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SPECIAL CLASSES OF POSITIVE PREORDERS

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Abstract. We study positive preorders relative to computable reducibility. An approach is suggested to lift well-known notions from the theory of ceers to positive preorders. It is shown that each class of positive preoders of a special type (precomplete, e-complete, weakly precomplete, effectively finite precomplete, and effectively inseparable ones) contains infinitely many incomparable elements and has a universal object. We construct a pair of incomparable dark positive preorders that possess an infimum. It is shown that for every non-universal positive preorder P, there are infinitely many pairwise incomparable minimal weakly precomplete positive preorders that are incomparable with P.

Keywords: positive preorder, ceer, computable reducibility, precomplete, weakly precomplete, minimal preorder.

Introduction

The work is dedicated to studying positive preorders on the set of natural numbers ω. By a positive preorder we mean a computably enumerable reflective and transitive binary relation. A special case of a positive preorder–positive equivalence (ceer) –is sufficiently well studied. Classic works in this area include the works by Yu.L. Ershov [1], K. Bernardi and A. Sorbi [2], A. Lachlan [3], S.A. Badaev [4], S. Gao and P. Gerdes [5]. The current state of study the ceers is given in [6], articles [7]–[10], and works by other authors.

If E is a ceer, then it is possible to define functions and relations on the factor set ω/E and obtain interesting structures with relatively computable properties. Our motivation for such approach was the work by F. Montagna and A. Sorbi [11], where

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for E the provable equivalence relation for propositions from Peano arithmetic is taken, and operations on ω/E are naturally generated by logical connectives. Some modern studies in the area of ceers go in the opposite direction, solving, for example, issues on representability of linear orders [9] or Boolean algebras [10] in the form of a quotient structure ω/E for a suitable ceer E. In this work, for every positive preorder P we introduce the notion of so called *support* E of preorder P, that is, the maximal ceer contained in the preorder P, and actually work with a partially ordered set ω/E. Based on the notion of support, we generalize some known notions and results of the theory of ceers to the positive preorders.

If R and S are preorders on ω, we say that R *is computably reducible* to S (denoted by $R \leq_c S$), if there exists a computable function f, such that $xRy \Leftrightarrow f(x)Sf(y)$ for all $x, y \in \omega$. Preorders R and S are said to be *equivalent* (denoted by $R \equiv_c S$), if $R \leq_c S$ and $S \leq_c R$. Preorders R and S are called *computably isomorphic*, if there exists a computable permutation of the set ω, that reduces R to S. A positive preorder is called *universal*, if any positive preorder is computably reducible to it. A well-known notion of *universal ceer* has been introduced in a similar way, as the maximal one among all ceers relative to computable reducibility.

In Section 1 of this work, we provide definitions of precompleteness, weak precompleteness, e-completeness, effectively finite precompleteness, and effective inseparability of positive preorders, and prove the existence of universal positive preorders of the above-mentioned types. In Section 2, we study the questions of existence of supremum and infimum of incomparable positive preorders. And in Section 3, we show for an arbitrary non-universal positive preorder P the existence of an infinite number of pairwise incomparable weakly precomplete minimal positive preorders, incomparable to P.

1. Special classes of positive preorders

Recall some notions that turned out to be helpful in the studies on universal ceers. The notion of precomplete equivalence was introduced by A.I. Maltsev in work [12]. It played a significant role in different areas of the computability theory, including the theory of ceers. An equivalence E is called *precomplete*, if it contains at least two equivalence classes and for every partial computable function φ there exists a computable function f such that $\varphi(x)Ef(x)$ for all $x \in \mathrm{dom}(\varphi)$.

According to the definition by Lachlan [3], a ceer S is called *e-complete* (from *extension complete*), if for every ceer R, every finite function f, which is an embedding of $\langle \mathrm{dom}(f), R \restriction \mathrm{dom}(f)\rangle$ into $\langle \omega, S\rangle$, and any number $i \in \omega \setminus \mathrm{dom}(f)$, by the number i, a canonical index of the function f, and a computably enumerable index of the relation R, it is possible to effectively find a number j such that $f \cup \{\langle i, j\rangle\}$ is an embedding of $\langle \mathrm{dom}(f) \cup \{i\}, R\rangle$ into $\langle \omega, S\rangle$. In [3], it was established that e-complete ceers and precomplete ceers are universal and form two different types of computable isomorphism.

Definition 1. *Let P be an arbitrary preorder. We will refer to the equivalence relation $E = \{(x, y) :\ xPy \ \&\ yPx\}$ as a support of the preorder P and denote the equivalence E by* $\mathrm{supp}(P)$.

Obviously, the support $\mathrm{supp}(P)$ of a positive preorder P is a ceer. However, the inverse is not correct. Indeed, if X is not a computably enumerable set, then the preorder $Q = \{(2x, 2x+1) :\ x \in X\}$ is not positive, but its support is the identity equivalence relation Id.

Remark 1. *Given any preorders P_1, P_2, if $P_1 \leq_c P_2$, then* $\text{supp}(P_1) \leq_c \text{supp}(P_2)$.

The inverse statement is not true. For example, $\text{supp}(Q) \leq_c \text{supp}(\text{Id})$, but $Q \nleq_c \text{Id}$.

Definition 2. *We refer to a preorder P as precomplete (weakly precomplete, e-complete, effectively finite precomplete, or effectively inseparable), if the support of the preorder P is a precomplete (weakly precomplete, e-complete, effectively finite precomplete, or effectively inseparable, respectively) equivalence.*

Theorem 1. *For every positive preorder P, there exists a precomplete positive preorder Q such that $P \leq_c Q$.*

Proof. Let P be an arbitrary positive preorder, and E be a universal precomplete ceer. Then $\text{supp}(P) \leq_c E$ by some computable function f. We define the positive preorder Q in the following way: for all $x, y \in \omega$,

$$xQy \Leftrightarrow xEy \vee (\exists u)(\exists v)[xEf(u) \ \& \ yEf(v) \ \& \ uPv]$$

It is easy to see that $P \leq_c Q$ and $\text{supp}(Q) = E$. Due to the choice of E, the preorder Q is precomplete. □

Corollary 1. *Universal precomplete positive preorders exist.*

Proof. It is sufficient to take any universal positive preorder for P in the proof of the theorem [13]. □

Precomplete ceers are universal [2], and therefore, they all are pairwise equivalent. Moreover, nontrivial precomplete ceers form the unique type of computable isomorphism [3]. For precomplete positive preorders, the picture is completely different, as it follows from the next corollary. In particular, precomplete positive preorders can be non-universal.

Corollary 2. *There are infinitely many non-equivalent precomplete positive preorders.*

Proof. Consider a computable sequence $\{P_k\}_{k\in\omega}$ of positive preorders, such that

- $\text{supp}(P_k) = \text{Id}_{k+1}$,
- $(\forall i \leq j \leq k)([i]_{\text{Id}_{n+1}} P_k [j]_{\text{Id}_{n+1}})$.

Every P_k is a positive preorder, whose quotient over the support is a linear order, isomorphic to the order $\{0 < 1 < 2 < \ldots < k\}$. Obviously, $P_j \nleq_c P_i$ for all $i < j$. We denote by Q_k a positive preorder which can be constructed similarly as in the proof of Theorem 1, where for P we take a positive preorder P_k. Then $Q_j \nleq_c Q_i$ for all $i < j$. □

It is well known that every of the following classes:

- weakly precomplete ceers,
- e-complete ceers,
- effectively finite precomplete ceers,
- effectively inseparable ceers,–

contains universal equivalences, see the survey [6]. Applying the construction from the proof of Theorem 1, we obtain the following

Corollary 3. *For every positive preorder P there exists a preorder Q from each of the classes listed above, such that $P \leq_c Q$.*

2. Suprema and infima

Relative to computable reducibility $\leq_c$, the set of positive preorders is a preorder structure whose quotient structure over the equivalence $\equiv_c$ forms a partially ordered set with respect to the ordering induced by $\leq_c$. Not entirely correctly, we will say that a positive preorder P consists of equivalence classes (to be exact, equivalence classes of its support $\mathrm{supp}(P)$), connected by the relation P.

One of the most important problems in studying of any ordered structure is the question of existence of the least upper bounds and greatest lower bounds. In paper [7], the questions of existence of suprema and infima with respect to computable reducibility were studied. The results of this work are presented in the tables below:

X	Y	$X \wedge Y$?
light	light	-sometimes NO -sometimes YES
dark	dark	NO
light	dark	NO

Table 1. *The existence of infima in the structure of ceers*

X	Y	$X \vee Y$?
light	light	-sometimes NO -sometimes YES
dark	dark	NO
light	dark	-sometimes NO -sometimes YES

Table 2. *The existence of suprema in the structure of ceers*

Of course, in these tables, incomparable ceers X and Y were considered.

Recall (see, for example, [6]) that by Id_n we denote an equivalence with n computable classes. By Id the identity equivalence is denoted. An equivalence E is called *dark*, if E is incomparable to Id. An equivalence E with infinitely many classes is called *light*, if it is not dark.

Using the natural approach outlined in Section 1, we will introduce the notions of dark, light, and finite preorders in the following way.

Definition 3. *A positive preorder P is called dark (finite, light), if the support* $\mathrm{supp}(P)$ *is a dark equivalence (consists of a finite number of classes; is neither finite nor dark equivalence).*

Note that in work [8] a slightly different notion of a dark positive preorder P was introduced for the case where P induces a linear order on $\mathrm{supp}(P)$.

Since ceers are positive preorders, for the questions of existence of suprema and infima in the tables provided above we do not have to consider cases with answers of the kind "sometimes YES, sometimes NO". In this paper, we will only consider the case where both preorders are dark.

Theorem 2. *There exist dark positive preorders that have the greatest lower bound.*

Proof. Let S be a simple set. Then $R_S = \{(x, y) : x = y \vee \{x, y\} \subseteq S\}$ is a dark ceer. We define preorders P and Q in the following way:

$$P = R_S \oplus \mathrm{Id}_1;$$
$$Q = R_S \oplus \mathrm{Id}_1 \cup \{(2x, 2y+1) : x \in S\}.$$

Obviously, the positive preorders P and Q are dark. We will show that they are incomparable, and R_S is their greatest lower bound. It is easy to see that the preorder P is a ceer and that the positive preorder Q has exactly two related equivalence classes $2S$ and $2\omega+1$, in particular, Q is not an equivalence. Obviously, each preorder that is computably reducible to an equivalence is an equivalence itself. Therefore, we directly obtain that $Q \nleq_c P$.

Let us prove that $P \nleq_c Q$. Assume that $P \leq_c Q$ by a computable function f. Due to reducibility, the image of the function f cannot intersect both Q-classes $2S$ and $2\omega + 1$. If $\mathrm{range}(f) \cap 2S = \emptyset$, then $2S$ is the preimage of a computable set, which is impossible. But if $\mathrm{range}(f) \cap 2S \neq \emptyset$, then the function $f([\frac{x}{2}])$ provides the non-surjective reducibility $R_S \leq_c R_S$, which is also impossible. Hence, the positive preorders P and Q are incomparable.

It is clear that R_S is a lower bound of P and Q. Suppose that a positive preorder T is a lower bound of the preorders P and Q. Since P is an equivalence and $T \leq_c P$, then the preorder T is also an equivalence. Let $T \leq_c Q$ by a computable function g. Similarly to the above, note that $\mathrm{range}(g)$ cannot intersect both related Q-classes $2S$ and $2\omega + 1$ and consider two cases.

Case 1. $\mathrm{range}(g) \cap 2S = \emptyset$. Since all Q-classes of equivalence, except for the class $2S$, are computable, then all equivalence classes T are also computable. We will prove by contradiction that the number of equivalence classes in T is finite. Assume that T has infinitely many equivalence classes. Then $\mathrm{range}(g)$ consists of an infinite number of Q-classes. Therefore, the computably enumerable set $\mathrm{range}(g) \setminus (2\omega+1 \cup 2S)$ is infinite, which is impossible since the set S is simple. Hence, T is a partition of ω into a finite number of computable sets, and therefore, $T \leq_c R_S$.

Case 2. $\mathrm{range}(g) \cap 2S \neq \emptyset$. Then $\mathrm{range}(g) \cap 2\omega + 1 = \emptyset$ and the computable function $\frac{g(x)}{2}$ provides the reducibility of the equivalence T to R_S.

Therefore, R_S is the greatest lower bound of the positive preorders P and Q. □

3. Weakly precomplete minimal positive preorders

In the paper [7] by U. Andrews and A. Sorbi the following fact was established for ceers.

Theorem 3. *Let R be a non-universal ceer. Then there exist infinitely many pairwise incomparable ceers $\{E_l\}_{l\in\omega}$, such that for all l and a ceer X*

(1) $E_l \nleq R$
(2) $X \leq E_l \Rightarrow (\exists n)[X \leq \mathrm{Id}_n]$

Note that in Theorem 3, there is not necessary to construct an infinite sequence of equivalences $\{E_l\}_{l\in\omega}$, it suffices to construct only one. Moreover, we can additionally require that this equivalence was weakly precomplete. Using the reasoning from the proof of Theorem 3, we will prove its analogue for positive preorders.

Definition 4. *A preorder P is called minimal, if for every preorder X the reducibility $X \leq_c P$ leads to the reducibility $P \leq_c X$ or finiteness of the number of equivalence classes of the support* $\mathrm{supp}(P)$.

Theorem 4. *For every non-universal positive preorder R, there exists a weakly precomplete minimal positive preorder P, which is not reducible to R.*

Proof. Let R be a non-universal positive preorder. We will construct a positive preorder P, fulfilling the following requirements for all $i, j, k, e \in \omega$:

$P_{i,j}$: If a set W_i intersects infinitely many equivalence classes of $\mathrm{supp}(P)$, then it intersects the class $[j]_{\mathrm{supp}(P)}$.

T_k : φ_k is not a reducing function of P to R.

WP_e : If φ_e is total, then $\exists x_e((x_e, \varphi_e(x_e)) \in \mathrm{supp}(P))$.

The main operation in our construction is the one of reflexive and transitive closure, denoted by the symbol $*$: for an arbitrary binary relation X, by X^* we denote the smallest preorder containing X. Every time when one or several pairs of numbers is added into the preorder under construction, for the obtained binary relation immediately the operation $*$ is performed.

We denote by $(R^s)_{s\in\omega}$ an uniformly computable sequence of positive preorders with the following properties: $R^0 = \mathrm{Id}, \bigcup_{s\in\omega} R^s = R$, for all s the following relation holds: $R^{s+1} = (R^s \cup \{(x_s, y_s)\})^*$ for some pair of numbers (x_s, y_s).

By a *large* number we mean the one that is larger than all numbers that have already been used in the construction.

An isolated strategy for satisfying the requirement $P_{i,j}$:

(1) Choose a large number t.

(2) If $W_i \cap [j]_{\mathrm{supp}(P)} = \emptyset$, then wait until $W_i \cap [k]_{\mathrm{supp}(P)} \neq \emptyset$ for some $k \geq t$.

(3) Add the pairs (j, k) and (k, j) into P.

An isolated strategy for satisfying the requirement T_k:

We fix a universal positive preorder U.

(1) Choose *fresh* (that is, never used before) numbers a_0, a_1 and assume that the parameter r of this strategy equals 1.

(2) For all distinct $i, j \leq r$, restrain adding the pairs (a_i, a_j) in P, if they are not already in P. This restraining applies only to the strategies distinct from T_k.

(3) Wait until all values of $\varphi_k(a_0), \varphi_k(a_1), ..., \varphi_k(a_r)$ become defined on some stage s.

(4) If for all $i, j \leq r$

$$a_i P^s a_j \Leftrightarrow \varphi_k(a_i) R^s \varphi_k(a_j), \qquad (\dagger)$$

then add the pairs (a_i, a_j) into P for all $i, j \leq r$ such that $iU^s j$, increase by 1 the value of the parameter r, choose a fresh element a_r and go to stage (2).

An isolated strategy for satisfying the requirement WP_e:

(1) Choose a fresh number x_e.

(2) Wait until the value of $\varphi_e(x_e)$ becomes defined.

(3) Add the pairs $(\varphi_e(x_e), x_e)$ and $(x_e, \varphi_e(x_e))$ into P.

It is clear that fulfillment of all the requirements $T_k, k \in \omega$, and $WP_e, e \in \omega$, ensures irreducibility of the preorder P to R and weak precompleteness of P. We will show that fulfillment of all requirements $P_{i,j}, i, j \in \omega$, ensures that the preorder P is minimal. Indeed, let X be an arbitrary positive preorder reducible to P by some computable function f. And suppose that W_i is the range(f). If the set W_i intersects only a finite number of equivalence classes of $\mathrm{supp}(P)$, then only a positive preorder whose support consists of a finite number of equivalence classes can be the preimage of f. But if W_i intersects infinitely many equivalence classes, then the strategies $P_{i,j}, j \in \omega$, in total yeild the equality $[\mathrm{range}(f)]_{\mathrm{supp}(P)} = \omega$, and due to positivity of the preorder P, it is easy to construct a computable reducing function for the reducibility $P \leq_c X$. Therefore, P is the minimal positive preorder.

Strategy conflicts. Obviously, the strategies $P_{i,j}$ and WP_e do not conflict with each other. Conflicts can arise if one of these strategies wants to add some pair of numbers in P, and this pair is already restrained by some strategy T_k. Note that restrains are only imposed by strategies T_k. It might seem that the strategy $T_{k'}$ for $k' \neq k$ can prevent the strategy T_k from enumerating in P the pair (a_i, a_j) when performing stage (4) by restraining this pair. But that does not happen, since the parameters $a_0, a_1, \ldots, a_r$ in both strategies $T_{k'}$ and T_k are chosen among fresh numbers. Due to the same reason, parameters x_e, chosen by the strategy WP_e, cannot be a component of any pair restrained by any of the strategies T_k. Therefore, in the construction of the preorder P, we only need to care about solving the conflicts between the strategies $P_{i,j}$ and T_k.

The way of solving conflicts of such kind is well known – that is the priority argument, in this case, the finite-injury priority argument. We assume that all the strategies are linearly ordered: $S_0, S_1, S_2, \ldots, S_k, \ldots$, moreover, by the number k we can effectively reconstruct the type of strategy and its indices. We will say that the strategy S_k has a higher priority compared to S_n, if $k < n$.

On each stage s of the construction, we consider exactly one of the strategies $S_l, l \in \omega$, denoted by S^s, moreover, S^s is a computable function of the argument s and for every l there exist infinitely many s such that $S^s = S_l$. By the end of any stage s, the strategy S_l can either be *active*, or *passive*. On stage 0, all strategies are passive. The strategy that is active (passive) on stage s remains so until on some of the following stages it is declared passive (active, respectively). An active strategy that turned into a passive one can be reactivated on some of the further stages.

Activation (reactivation) of a strategy S_l means choosing its parameters: for the strategies $P_{i,j}$, T_k, and WP_e, that is performing point (1) of these strategies, and the parameters are the numbers t, r, a_0, a_1, and x_e, respectively. In the course of construction, the value of the parameter r of the strategy T_k can be increased, and fresh numbers a_r can be added to the list of its parameters, as it is described at stage (4) of this strategy. Apart from the parameters r, a_0, a_1, the strategy T_k during its activation forms the set $\mathrm{res}(k)$, consisting of pairs of numbers which are restrained from adding to P by the strategy T_k to all strategies of lower priority. During activation (reactivation), $\mathrm{res}(k)$ consists of two pairs (a_0, a_1) and (a_1, a_0), to which new pairs are added as the parameter r increases, based on stage (4) of the description of the strategy T_k.

Declaring a strategy to be passive is accompanied by turning its parameters into undefined ones, and the set of restraints $\mathrm{res}(k)$ of the strategy T_k is emptied. After that, the strategy reactivates at a larger stage.

Construction. The preorder P is constructed by stages. By P^s we denote a preorder on ω, consisting of a finite number of pairs (x, y) with $x \neq y$, constructed by the end of stage s.

We will say that an active strategy S^{s+1} *requires attention*, if

(1) $S^{s+1} = S_l$ and S_l is the strategy $P_{i,j}$ for some i, j and the following condition is fulfilled: if $W_i^s \cap [j]_{\mathrm{supp}(P^s)} = \emptyset$, then $W_i^s \cap [m]_{\mathrm{supp}(P^s)} \neq \emptyset$ for some $m \geq t$ such that $\{(j,m),(m,j)\} \cap \mathrm{res}(l') = \emptyset$ for all $l' < l$, or

(2) $S^{s+1} = S_l$ and S_l is the strategy T_k and the values of

$$\varphi_k(a_0), \varphi_k(a_1), \ldots, \varphi_k(a_r)$$

are defined on stage s for all parameters $a_0, a_1, \ldots, a_r$ of the strategy T_k and (†) is fulfilled, or

(3) $S^{s+1} = S_l$ and S_l is the strategy WP_e for some e, $(y, \varphi_e(y)) \notin \mathrm{supp}(P^s)$ for every $y \in \mathrm{dom}(\varphi_e^s)$ and $\varphi_e^s(x_e)$ is defined.

Stage 0. Put $P^0 = \mathrm{Id}, \mathrm{res}(l) = \emptyset$ for all l. Declare all strategies on stage 0 to be passive. Go to the next stage.

Stage $s+1$**.** If the strategy S^{s+1} is passive, then activate (reactivate) it and go to the next stage. If the strategy S^{s+1} is active but does not require attention, then go to the next stage without any changes. Finally, if the strategy S^{s+1} requires attention, then consider three cases.

(1) $S^{s+1} = S_l$ and S_l is the strategy $P_{i,j}$ for some i, j. If $W_i^s \cap [j]_{\mathrm{supp}(P^s)} \neq \emptyset$, then go straitly to the next stage. Otherwise, choose the smallest number $m \geq t$ such that the equivalence class $[m]_{\mathrm{supp}(P^s)}$ contains a number from W_i^s and both pairs $(j, m), (m, j)$ do not belong to any of the sets $\mathrm{res}(l'), l' < 1$; set $P^{s+1} = (P^s \cup \{(j, m), (m, j)\})^*$; declare all the strategies $S'_l, l > l'$, to be passive.

(2) $S^{s+1} = S_l$ and S_l is the strategy WP_e for some e. Let x_e be a parameter of the strategy WP_e. Set $P^{s+1} = (P^s \cup \{(x_e, \varphi_e(x_e)), (\varphi_e(x_e), x_e)\})^*$. Declare all strategies $S'_l, l > l'$, to be passive.

(3) $S^{s+1} = S_l$ and S_l is the strategy T_k for some k. Let $r, a_0, a_1, \ldots, a_r$ be its parameters. Set $P^{s+1} = (P^s \cup \{(a_i, a_j) : i, j \leq r \,\&\, (i.j) \in U^s\})^*$. Increase the value of the parameter r by one. Let the parameter a_{r+1} be equal to the smallest of the fresh numbers. Enumerate in res_k all pairs of numbers $(a_1, a_{r+1}), (a_{r+1}, a_i), i \leq r$. Declare all strategies $S'_l, l > l'$, to be passive.

Go to the next stage.

Verification. Obviously, for all s, every strategy that is passive on stage s will be activated (reactivated) on some larger stage.

Lemma 1. *Every strategy requires attention for only a finite number of times.*

Proof. To simplify the proof, we assume that S_0 is T_0, and φ_0 is a nowhere defined function. Then the strategy S_0 is activated exactly one time and never requires attention.

Inductive step. Suppose that after some stage s_0 none of the strategies

$$S_0, S_1, \ldots, S_{l-1}$$

requires attention. We will prove that after some stage $s_1 \geq s_0$ the strategy S_l will not require attention either. It is easy to see that if S_l is one of the strategies $P_{i,j}$ or WP_e, then after the stage s_0 the strategy S_l can require attention at most once.

Now assume that S_l is the strategy T_k for some k and S_l requires attention on infinitely many stages $s > s_0$. Due to the choice of s_0, the strategy S_l cannot be declared passive and the relation (†) is fulfilled on all of these stages. Then the sequence of parameters $a_0, a_1, a_2, \ldots$ is infinite, the function φ_e is total, and therefore, the computable function $\varphi_e(a_i)$ in variable i reduces the universal positive preorder U to the preorder R, which is impossible. □

Lemma 2. *All the requirements $P_{i,j}$, WP_e, T_k are satisfied.*

Proof. Suppose that S_l is any of the strategies $P_{i,j}$, WP_e, T_k and s_0 is the smallest stage after which none of the strategies $S_0, S_1, \dots, S_l$ requires attention. Existence of such stage follows from Lemma 1. We will show that the strategy S_l will be satisfied during or after the stage s_0. Consider three possible cases.

Case 1. S_l is the strategy $P_{i,j}$ for some i, j, moreover, W_i intersects infinitely many equivalence classes $\mathrm{supp}(P)$, and the number j is arbitrary. Due to the choice of s_0, the set $\bigcup_{l'<l} \mathrm{res}_{l'}$ is finite. Therefore, there are infinitely many numbers $m \in W_i$, different from components of all pairs of this set. Let $s_1 + 1 > s_0$ be some stage for which one of such sufficiently large numbers m belongs to $W_i^{s_1}$, and S^{s_1+1} is the strategy $P_{i,j}$. Then $W_i^{s_1} \cap [j]_{\mathrm{supp}(P^s)} \neq \emptyset$, otherwise, on stage $s_1 + 1$ the strategy S_l will require attention, which contradicts the choice of the stage s_0.

Case 2. S_l is the strategy WP_e for some e, moreover, the function φ_e is total. Suppose that on stage $s_2 + 1 > s_0$, the parameter x_e and the value of $\varphi_e(x_e)$ are defined, and S^{s_2+1} is the strategy WP_e. Then $(y, \varphi_e(y)) \in \mathrm{supp}(P^{s_2})$ for some $y \in \mathrm{dom}(\varphi_e^{s_2})$, otherwise, on stage $s_2 + 1$ the strategy S_l would require attention, which contradicts the choice of the stage s_0.

Case 3. S_l is the strategy T_k for some k, moreover, the function φ_k is total. Since the strategy T_k does not require attention after the stage s_0, the list of its parameters stabilises on some finite set $a_0, a_1, \dots, a_r$, the set of restraints res_k stabilises, and by some stage $s_3 + 1 > s_0$, for which S^{s_3+1} is T_k, all the values of $\varphi_k(a_0), \varphi_k(a_1), \dots, \varphi_k(a_r)$ are defined, and for all $i, j \leq r$ the following relations hold:

$$\varphi_k(a_i) R \varphi_k(a_j) \iff \varphi_k(a_i) R^{s_3} \varphi_k(a_j), \quad iUj \iff iU^{s_3} j.$$

Therefore, on the stage $s_3 + 1$, the relation (†) fails. Due to the choice of s_0, none of the pairs $(a_i, a_j), i, j \leq r$, can be enumerated in P by strategies of higher priority and cannot be enumerated in P by the strategies of lower priority than the priority of S_l. Hence, it follows that $a_i P^{s_3} a_j \iff a_i P a_j$ for all $i, j \leq r$. Therefore, reducibility $P \leq_c R$ by the computable function φ_k fails for at least on one pair of numbers $(a_i, a_j) \in \mathrm{res}_k$.

Therefore, in each of three considered cases, the construction provides the fulfillment of the corresponding requirement. □

The theorem is proved.

□

Corollary 4. *For every positive non-universal preorder R, there exist inifinitely many pairwise incomparable weakly precomplete minimal positive preorders, each of which does not reduce to R.*

Proof. By Theorem 4, for preorder R there exists a weakly precomplete minimal preorder P_0, incomparable to R. Consider a positive preorder $R_1 = R \oplus P_0$. Then $\mathrm{supp}(R_1)$ is not a universal ceer, since a universal ceer does not decompose into a direct sum of incomparable equivalences, [1]. Therefore, R_1 is not a universal positive preorder. Applying Theorem 4, we obtain a weakly precomplete minimal preorder P_1, incomparable to R_1, and therefore, incomparable to R and P_0. Iterating this process, we obtain the required sequence of weakly precomplete minimal preorders. □

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