_{-\infty}^{\infty} \mu_{22}(t) e^{ixt}\,dt\Big), \end{gathered} \tag{1.1}$$

where $\mu_{jl} \in L_1(R),\ j,l = 1,2.$

It is easy to see that, given $\tau = \infty$, the matrix $C\,G(x)$ has a general form in the group of all non-special matrix functions from the Wiener algebra $\mathbb{W}^{2\times 2}$, where C is a constant non-special matrix of order 2.

The first condition in (0.2) implies that the determinant of the matrix $G(x)$ admits effective factorization (see, for example, [8, § 38]):

$$\det G(x) = d_G^+(x)\, p^\kappa d_G^-(x),\ x \in \mathbb{R},$$

where

$$d_G^\pm \in \mathcal{G}\mathbb{W}_\pm,\ d_G^\pm(\infty) = 1,\ p = \frac{x-i}{x+i}.$$

We have the following

Lemma 1. *The solution of factorization problem (0.1)–(0.2) under restriction (1.1) possesses the following property:*

$$P_0^+\{e^{-ix\tau}\big(\Phi_1^+(x) - C_1\big)\} = 0,\ P_0^-\{e^{ix\tau} p^{\kappa_0} d_G^-(x)\Phi_2^-(x)\} = 0, \tag{1.2}$$

where

$$x \in \mathbb{R},\ C_1 = \Phi_1^\pm(\infty);\ \kappa_0 = \kappa \textit{ for } \kappa \ge 0,\ \kappa_0 = 0 \textit{ for } \kappa < 0.$$

Proof. From (0.1), given $x = \infty$, we obtain

$$\Phi_1^+(\infty) = \Phi_1^-(\infty) = C_1,\ \Phi_2^+(\infty) = \Phi_2^-(\infty) = C_2 = 0.$$

We will define the linear projector operators $P^\pm$ on the functions $e^{\pm ix\tau}$ by the formulas:

$$P^\pm\{e^{\pm ix\tau}\} = e^{\pm ix\tau},\ P^\pm\{e^{\mp ix\tau}\} = 0.$$

For functions from the algebra $\mathbb{W}_0$, we assume that on such the operators $P^\pm$ and $P_0^\pm$ are identically equal respectively, in other words,

$$P^\pm F(x) = P_0^\pm F(x),\ F \in \mathbb{W}_0.$$

Multiplying left and right-hand sides of the first equality from system of equalities (0.1) by the multiplier $e^{-ix\tau}$ and applying the operator P^+ to the obtained equality (to its left and right-hand sides), taking into account (1.1) and the properties of the operators $P^\pm$ and $P_0^\pm$, we obtain the first equality in (1.2) due to the fact that

$$P^+\{e^{-ix\tau} g_{1l}(x)\} = P_0^+\{e^{-ix\tau} g_{1l}(x)\} = 0,\ l = 1,2,$$

by the condition of Lemma 1.

It remains to prove that the second equality in (1.2) holds. Multiplying on the left the left and right-hand sides of boundary condition (0.1) by the multiplier $e^{ix\tau}\,p^{\kappa_0}d_G^-(x)\,G^{-1}$, where

$$G^{-1}(x) = \frac{p^{-\kappa}}{d_G^+(x)\,d_G^-(x)}\begin{pmatrix} 1+g_{22}(x) & -g_{12}(x) \\ -g_{21}(x) & 1+g_{11}(x) \end{pmatrix}, \tag{1.3}$$

we obtain

$$e^{ix\tau}\,p^{\kappa_0}d_G^-(x)G^{-1}(x)\,\Phi^+(x) = e^{ix\tau}\,p^{\kappa_0}d_G^-(x)\,\Phi^-(x),\ \ x\in\mathbb{R}. \tag{1.4}$$

We apply the operator P^- to the left and right-hand sides of the second equality of system of equalities (1.4), taking into account (1.1), the properties of the projecting operators P^- and P_0^- and the following obvious inclusions

$$e^{ix\tau}g_{j1}\in\mathbb{W}_{0+}\ (j=1,2),$$

we obtain the second equality in (1.2). □

From Lemma 1, it follows that the functions Φ_1^+ and Φ_2^- have the following general form:

$$\Phi_1^+(x) = \int_0^\tau \alpha_1(t)e^{ixt}\,dt + C_1,\ \ \Phi_2^-(x) = \frac{p^{-\kappa_0}}{d_G^-(x)}\int_{-\tau}^0 \alpha_2(t)e^{ixt}\,dt,$$

where $\alpha_1\in L_1(0,\tau),\ \ \alpha_2\in L_1(-\tau,0)$.

Now consider the transform of the Riemann problem (0.1)–(0.2),(1.1). It is easy to see that the boundary condition (0.1) can be written in the following equivalent form:

$$\Psi^+(x) = M_\tau(x)\,\Phi^-(x),\ \ x\in\mathbb{R}, \tag{1.5}$$

where

$$M_\tau(x) = \begin{pmatrix} 1+g_{11}(x) & g_{12}(x)+e^{ix\tau}\,p^{\kappa_0}d_G^-(x) \\ g_{21}(x) & 1+g_{22}(x) \end{pmatrix}, \tag{1.6}$$

$$\Psi_1^+(x) = \Phi_1^+(x) + e^{ix\tau}p^{\kappa_0}\,d_G^-(x)\,\Phi_2^-(x)\in\mathbb{W}_+,\ \ \Psi_2^+(x) = \Phi_2^+(x). \tag{1.7}$$

We put

$$w_\tau^-(x) := d_G^-(x) + e^{-ix\tau}\,p^{-\kappa_0}g_{12}(x),\ \ w_\tau^+(x) := d_G^+(x) - e^{ix\tau}p^{\kappa_0-\kappa}g_{21}(x). \tag{1.8}$$

From (1.8), (1.6), and (1.1), it follows that

$$w_\tau^\pm\in\mathbb{W}_\pm,\ \det M_\tau(x) = d_G^-(x)p^\kappa\,w_\tau^+(x).$$

We will assume that

$$w_\tau^\pm(x)\neq 0,\ \ x\in R. \tag{1.9}$$

Note that the case when condition (1.9) is not fulfilled is considered in [2, inequality (1.19)].

By the uniqueness theorem for analytical functions, the number of zeros of the functions $w_\tau^\pm(z)$ on the semiplanes $\pm\,\mathrm{Im}\,z>0$, respectively, is finite. We denote by $z_j^\pm$ $(j=1,...,J^\pm)$ the zeros of the functions $w_\tau^\pm(z)$ on the semiplanes $\pm\,\mathrm{Im}\,z>0$

respectively. Consider two following rational functions with poles at the points z_j^+, $j = 1, ..., J^+$ and z_j^-, $j = 1, ..., J^-$, respectively:

$$Q^{\pm}(x) := \sum_{j=1}^{J^{\mp}} \sum_{l=1}^{n_j^{\mp}} c_{lj}^{\mp}(x - z_j^{\mp})^{-l}, \ (Q^{\pm} = 0, \ J^{\mp} = 0), \tag{1.10}$$

where $n_j^{\mp}$ is multiplicity of the $z_j^{\mp}$ – th zero, $c_{lj}^{\mp}$ $(l = 1, ..., n_j^{\mp}, j = 1, ..., J^{\mp})$ are some constants defined by the inclusions:

$$\frac{\Phi_2^+}{w_\tau^+} - Q^- \in \mathbb{W}_{0+}, \ \frac{\Phi_1^- - C_1}{w_\tau^-} - Q^+ \in \mathbb{W}_{0-}, \tag{1.11}$$

where $\Phi^{\pm}$ is the solution (assuming its existence) of factorization problem (0.1)–(0.2) under restrictions (1.1), (1.9). By the condition, $\Phi_2^+(z), \Phi_1^-(z)$ are regular analytic functions on the semiplanes $\pm \operatorname{Im} z > 0$ respectively, therefore, the constants $c_{lj}^{\mp}$ $(l = 1, ..., n_j^{\mp}, j = 1, ..., J^{\mp})$ can be defined from (1.11) by the Laurent theorem on decomposition of an analytic function in the neighbourhood of its poles.

§2. The method of reducing factorization problem (0.1)–(0.2) to truncated Wiener–Hopf equation (0.5)

We set

$$\hat{k}(x) := 1 - p^{-\kappa} \frac{1 + g_{22}(x)}{w_\tau^+(x) w_\tau^-(x)}. \tag{2.1}$$

From (2.1) and (1.9), by Wiener theorem we have $\hat{k} \in \mathbb{W}_0$, since $\hat{k}(\infty) = 0$. Therefore,

$$k(t) := \mathcal{F}^{-1}\{\hat{k}(x)\}(t) \in L_1(R). \tag{2.2}$$

The factorization problem (0.1)–(0.2) with restrictions (1.1) and (1.9) will be associated with the Wiener–Hopf equation (0.4) with condition (0.5). We will define the right-hand side of the equation by the formulas:

$$f(t) = \mathcal{F}^{-1}\{\hat{f}(x)\}(t), \ t \in (0, \tau), \ f(t) = 0, \ t \notin (0, \tau), \tag{2.3}$$

where

$$\hat{f}(x) = C_1\big(\hat{k}(x) + e^{-ix\tau} g_{12}(x) \frac{p^{-k_0}}{w_\tau^-(x)}\big) + d_G^-(x) Q^+(x) - e^{ix\tau} Q^-(x) \in \mathbb{W}_0. \tag{2.4}$$

The following theorem on the relationship of factorization problem (0.1)–(0.2) and truncated Wiener–Hopf equation (0.4) holds.

Theorem 2. *Suppose that for factorization problem (0.1)–(0.2), restrictions (1.1), (1.9) hold, and for the coefficients (k and f) of the equation in convolutions (0.4), equalities (2.1)–(2.4) hold.*

Then given $\kappa \geq 0$, the solution of the equation in convolutions (0.4) exists, as well as the solution of factorization problem (0.1)–(0.2), and can be expressed in terms of the solution of the factorization problem by the formula:

$$u(t) = \mathcal{F}^{-1}\{\Phi_1^+(x) - C_1 + e^{ix\tau} p^{\kappa_0} d_G^-(x) \Phi_2^-(x)\}(t), \ t \in (0, \tau). \tag{2.5}$$

If $\kappa < 0$ and the solution of factorization problem (0.1)–(0.2) exists, then the solution of the equation in convolutions (0.4) also exists, the latter is expressed in terms of the solution of the factorization problem by formula (2.5).

Proof. Assume that the solution of factorization problem (0.1)–(0.2) exists.

Multiplying on the left the left and right-hand sides of system (1.5) by the matrix M_τ^{-1}, we obtain

$$\frac{p^{-\kappa}}{w_\tau^+(x)d_G^-(x)}\begin{pmatrix} 1+g_{22}(x) & -g_{12}(x)-e^{ix\tau}p^{\kappa_0}d_G^-(x) \\ -g_{21}(x) & 1+g_{11}(x)\end{pmatrix}\Psi^+(x)=\Phi^-(x). \tag{2.6}$$

Dividing the left and right-hand sides of the first equality from system (2.6) by w_τ^-/d_G^-, taking into account the equalities

$$w_\tau^-(x)e^{ix\tau}=p^{-\kappa_0}g_{12}(x)+e^{ix\tau}d_G^-(x),\ \Psi_2^+=\Phi_2^+,$$

we have

$$p^{-\kappa}\frac{(1+g_{22}(x))\Psi_1^+(x)}{w_\tau^+(x)w_\tau^-(x)}-\frac{\Phi_2^+(x)}{w_\tau^+(x)}e^{ix\tau}=d_G^-(x)\frac{\Phi_1^-(x)}{w_\tau^-(x)}.$$

Taking into account formula (2.1) and the inclusions in (1.11), we write the obtained equality in the following equivalent form:

$$\begin{gathered}\big(1-\mathcal{F}k(x)\big)\big(\Psi_1^+(x)-C_1\big)+\big(\frac{\Phi_2^+(x)}{w_\tau^+(x)}-Q^-(x)\big)e^{ix\tau}+\\ C_1\big(1-\mathcal{F}k(x)\big)+e^{ix\tau}Q^-(x)=\\ =d_G^-(x)\big(\frac{\Phi_1^-(x)-C_1}{w_\tau^-(x)}-Q^+(x)\big)+C_1\frac{d_G^-(x)}{w_\tau^-(x)}+d_G^-(x)Q^+(x).\end{gathered} \tag{2.7}$$

Now consider the expression $\Psi_1^+(x)-C_1$. From the first equality in (1.7) and Lemma 1, we have

$$\Psi_1^+(x)-C_1\in\mathbb{W}_{0+},\ e^{-ix\tau}\big(\Psi_1^+(x)-C_1\big)\in\mathbb{W}_{0-}.$$

Therefore, there exists a function $u\in L_1(R),\ u(t)=0,\ t\notin(0,\tau)$, such that

$$\mathcal{F}u(x)=\Psi_1^+(x)-C_1. \tag{2.8}$$

Now we return to equation (2.7). From (2.7), taking into account (2.8) and (2.4), we obtain

$$\begin{gathered}\big(1-\mathcal{F}k(x)\mathcal{F}u(x)+\big(\frac{\Phi_2^+(x)}{w_\tau^+(x)}-Q^-(x)\big)e^{ix\tau}=\\ =d_G^-(x)\big(\frac{\Phi_1^-(x)-C_1}{w_\tau^-(x)}-Q^+(x)\big)+\hat{f}(x).\end{gathered} \tag{2.9}$$

Left and right-hand sides of equality (2.9) belong to the algebra $\mathbb{W}_0$ by construction. Applying to the left and right-hand sides of equality (2.9) the inverse Fourier transform, we obtain the required integral equation (0.4) due to the fact that

$$\mathcal{F}^{-1}\big\{\big(\frac{\Phi_2^+(x)}{w_\tau^+(x)}-Q^-(x)\big)e^{ix\tau}\big\}=0,\ t<\tau,$$

$$\mathcal{F}^{-1}\big\{d_G^-(x)\big(\frac{\Phi_1^-(x)-C_1}{w_\tau^-(x)}-Q^+(x)\big)\big\}(t)=0,\ t>0,$$

$$\begin{gathered}\mathcal{F}^{-1}\big\{\big(1-\mathcal{F}k(x)\big)\mathcal{F}u(x)\big\}(t)=\\ =u(t)-\int_0^\tau k(t-s)u(s)\,ds,\ t\in R.\end{gathered}$$

To complete the proof of the theorem 2, we will prove the following lemma.

Lemma 2. *Suppose that the total index is $\kappa \geq 0$, then a non-trivial solution to Riemann problem (0.1)–(0.2) certainly exists.*

If $\Phi_1^+(\infty) = 0$, then the dimension of the solution space of Riemann problem (0.1)–(0.2) equals $q = \kappa_1$ for $\kappa_2 \leq 0$, $q = \kappa_1 + \kappa_2$ for $\kappa_2 > 0$. In this case, the problem only has a trivial solution when $\kappa_1 = \kappa_2 = 0$.

Proof. Indeed, we multiply on the left the left and right-hand sides of boundary condition (0.1) by G_+^{-1} and, taking into account (0.3), we obtain

$$G_+^{-1}(x)\Phi^+(x) = \{p^{\kappa_1}, p^{\kappa_2}\}G_-(x)\,\Phi^-(x),\ x \in \mathbb{R}, \tag{2.10}$$

where $\kappa_1 + \kappa_2 \geq 0$, $\kappa_2 \leq \kappa_1 \geq 0$ by condition.
The left-hand (right-hand) side of equation (2.10) is an analytic function on the upper (lower) semiplane and is continuous up to the boundary. Then from equation (2.10) by (generalized) Liouville's theorem, we obtain the following chain of equalities to solve the problem:

$$G_+^{-1}(x)\Phi^+(x) = \big(Q_1^+(x), Q_2^+(x)\big)^T = \{p^{\kappa_1}, p^{\kappa_2}\}G_-(x)\,\Phi^-(x), \tag{2.11}$$

where Q_j^+ $(j = 1, 2)$ are rational functions. Moreover,

$$\begin{gathered} Q_1^+(x) = (x+i)^{-\kappa_1}P_{\kappa_1}(x), \\ Q_2^+(x) = (x+i)^{-\kappa_2}P_{\kappa_2}(x) \text{ given } \kappa_2 \geq 0,\ Q_2^+(x) = 0 \text{ given } \kappa_2 < 0, \end{gathered} \tag{2.12}$$

where P_m is a polynomial of degree at most m with arbitrary coefficients.
From (2.12), it follows that

$$Q_1^+(\infty) = c_1;\ Q_2^+(\infty) = c_2 \text{ given } \kappa_2 > 0,\ Q_2^+(\infty) = 0 \text{ given } \kappa_2 < 0, \tag{2.13}$$

where c_j are arbitrary constants, $j = 1, 2$.
From the first equation of the chain of equations (2.11), given $x = \infty$ and condition (2.13), follows the existence of constants c_1, c_2 $\big($for any matrix $G_+(\infty)\big)$, such that $\Phi_1^+(\infty) = 0$. The first part of Lemma 2 is proved, and its second part follows from the chain of equalities (2.11), and is a well-known result (see, for example, [8, Ch. 6]). □

Thus the theorem 2 is proved.

□

It is easy to see that

$$f(t) = C_1 k(t),\ t \in (0, \tau)$$

given that the condition

$$|w_\tau^\pm(z)| > 0,\ \pm\mathrm{Im}\, z \geq 0 \tag{2.14}$$

is fulfilled. Indeed, from (2.14) by definition of functions $Q^\pm$, we have $Q^\pm = 0$. Then, applying to the left and right-hand sides of equality (2.4) the projector P_0^+, we obtain

$$P_0^+\{\hat{f}(x)\} = C_1 P_0^+\{\hat{k}(x)\},$$

since

$$e^{-ix\tau}g_{12}(x)\frac{p^{-k_0}}{w_\tau^-(x)} \in \mathbb{W}_{0-}.$$

Note that Theorem 2, in fact, contains the method of reducing factorization problem (0.1)–(0.2) to truncated Wiener–Hopf equation (0.4).

§3. Applying Theorem 2 to the study of equation (0.4)

We set

$$k(t) := 0,\ t \notin (-\tau, \tau),\ k_{\pm}(t) := \theta(\pm t) k(t),\ t \in R,\ \Lambda^{\pm}(x) := 1 - \mathcal{F}k_{\pm}(x).$$

Consider the case when

$$f(t) = C_1\, k(t), \quad t \in (0, \tau),\ C_1 \in \{1, 0\}. \tag{3.1}$$

We have the following

Theorem 3. *Suppose that condition (3.1) is fulfilled and*

$$\Lambda^{\pm}(x) \neq 0,\ x \in R,\ \operatorname{Ind} \Lambda^{\pm}(x) = 0. \tag{3.2}$$

Then at least one of two following statements holds:
(i) given $C_1 = 1$, equation (0.4) is unconditionally solvable in $L_1(0, \tau)$;
(ii) given $C_1 = 0$, equation (0.4) has a non-trivial solution in $L_1(0, \tau)$.

Proof. With the integral equation (0.4) we associate the factorization problem (0.1)–(0.2), where

$$G(x) = \begin{pmatrix} 1 & e^{ix\tau} m^{-}(x) \\ e^{-ix\tau} m^{+}(x) & 1 + m^{+}(x) m^{-}(x) \end{pmatrix},\ m^{\pm} \in \mathbb{W}_{0\pm}. \tag{3.3}$$

We set

$$m^{+}(x) := -\frac{\mathcal{F}k_{+}(x)}{\Lambda^{+}(x)},\ m^{-}(x) := \frac{\mathcal{F}k_{-}(x)}{\Lambda^{-}(x)},$$

and, taking into account (3.2), directly obtain

$$\frac{1}{1 - m^{+}(x)} = \Lambda^{+}(x) \in \mathcal{G}\mathbb{W}_{+},\ \frac{1}{1 + m^{-}(x)} = \Lambda^{-}(x) \in \mathcal{G}\mathbb{W}_{-}. \tag{3.4}$$

We check that the conditions of Theorem 2 are fulfilled. From (3.3)–(3.4), we have

$$\det\, G(x) = 1,\ d_G^{\pm}(x) = 1,\ \kappa = 0,$$

$$g_{22}(x) = -\frac{\mathcal{F}k_{+}(x)\mathcal{F}k_{-}(x)}{\Lambda^{+}(x)\Lambda^{-}(x)},\ w_{\tau}^{\pm}(x) = \frac{1}{\Lambda^{\pm}(x)} \in \mathcal{G}\mathbb{W}_{\pm}. \tag{3.5}$$

Then from (2.1) and (2.5), taking into account (3.5), we obtain respectively:

$$\hat{k}(x) = \mathcal{F}k_{-}(x) + \mathcal{F}k_{+}(x),\ \hat{f}(x) = C_1\big(\mathcal{F}k_{+}(x) + \mathcal{F}k_{-}(x)\big). \tag{3.6}$$

From (3.1)–(3.6), it follows that all conditions of Theorem 2 are satisfied. By Theorem 2 and Lemma 2, given $\kappa_2 \neq 0$, we obtain that statements (i),(ii) from Theorem 3 hold, given $\kappa_1 = \kappa_2 = 0$, the homogeneous equation (0.4) (when $C_1 = 0$) only has a trivial solution, therefore, only statement (i) of Theorem 3 holds. $\square$

Note that given

$$||k_{\pm}||_{L_1} < 1,$$

condition (3.2) (the condition of Theorem 3) will be apriori fulfilled.

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