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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 18, №2, стр. 1639–1650 (2021) УДК 517.958
DOI 10.33048/semi.2021.18.123 MSC 35Q70

INVARIANT SOLUTIONS OF THE GAS DYNAMICS EQUATIONS FROM 4-PARAMETER THREE-DIMENSIONAL SUBALGEBRAS CONTAINING ALL TRANSLATIONS IN SPACE AND PRESSURE TRANSLATION

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Abstract. The gas dynamics equations with pressure being of the sum of density and entropy functions are considered. The admissible group of transformations is expanded due to the pressure translation. The Lie algebra corresponding to the group is 12-dimensional. Invariant submodels of rank 1 generated by 3-dimensional 4-parameter subalgebras consisting of all translations in space and pressure translation are constructed. Three families of exact solutions are found which describe the motion of particles with a linear velocity field with inhomogeneous deformation. The moment of time of the presence or absence of collapse of particles for each family of solutions are found. In a particular case, the trajectories of particles motion are constructed. The volume of particles at the initial moment of time restricted by the sphere is isolated. It is proved that at any other time moments the volume turns into an ellipsoid and the particles volume value does not change with time.

Keywords: gas dynamics equations, equation of state, admissible subalgebra, invariant submodel, exact solution, linear velocity field.

Siraeva, D.T., Invariant solutions of the gas dynamics equations from 4-parameter three-dimensional subalgebras containing all translations in space and pressure translation.

The author was supported by the Russian Foundation for Basic Research (project no. 18-29-10071) and partially from the Federal Budget by the State Target (project no. 0246-2019-0052).
Received February, 15, 2021, published December, 21, 2021.

1. Introduction

The expediency of studying the equations of continuum mechanics by means of group analysis was substantiated in the work of outstanding scientist L.V. Ovsyannikov «The "podmodeli" program. Gas dynamics» [1]. The gas dynamics equations are considered within the framework of this program

$$D\vec{u}+\rho^{-1}\nabla p=0,\quad D\rho+\rho\,\mathrm{div}\vec{u}=0,\quad Dp+\rho f_\rho \mathrm{div}\vec{u}=0, \tag{1}$$

where $D=\partial_t+(\vec{u}\cdot\nabla)$ is the total differentiation operator; t is the time; $\nabla=\partial_{\vec{x}}$ is the gradient by spatial independent variables $\vec{x}$; $\vec{u}$ is the velocity vector; ρ is the density; p is the pressure. In the Cartesian coordinate system we have

$$\vec{x}=x\vec{i}+y\vec{j}+z\vec{k},\quad \nabla=\vec{i}\partial_x+\vec{j}\partial_y+\vec{k}\partial_z,\quad \vec{u}=u\vec{i}+v\vec{j}+w\vec{k},$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ is an orthonormal basis.

It is known that the system (1) with a general equation of state

$$p=f(\rho,S)$$

admits an 11-dimensional Lie algebra L_{11}. The equations of state for which an extension of the 11-dimensional Lie algebra occurs were listed in [1]. All nonisomorphic Lie algebras of the group classification according to the equation of state were given in [2], for each of which the method of enumerating nonsimilar subalgebras was finally formulated in [3]. Invariant submodels of rank 3, 1 [4, 5], and 2 [6, 7] were constructed for an 11-dimensional Lie algebra.

In this article we consider the system (1) with the special equation of state [1]

$$p=f(\rho)+h(S),\quad f=\rho^2F'(\rho). \tag{2}$$

In this case , the thermodynamic parameters of the ideal medium of specific internal energy and temperature are given by the formulas

$$\varepsilon=F(\rho)-\rho^{-1}h(S)+g(S),\quad T=g'(S)-\rho^{-1}h'(S).$$

The system (1), (2) admits an equivalence transformation for the function $h(S)$. Let the transformation act so that $\tilde{S}=h(S)$. For the inverse function $S=\tilde{h}(\tilde{S})$ the following formulas are valid

$$\tilde{S}=h(\tilde{h}(\tilde{S})),\quad T(\rho,S)=T(\rho,\tilde{h}(\tilde{S}))=\tilde{T}(\rho,\tilde{S}),$$
$$\varepsilon(\rho,S)=\varepsilon(\rho,\tilde{h}(\tilde{S}))=\tilde{\varepsilon}(\rho,\tilde{S}),\quad g(S)=\tilde{g}(\tilde{S}),$$
$$p(\rho,S)=p(\rho,\tilde{h}(\tilde{S}))=\tilde{p}(\rho,\tilde{S}).$$

We obtain the equations of state

$$\tilde{p}=f(\rho)+\tilde{S},\quad \tilde{\varepsilon}=F(\rho)-\rho^{-1}\tilde{S}+\tilde{g}(\tilde{S}),\quad \tilde{T}=\frac{\tilde{g}'(\tilde{S})}{\tilde{h}'(\tilde{S})}-\frac{1}{\rho\tilde{h}'(\tilde{S})}.$$

The entropy S is determined from the equality (2). The last equation of the system (1) can be replaced by an equation for entropy

$$DS=0.$$

The equations (1) with a general equation of state are invariant under the action of the Galilean group extended by uniform dilatation [4]:

(3)
1^o. $\vec{x}^* = \vec{x} + \vec{a}$ (space translations),
2^o. $t^* = t + a_0$ (time translation),
3^o. $\vec{x}^* = O\vec{x}, \vec{u}^* = O\vec{u}, OO^T = E, \det O = 1$ (rotations),
4^o. $\vec{x}^* = \vec{x} + t\vec{b}, \vec{u}^* = \vec{u} + \vec{b}$ (Galilean translations),
5^o. $t^* = ct, \vec{x}^* = c\vec{x}$ (uniform dilatation).

The system (1), (2) is also invariant under the action of the pressure translation [4]:

$$6^o.\ p^* = p + p_0. \tag{4}$$

Each solution of the system (1), (2) up to transformations (3), (4) is a solution again. Therefore, in what follows, the solutions are considered up to the transformations (3), (4).

The 12-dimensional Lie algebra L_{12} corresponds to the transformation group (3), (4). The basis operators of L_{12} in the Cartesian coordinate system have the form [1]

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z,$$

$$X_4 = t\partial_x + \partial_u, \quad X_5 = t\partial_y + \partial_v, \quad X_6 = t\partial_z + \partial_w,$$

$$X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v, \quad X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,$$

$$X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u, \quad X_{10} = \partial_t,$$

$$X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z, \quad Y_1 = \partial_p.$$

All subalgebras of the Lie algebra L_{12} up to internal automorphisms were listed in the optimal system of nonsimilar subalgebras in [8]. Using one-dimensional, two-dimensional and three-dimensional subalgebras, it is possible to construct invariant submodels of the system (1), (2) of rank 3, 2, 1, respectively. The rank of the submodel is the number of independent variables. Two two-dimensional subalgebras of the Lie algebra L_{12} define partially invariant submodels of rank 3 of defect 1. Their reduction to invariant submodels was proved in [9]. Invariant submodels of rank 2 in canonical form for the remaining two-dimensional subalgebras of the Lie algebra L_{12} were constructed in [10, 11]. An example of a description of particles motion for solution from invariant submodel of rank 2 is given in [12]. Invariant submodels of rank 1 are constructed from 3-dimensional subalgebras $3.N$ of Lie algebra L_{12}, where N is the number of the subalgebra.

For two rank 1 submodels constructed from 3-dimensional subalgebras 3.42 ($a \neq 0$, $b = 0$) and 3.47 from L_{12}, exact solutions were obtained [13].

Invariant submodels of rank 1 were constructed in [14, 15] for the gas dynamics equations with the equation of state with the density separated into a product. In the case of a polytropic gas, the invariants of three-dimensional subalgebras were calculated and the submodels were classified [16], 37 of which were investigated in [17], and 95 invariant submodels were considered in [18]. For gas dynamics equations with the equation of state in the form of pressure being equal to the sum of the power density function and the entropy function, invariant submodels of rank 1 were classified in [19]. In the case of a monatomic gas, all three-dimensional subalgebras containing the projective operator were considered in [20, 21]. For 9 of them, invariant submodels of rank 1 were constructed, which are systems of ordinary differential equations. For the remaining 3 subalgebras, regular partially invariant submodels were constructed and their compatibility was investigated.

This article is devoted to the study of invariant submodels of rank 1 constructed from a three-dimensional 4-parameter subalgebra 3.30 of the Lie algebra L_{12} [8], containing all translations in space and pressure translation. Submodels define solutions with a linear velocity field with inhomogeneous deformation. In [22] submodels of particles motion with a linear velocity field, including with a time-dependent density, were written, but explicit representation of solutions wasn't obtained.

2. Invariant submodel of rank 1 for the subalgebra 3.30

The basis operators of the subalgebra 3.30 [8] in the Cartesian coordinate system t, x, y, z, u, v, w have the form

$$\begin{gathered} aX_1 + X_2 = a\partial_x + \partial_y, \quad X_3 + X_4 = t\partial_x + \partial_z + \partial_u, \\ Y_1 + bX_1 + cX_3 + dX_5 + eX_6 = \\ = \partial_p + b\partial_x + dt\partial_y + (et+c)\partial_z + d\partial_v + e\partial_w, \\ b^2 + c^2 + d^2 + e^2 = 1. \end{gathered} \tag{5}$$

Invariants are functions that are vanished by the action of subalgebra operators [23]. The invariants of the subalgebra 3.30 (5) for $b^2 + e^2 + (ad+c)^2 \neq 0$ are as follows

$$\begin{gathered} t, \quad u + \frac{(et+c)(x-ay) + (adt-b)z}{b - t(et+ad+c)}, \quad v + \frac{d(x-ay-tz)}{t(et+ad+c)-b}, \\ w + \frac{e(x-ay-tz)}{t(et+ad+c)-b}, \quad \rho, \quad p + \frac{x-ay-tz}{t(et+ad+c)-b}. \end{gathered} \tag{6}$$

The representation of the invariant solution is chosen as follows. Invariants (6) containing gas-dynamic functions are assigned new functions depending on the invariant of the independent variable. In the representation of the solution, the coefficient γ is introduced to distinguish the submodel from the well-known submodel of the Lie algebra L_{11} [5] ($\gamma = 1$ in the case of L_{12} and $\gamma = 0$ in the case of L_{11}). Representation of an invariant solution with a linear velocity field and inhomogeneous deformation has the form

$$\begin{gathered} u = u_1(t) + \frac{(c+et)(x-ay) + (adt-b)z}{t(et+ad+c)-b}, \\ v = v_1(t) + \frac{d(x-ay-tz)}{b - t(et+ad+c)}, \\ w = w_1(t) + \frac{e(x-ay-tz)}{b-t(et+ad+c)}, \quad \rho = \rho(t), \\ p = p_1(t) + \gamma\frac{x-ay-tz}{b-t(et+ad+c)}, \quad S = S_1(t) + \gamma\frac{x-ay-tz}{b-t(et+ad+c)}, \\ p_1 = f(\rho) + S_1. \end{gathered} \tag{7}$$

Substituting the solution representation (7) into the system (1), (2) leads to an invariant submodel 3.30 of rank 1

$$
\begin{aligned}
u_{1t} &= \frac{1}{m}[(c+et)(u_1 - av_1) - m - t(c+et)w_1 - \gamma\rho^{-1}],\\
v_{1t} &= \frac{1}{m}[-d(u_1 - av_1) + dtw_1 + a\gamma\rho^{-1}],\\
w_{1t} &= \frac{1}{m}[-e(u_1 - av_1) + etw_1 + \gamma t\rho^{-1}],\\
\rho_t &= -\frac{\rho m_t}{m},\\
S_{1t} &= -\frac{\gamma}{m}[u_1 - av_1 - tw_1], \quad p_1 = f(\rho) + S_1,
\end{aligned}
\tag{8}
$$

where $m = b - t(et + ad + c)$.

The motion of the particles is given by the equation [24]:

$$
\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t).
\tag{9}
$$

Integral curves of the equation (9) are the world lines of particles in space $\mathbb{R}^4(t, \vec{x})$, the projections of which in $\mathbb{R}^3(\vec{x})$ are trajectories of particles.

3. Solution from an invariant submodel of rank 1 for $e \neq 0$

The system (8) for $e \neq 0$ has the following integrals

$$
\begin{aligned}
&\rho = \frac{\rho_0}{m}, \quad ev_1 - dw_1 = \frac{\gamma}{\rho_0}\left(aet - d\frac{t^2}{2}\right) + C_0,\\
&eu_1 + (et+c)w_1 = \frac{\gamma t}{\rho_0}\left(\frac{ct}{2} + \left(\frac{t^2}{3} - 1\right)e\right) + cC_1,\\
&etu_1 - aetv_1 + bw_1 = \frac{\gamma t^2}{2\rho_0}\left(b - e(a^2+1)\right) + C_2.
\end{aligned}
\tag{10}
$$

With $e = 0$ from each formula from (10) with $C_0 = -dC_1$, $C_2 = bC_1$ we obtain the integral $w_1 = \dfrac{\gamma t^2}{2\rho_0} + C_1$. Using Galilean translations 4^o from (3) with $\vec{b} = (0, 0, -C_1)$ we obtain $C_0 = C_1 = C_2 = 0$ in integrals (10).

Thus, from the representation of the solution (7) for $e \neq 0$, from the equation for S_1 from (8) and integrals (10) the exact solution of the system of equations (1), (2) has the form

$$
\begin{aligned}
u &= -\frac{1}{m}\left((c+et)(x-ay) + (adt-b)z\right) - \\
&\quad -\frac{\gamma t}{6\rho_0}\left[t^2 + 6 + t(c+et)\frac{K}{m}\right],\\
v &= \frac{d}{m}(x - ay - tz) + \frac{\gamma t}{6\rho_0}\left[6a + \frac{dtK}{m}\right],\\
w &= \frac{e}{m}(x - ay - tz) + \frac{\gamma t^2}{6\rho_0}\left[3 + \frac{eK}{m}\right],\\
S &= \frac{\gamma}{m}(x - ay - tz) + \\
&\quad + \frac{\gamma^2}{6\rho_0 e^3}\left[(ad+c)^2 + be + 3e^2(a^2+1) + \frac{e^3t^2K}{m}\right],\\
\rho &= \frac{\rho_0}{m}, \quad p = S + f(\rho),
\end{aligned}
\tag{11}
$$

where $K = t^2 + 3(a^2+1)$, in the expression for pressure the constant is vanished using (4).

The density from the solution (11) has a singularity at two moments of time $t = t_-$ and $t = t_+$:

$$t_\pm = \frac{ad + c \pm \sqrt{(ad+c)^2 + 4eb}}{-2e}.$$

The formulas (11) for $(ad+c)^2 + 4eb > 0$ define 3 solutions: for $t < t_-$ with collapse of particles, for $t_- < t < t_+$ with source and collapse of particles and for $t > t_+$ with source of particles. For $(ad+c)^2 + 4eb = 0$ we have $t_- = t_+$. Consequently, the formulas (11) specify 2 solutions for $t < t_-$ with collapse of particles and for $t > t_-$ with source of particles. For $(ad+c)^2 + 4eb < 0$ there is one solution for any t, motion of particles doesn't have singularities.

4. Solution from an invariant submodel of rank 1 for $e = 0,\ c + ad \neq 0$

The system (8) for $e = 0, c + ad \neq 0$ has the following integrals

$$\begin{gathered} du_1 + cv_1 = \frac{\gamma t}{\rho_0}\left(ac - d - \frac{dt^2}{6}\right) - dtC_1 + C_3, \\ (b - t(ad+c))(u_1 - av_1) = -bC_1 t - \\ -\frac{\gamma}{\rho_0}\left[\frac{b}{6}t^3 + t(a^2+1)\left(b - \frac{t}{2}(ad+c)\right)\right] + C_4, \\ \rho = \rho_0(b - t(ad+c))^{-1}, \quad w_1 = \frac{\gamma t^2}{2\rho_0} + C_1. \end{gathered} \tag{12}$$

Using Galilean translations 4^o from (3) with

$$\vec{b} = \left(-\frac{abC_3 + cC_4}{b(c+ad)}, \frac{dC_4 - bC_3}{b(c+ad)}, -C_1\right)$$

we obtain $C_1 = C_3 = C_4 = 0$ in the integrals (12). The exact solution of the system of equations (1), (2) from the submodel 3.30 with $e = 0$, $c + ad \neq 0$ up to the transformation 6^o from (4) has the form

$$\begin{gathered}
u=-\frac{c(x-ay)+(adt-b)z}{b-t(ad+c)}+\frac{\gamma t}{\rho_0(c+ad)}\times\\
\times\left[a\left(ac-d-\frac{dt^2}{6}\right)-\frac{c}{6}\frac{bt^2+(a^2+1)(6b-3t(ad+c))}{b-t(ad+c)}\right],\\
v=\frac{d}{b-t(ad+c)}(x-ay-tz)+\frac{\gamma t}{\rho_0(ad+c)}\times\\
\times\left[\frac{d}{6}\frac{bt^2+(a^2+1)(6b-3t(ad+c))}{b-t(ad+c)}+ac-d-\frac{dt^2}{6}\right],\\
w=\frac{\gamma t^2}{2\rho_0},\quad \rho=\rho_0(b-t(ad+c))^{-1},\\
S=\frac{\gamma}{b-t(ad+c)}(x-ay-tz)-\\
-\frac{\gamma^2}{6\rho_0}\left[\frac{t^3}{ad+c}+\frac{b}{(ad+c)^2}t^2+\frac{b^2+3(ad+c)^2(a^2+1)}{(ad+c)^3}t-\right.\\
\left.-\frac{b^2(b^2+3(ad+c)^2(a^2+1))}{(ad+c)^4(b-t(ad+c))}\right],\\
p=S+f(\rho).
\end{gathered}\tag{13}$$

The density in (13) has a singularity at $t=t_0=\dfrac{b}{ad+c}$. The formulas (13) specify two solutions: for $t<t_0$ with collapse of particles and for $t>t_0$ with source of particles.

The world lines of particles on the solution (13) are obtained by integrating (9)

$$\begin{gathered}
x=-\frac{\gamma t^2}{2\rho_0}+(C_3-cC_1)\,t+aC_2+bC_1,\\
y=\frac{a\gamma}{2\rho_0}t^2+dC_1t+C_2,\quad z=\frac{\gamma}{6\rho_0}t^3+C_3,
\end{gathered}\tag{14}$$

where C_1, C_2, C_3 are global Lagrangian coordinates. The Jacobian of the change of variables (14) has the form

$$J=b-t(c+ad).$$

Since the rank of the Jacobi matrix at the moment of collapse is 2, the manifold of collapse of particles is the plane

$$x-ay-\frac{b}{ad+c}z=-\frac{\gamma b^2}{2(ad+c)^2\rho_0}\left[1+a^2+\frac{b^2}{3(ad+c)^2}\right].$$

5. Solution on an invariant submodel of rank 1 for $e=0$, $c+ad=0$

The integrals of the submodel 3.30 (8) for $e=0$, $c+ad=0$ are as follows

$$\begin{gathered}
\rho=\frac{\rho_0}{b},\quad w_1=\frac{\gamma t^2}{2\rho_0}+C_1,\\
u_1-av_1=-\frac{\gamma t^3}{6\rho_0}-C_1t-\frac{\gamma}{\rho_0}(a^2+1)t+C_5,\\
v_1=\frac{\gamma dt^4}{6b\rho_0}+\frac{d}{b}\left((a^2+1)\frac{\gamma}{2\rho_0}+C_1\right)t^2+\left(\frac{a\gamma}{\rho_0}-\frac{d}{b}C_5\right)t+C_6.
\end{gathered}\tag{15}$$

Using Galilean translations 4^o from (3) with $\vec{b} = (-C_5 - aC_6, -C_6, -C_1)$ we obtain $C_1 = C_5 = C_6 = 0$ in the integrals (15).

The exact solution of the system of equations (1), (2) from the submodel 3.30 for $e = 0$, $c + ad = 0$ up to the transformation (4) has the form

$$
\begin{aligned}
u &= \frac{1}{b}(ad(x - ay) - (adt - b)z) + \\
&+ \frac{\gamma}{\rho_0}\left[\frac{ad}{6b}t^3 - \frac{1}{6}t^2 + \frac{ad}{2b}(a^2+1)t - 1\right]t, \\
v &= \frac{d}{b}(x - ay - tz) + \frac{\gamma}{\rho_0}\left[\frac{d}{6b}t^3 + \frac{d}{2b}(a^2+1)t + a\right]t, \\
w &= \frac{\gamma t^2}{2\rho_0}, \quad \rho = \frac{\rho_0}{b}, \quad S = \frac{\gamma}{b}(x - ay - tz) + \frac{\gamma^2 t^2}{2b\rho_0}\left[\frac{t^2}{3} + a^2 + 1\right], \\
p &= S + f(\rho).
\end{aligned} \tag{16}
$$

The solution (16) is the solution with a linear velocity field with inhomogeneous deformation and isochoric.

6. Trajectories of particles motion and the motion of particles volume

The world lines of particles on the solution (16) are given by the formulas

$$
\begin{aligned}
x &= -\frac{\gamma t^2}{2\rho_0} + C_1 t + C_2, \\
y &= \gamma\frac{at^2}{2\rho_0} + \frac{t}{a}(C_1 - C_3) + \frac{C_2}{a} - \frac{b}{a^2 d}(C_1 - C_3), \quad z = \gamma\frac{t^3}{6\rho_0} + C_3,
\end{aligned} \tag{17}
$$

where C_1, C_2, C_3 are global Lagrangian coordinates; C_1 is the particle velocity along x at $t = 0$; C_3 is projection of a particle onto an axis z at $t = 0$; C_2 is projection of a particle onto an axis x at $t = 0$. The Jacobian of the change of variables (17) is constant $b/(a^2 d)$, therefore, there is no collapse of particles.

Proposition 1. *The world lines of particles* (17) *do not intersect.*

Proof. The world lines of particles (17) in vector form are as follows

$$
\vec{x} = \vec{a}(t) + \Omega(t)\vec{C}, \quad \Omega = \begin{Vmatrix} t & 1 & 0 \\ t/a - b/(a^2 d) & 1/a & b/(a^2 d) - t/a \\ 0 & 0 & 1 \end{Vmatrix},
$$

$$
\vec{a} = \begin{pmatrix} -\gamma t^2/(2\rho_0) \\ \gamma a t^2/(2\rho_0) \\ \gamma t^3/(6\rho_0) \end{pmatrix}, \quad \vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix},
$$

where Ω is the Jacobi matrix, $|\Omega| \neq 0$,

$$
\Omega^{-1} = \begin{Vmatrix} ad/b & -a^2 d/b & -adt/b + 1 \\ -adt/b + 1 & a^2 dt/b & adt^2/b - t \\ 0 & 0 & 1 \end{Vmatrix}.
$$

Let for $t = t_0$ the particles are at different points in the space $\vec{x}_1$ and $\vec{x}_2$, $\vec{x}_1 \neq \vec{x}_2$.

The coordinates of the points $\vec{x}_1$, $\vec{x}_2$ at the moment of time $t = t_0$ have the form

$$
\vec{x}_1 = \vec{a}(t_0) + \Omega(t_0)\vec{C}_1, \quad \vec{x}_2 = \vec{a}(t_0) + \Omega(t_0)\vec{C}_2,
$$

from which follows

$$
\vec{C}_i = \Omega^{-1}(t_0)(\vec{x}_i - a(t_0)), \quad i = 1, 2.
$$

World lines of particles passing at time $t = t_0$ through points with coordinates $\vec{x}_1$ и $\vec{x}_2$, have the form

$$\vec{x}_i(t) = \vec{a}(t) + \Omega(t)\Omega^{-1}(t_0)(\vec{x}_i - a(t_0)), \quad i = 1, 2. \tag{18}$$

Let the world lines of the particles (18) intersect at the time $t = t_1 \neq t_0$, then the following relation is valid

$$\Omega(t_1)(\Omega^{-1}(t_0)(\vec{x}_1 - a(t_0))) = \Omega(t_1)(\Omega^{-1}(t_0)(\vec{x}_2 - a(t_0))). \tag{19}$$

Since $|\Omega| \neq 0$ for any t, from (19) it follows that

$$\vec{x}_1 = \vec{x}_2,$$

which contradicts the original assumption. This means that the world lines of the particles (17) do not intersect. □

Let further $b = d \neq 0$, $a = 1$, $\rho_0 = 1/2$, $\gamma = 1$. The initial conditions are as follows $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$. The motion of particles is vortex, since $\mathrm{rot}\vec{u} = (t, 1 - t, 2)$. The trajectories (17) have the form

$$\begin{gathered} x = -t^2 + (x_0 - y_0 + z_0)t + x_0, \quad y = t^2 + (x_0 - y_0)t + y_0, \\ z = \frac{t^3}{3} + z_0, \end{gathered} \tag{20}$$

where x_0, y_0, z_0 are local Lagrangian coordinates. The trajectories of motion of particles (20) at $t = 0$ located on a square with a side of unit length are shown in Figure 1.

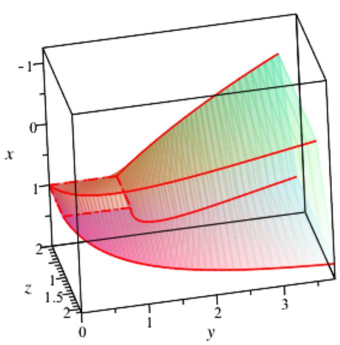

FIGURE 1. A surface of trajectories of particles (20) for $x_0 = 1$, $0 < y_0 < 1$, $0 < z_0 < 1$, $t = 0..1.5$

Let for $t = 0$ particles with trajectories (20) are on the sphere

$$x_0^2 + y_0^2 + z_0^2 = r^2. \tag{21}$$

Proposition 2. *For $t > 0$ the sphere (21) turns into an ellipsoid (Figure 2). Over time, the value of the moving volume consisting of the same particles does not change.*

Proof. On expressing x_0, y_0, z_0 from (20) and substituting into the equation of the sphere (21), we get a quadratic form specifying the location of particles for any t. The invariants of the second-order surface [25] show that for $t > 0$ the location of the particles is an ellipsoid. Since the Jacobian of the change of variables (20) is equal to 1, over time, the value of the moving volume consisting of the same particles does not change. □

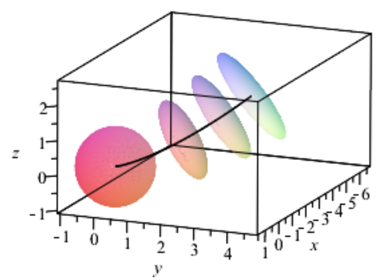

FIGURE 2. The motion of the particles volume for $r = 1$, $t = 0; 1.2; 1.5; 1.7$; the curve is the location of centers of the ellipsoids (20) for $x_0 = y_0 = z_0 = 0$.

Calculations in the Maple 2018.2 computer mathematics system show that one of the axes of the ellipsoid tends to zero (Figure 2).

7. CONCLUSION

In this work, for 4-parameter three-dimensional subalgebras consisting of all translations in space and pressure translation 3.30 of a 12-dimensional Lie algebra admitted by the gas dynamics equations with pressure being the sum of density and entropy functions, invariants have been calculated, invariant submodels of rank 1 have been constructed, and three families of exact solutions have been obtained. The solutions define the motion of particles in a space with a linear velocity field with inhomogeneous deformation. The first family of solutions have two moments of time of density collapse. The second family of solutions has one moment of time of collapse of particles onto a plane. The third family of solutions has no collapse, the world lines of particles do not intersect. The spherical volume of the particles isolated at the initial moment of time turns into an ellipsoid at subsequent moments of time.

References

[1] L.V. Ovsyannikov, *The "PODMODELI" program. Gas dynamics*, J. Appl. Math. Mech., **58**:4 (1994), 601–627. Zbl 0890.76070

[2] S.V. Khabirov, *Nonisomorphic Lie algebras admitted by gas dynamic models*, Ufim. Mat. Zh., **3**:2 (2011), 87–90. Zbl 1249.76076

[3] S.V. Khabirov, *Optimal system for sum of two ideals admitted by hydrodynamic type equations*, Ufim. Mat. Zh., **6**:2 (2014), 94–103. Zbl 1374.76189

[4] Yu.A. Chirkunov, S.V. Khabirov, *Elements of symmetry analysis of differential equations of continuum mechanics*, NSTU, Novosibirsk, 2012.

[5] S.V. Khabirov, *Lectures analytical methods in gas dynamics*, BSU, Ufa, 2013.

[6] E.V. Mamontov, *Invariant submodels of rank two to the equations of gas dynamics*, J. Appl. Mech. Tech. Phys., **40**:2 (1999), 232–237. Zbl 0935.76075

[7] E.V. Mamontov, *Group properties of 2-submodels for the stationary class of gas-dynamic equations*, Vestn. Novosib. Gos. Univ., Ser. Mat. Mekh. Inform., **7**:1 (2007), 72–84. Zbl 1249.35008

[8] D.T. Siraeva, *Optimal system of non-similar subalgebras of sum of two ideals*, Ufim. Mat. Zh., **6**:1 (2014), 94–107. Zbl 1299.35018

[9] D.T. Siraeva, *Reduction of partially invariant submodels of rank 3 of defect 1 to invariant submodels*, Multiphase Systems, **13**:3 (2018), 59–63.

[10] D.T. Siraeva, *Classification of rank 2 stationary submodels of ideal hydrodynamics*, Chelyabinskii Fiz.-Mat. Zh., **4**:1 (2019), 18–32. Zbl 7405303

[11] D.T. Siraeva. *The canonical form of the rank 2 invariant submodels of evolutionary type in ideal hydrodynamics*, J. Appl. Ind. Math., **13**:2 (2019), 340–349. Zbl 1438.76028

[12] D.T. Siraeva, S.V. Khabirov, *Invariant submodel of rank 2 on subalgebra of translations linear combinations for a hydrodynamic type model*, Chelyabinskii Fiz.-Mat. Zh., **3**:1 (2018), 38–57. Zbl 1465.35341

[13] D.T. Siraeva, *Two invariant submodels of rank 1 of the hydrodynamic type equations and exact solutions*, J. Phys.: Conf. Ser., **1666** 012049 (2020). https://doi.org/10.1088/1742-6596/1666/1/012049

[14] E.V. Makarevich, *Invariant and partially invariant solutions with respect to Galilean shifts and dilatation*, Ufim. Mat. Zh., **5**:3 (2013), 121–129. Zbl 1299.76221

[15] E.V. Makarevich, *Gasdynamics equations submodels hierarchy in case of state equation with separated density*, Sib. Èlektron. Mat. Izv., **9** (2012), 306–328. Zbl 1330.76121

[16] A.A. Cherevko, *Group-theoretical solutions to gas dynamic equations generated by three-dimensional Lie subalgebras*, Sib. Èlektron. Mat. Izv., **4** (2007), 553–595. Zbl 1132.76307

[17] A.I. Golod, A.P. Chupakhin, *Invariant solution of dynamics of polytropic gas generated by three-dimensional algebras of symmetry*, Sib. Èlektron. Mat. Izv., **5** (2008), 229–250. Zbl 1299.76209

[18] E.V. Mamontov, *Invariant solutions to the dynamic of polytropic gas generated by a three-dimensional Lie subalgebras*, Sib. Èlektron. Mat. Izv., **6** (2009), 53–109. Zbl 1299.35203

[19] L.Z. Urazbakhtina, *Invariant submodels of the rank one of gas dynamics with special state equation*, Ufim. Mat. Zh., **1**:3 (2009), 139–153. Zbl 1228.76127

[20] R.F. Nikonorova, *The lowest-rank monatomic gas submodels constructed on the basis of three-dimensional symmetry subalgebras*, Sib. Èlektron. Mat. Izv., **15** (2018), 1216–1226. Zbl 1404.35369

[21] R.F. Shayakhmetova, *Vortex scattering of monatomic gas along plane curves*, J. Appl. Mech. Tech. Phys., **59**:2 (2018), 241–250. Zbl 1443.76196

[22] Yu.V. Tarasova, *Classification of submodels with a linear velocity field in gas dynamics*, J. Appl. Ind. Math., **4**:4 (2010), 570–577. Zbl 1228.76125

[23] L.V. Ovsyannikov, *Group analysis of differential equations*, Academic press, New York etc., 1982. Zbl 0485.58002

[24] L.V. Ovsyannikov, *Lectures on the fundamentals of gas dynamics*, Institut komp'yuternykh issledovaniy, Moskva-Izhevsk, 2003.

[25] I.N. Bronshtein, K.A. Semendyayev, *A guide-book to mathematics for technologists and engineers*, Pergamon Press, Oxford etc., 1964. Zbl 0114.00201

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