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ESTIMATES FOR SOLUTIONS TO ONE CLASS OF NONLINEAR NONAUTONOMOUS SYSTEMS WITH TIME-VARYING CONCENTRATED AND DISTRIBUTED DELAYS

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Abstract. We consider a class of nonlinear systems of nonautonomous differential equations with time-varying concentrated and distributed delays than can be unbounded. Using a Lyapunov – Krasovskii functional, some estimates of solutions are established. The obtained estimates allow us to conclude whether the solutions are stable. In the case of exponential and asymptotic stability, stabilization rates of the solutions at infinity are pointed out.

Keywords: time-varying delay systems, variable coefficients, estimates for solutions, stability, Lyapunov – Krasovskii functional.

1. Introduction

We consider the systems of delay differential equations of the following form:

$$\frac{d}{dt}y(t) = A(t)y(t) + B(t)y(t-\tau(t)) + \int\limits_{t-\tau(t)}^{t} D(t,t-s)y(s)\,ds$$
$$+F\left(t, y(t), y(t-\tau(t)), \int\limits_{t-\tau(t)}^{t} D(t,t-s)y(s)\,ds\right), \qquad t \geq 0, \tag{1.1}$$

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where $A(t)$, $B(t)$, $D(t,s)$ are $(n\times n)$-matrices with continuous real-valued entries, i.e.,

$$a_{ij}(t),\ b_{ij}(t)\in C(\overline{\mathbb{R}}_+),\quad d_{ij}(t,s)\in C(\overline{\mathbb{R}}^2_+),\quad i,j=1,\dots,n,$$

$\tau(t)$ is the delay function, $\tau(t)\in C^1([0,\infty))$,

$$0<\tau_0\le\tau(t)\le\tau_1 t+\tau_2,\quad 0\le\tau_1\le1,\quad \tau_2>0,\quad \frac{d\tau(t)}{dt}\le\tau_3<1, \tag{1.2}$$

$F(t,u,v,w)$ is a continuous real-valued vector-function. We assume that $F(t,u,v,w)$ is a Lipschitz function of u, w on every compact set $G\subset[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ and satisfies the inequality

$$\|F(t,u,v,w)\|\le q\|u\|^{1+\omega},\qquad t\ge0,\quad u,v,w\in\mathbb{R}^n, \tag{1.3}$$

with q, $\omega\ge0$.

Our aim is to obtain estimates for solutions to (1.1) on the whole half-axis $\{t\ge 0\}$, on the base of which we can make conclusions about stability of the solutions (in particular, exponential or asymptotic stability) and point out stabilization rates.

There is a large number of works devoted to the study of stability of solutions to delay differential equations (for example, see [1]–[5] and the bibliography therein). Researchers often use Lyapunov – Krasovskii functionals in order to obtain stability conditions. However, not every Lyapunov – Krasovskii functional allows us to obtain estimates characterizing decay rate at infinity. In recent years, investigations in this direction have been actively developed. A lot of works are devoted to delay differential equations with constant coefficients. In the nonautonomous case the number of relevant articles is significantly less.

This article continues our research of stability of solutions to nonautonomous delay differential equations. In particular, we investigated time-delay systems with periodic coefficients in linear terms. Conditions for exponential stability of the zero solution were established and estimates of exponential decay of solutions at infinity were obtained by using suitable Lyapunov–Krasovskii functionals. In [6] some nonlinear systems with variable coefficients and time-varying concentrated delay were studied.

In this article we consider nonautonomous systems of the from (1.1) with time-varying concentrated and distributed delays. Note that the delay can be unbounded if $\tau_1>0$. We establish estimates for solutions that allow us to conclude whether the solutions are stable. In the case of exponential and asymptotic stability, we point out stabilization rates of the solutions at infinity. In Section 2 we establish estimates for solutions to linear systems (i.e. $F(t,u,v,w)\equiv0$). In Section 3 we consider nonlinear systems of the form (1.1) when the vector-function $F(t,u,v,w)$ satisfies (1.3) with $q>0$.

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2. Estimates for solutions to linear systems

At first we introduce some notations. Define the $(n\times n)$-matrices $H(t)$, $K(t,s)$ such that

$$H(t)\in C^1(\overline{\mathbb{R}}_+),\qquad H(t)=H^*(t)>0,\quad t\ge0, \tag{2.1}$$

the minimal eigenvalue $h(t)$ of $H(t)$ satisfies the inequality

$$h(t)\ge h_0>0, \tag{2.2}$$

$$K(t,s)\in C^1(\overline{\mathbb{R}}^2_+),\quad K(t,s)=K^*(t,s),\quad K(t,s)\ge0,\quad (t,s)\in\overline{\mathbb{R}}^2_+. \tag{2.3}$$

Hereinafter, the matrix inequality $S > 0$ (or $S < 0$) means that S is a positive (or negative) definite Hermitian matrix. We use the spectral norm of matrices.

Define the matrix

$$Q(t,s) = \begin{pmatrix} Q_{11}(t,s) & Q_{12}(t,s) & Q_{13}(t,s) \\ Q_{12}^*(t,s) & Q_{22}(t,s) & Q_{23}(t,s) \\ Q_{13}^*(t,s) & Q_{23}^*(t,s) & Q_{33}(t,s) \end{pmatrix} \tag{2.4}$$

with the entries

$$\begin{aligned} Q_{11}(t,s) = & \quad -\frac{d}{dt}H(t) - H(t)A(t) - A^*(t)H(t) - K(t,0), \\ Q_{12}(t,s) = & \quad -H(t)B(t), \\ Q_{13}(t,s) = & \quad -\tau(t)H(t)D(t,s), \\ Q_{22}(t,s) = & \quad \left(1 - \frac{d}{dt}\tau(t)\right) K(t,\tau(t)), \\ Q_{23}(t,s) = & \quad 0, \\ Q_{33}(t,s) = & \quad -\tau(t)\left(\frac{\partial}{\partial t}K(t,s) + \frac{\partial}{\partial s}K(t,s)\right). \end{aligned} \tag{2.5}$$

Consider the initial value problem for the linear systems ($F(t,u,v,w) \equiv 0$)

$$\begin{aligned} & \frac{d}{dt}y(t) = A(t)y(t) + B(t)y(t-\tau(t)) + \int\limits_{t-\tau(t)}^{t} D(t,t-s)y(s)\,ds, \qquad t > 0, \\ & y(t) = \varphi(t), \quad t \in [-\tau_2, 0], \\ & y(+0) = \varphi(0), \end{aligned} \tag{2.6}$$

where $\varphi(t) \in C([-\tau_2, 0])$ is a given real-valued vector-function. Below we establish some estimates for solutions to (2.6).

Theorem 1. *Suppose that there are matrices $H(t)$, $K(t,s)$ satisfying (2.1)–(2.3) such that*

$$\left\langle Q(t,s)\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right\rangle \geq p(t)\langle H(t)u,u\rangle + k(t)\tau(t)\langle K(t,s)w,w\rangle, \tag{2.7}$$

$$u, v, w \in \mathbb{R}^n, \qquad (t,s) \in \overline{\mathbb{R}}_+^2,$$

where $p(t)$, $k(t) \in C(\overline{\mathbb{R}}_+)$. Then, for a solution $y(t)$ to (2.6), the following estimate holds

$$\|y(t)\| \leq \sqrt{\frac{V(0,\varphi)}{h(t)}} \exp\left(-\frac{1}{2}\int_0^t \gamma(\xi)d\xi\right), \quad t > 0, \tag{2.8}$$

where

$$V(0,\varphi) = \langle H(0)\varphi(0), \varphi(0)\rangle + \int\limits_{-\tau(0)}^{0} \langle K(0,-s)\varphi(s), \varphi(s)\rangle ds, \tag{2.9}$$

$$\gamma(t) = \min\{p(t), k(t)\}. \tag{2.10}$$

Proof. Let $y(t)$ be a solution to (2.6). Using the matrices $H(t)$, $K(t,s)$, satisfying the conditions of Theorem 1, by analogy with [7], we consider the following Lyapunov–Krasovskii functional on this solution

$$V(t,y) = \langle H(t)y(t), y(t)\rangle + \int\limits_{t-\tau(t)}^{t} \langle K(t,t-s)y(s), y(s)\rangle ds. \qquad (2.11)$$

Differentiating this functional, we have

$$\frac{d}{dt}V(t,y) = \left\langle \frac{d}{dt}H(t)y(t), y(t)\right\rangle$$
$$+\left\langle H(t)\frac{d}{dt}y(t), y(t)\right\rangle + \left\langle H(t)y(t), \frac{d}{dt}y(t)\right\rangle$$
$$+\langle K(t,0)y(t), y(t)\rangle - \left(1-\frac{d}{dt}\tau(t)\right)\langle K(t,\tau(t))y(t-\tau(t)), y(t-\tau(t))\rangle$$
$$+\int\limits_{t-\tau(t)}^{t}\left\langle \frac{d}{dt}K(t,t-s)y(s), y(s)\right\rangle ds.$$

Taking into account that $y(t)$ satisfies (2.6), we obtain

$$\frac{d}{dt}V(t,y) = \left\langle \left[\frac{d}{dt}H(t) + H(t)A(t) + A^*(t)H(t) + K(t,0)\right] y(t), y(t)\right\rangle$$
$$+\langle H(t)B(t)y(t-\tau(t)), y(t)\rangle + \langle B^*(t)H(t)y(t), y(t-\tau(t))\rangle$$
$$+\left\langle H(t)\int\limits_{t-\tau(t)}^{t} D(t,t-s)y(s)\,ds, y(t)\right\rangle + \left\langle H(t)y(t), \int\limits_{t-\tau(t)}^{t} D(t,t-s)y(s)\,ds\right\rangle$$
$$-\left(1-\frac{d}{dt}\tau(t)\right)\langle K(t,\tau(t))y(t-\tau(t)), y(t-\tau(t))\rangle + \int\limits_{t-\tau(t)}^{t}\left\langle \frac{d}{dt}K(t,t-s)y(s), y(s)\right\rangle ds.$$

Using the matrix $Q(t,s)$ defined in (2.4), (2.5), we have

$$\frac{d}{dt}V(t,y) = -\frac{1}{\tau(t)}\int\limits_{t-\tau(t)}^{t}\left\langle Q(t,t-s)\begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ y(s)\end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ y(s)\end{pmatrix}\right\rangle ds.$$

By (2.7), we arrive at the inequality

$$\frac{d}{dt}V(t,y) \le -p(t)\langle H(t)y(t), y(t)\rangle - k(t)\int\limits_{t-\tau(t)}^{t}\langle K(t,t-s)y(s), y(s)\rangle ds.$$

According to the definition of (2.11), we have

$$\frac{d}{dt}V(t,y) \le -\gamma(t)V(t,y),$$

where $\gamma(t)$ is given in (2.10). This differential inequality yields the estimate

$$V(t,y) \le V(0,\varphi)\exp\left(-\int_0^t \gamma(\xi)d\xi\right),$$

where $V(0,\varphi)$ is defined in (2.9). Obviously,

$$h(t)\|y(t)\|^2 \le \langle H(t)y(t), y(t)\rangle \le \|H(t)\|\|y(t)\|^2, \tag{2.12}$$

where $h(t)$ is the minimal eigenvalue of $H(t)$. Then,

$$\|y(t)\|^2 \le \frac{1}{h(t)}\langle H(t)y(t), y(t)\rangle \le \frac{V(t,y)}{h(t)} \le \frac{V(0,\varphi)}{h(t)} \exp\left(-\int_0^t \gamma(\xi)d\xi\right),$$

whence (2.8) follows.

Theorem 1 is proven. □

Corollary 1. *Let the conditions of Theorem 1 hold. If*

$$\int_0^t \gamma(s)ds \ge 0,$$

then the zero solution to (1.1) is stable; moreover, for a solution $y(t)$ to (2.6), we have the estimate

$$\|y(t)\| \le \sqrt{\frac{V(0,\varphi)}{h_0}}, \qquad t > 0,$$

where h_0 is defined in (2.2).

Corollary 2. *Let the conditions of Theorem 1 hold. If*

$$\int_{t_0}^t \gamma(s)ds \to \infty, \quad t \to \infty,$$

for some $t_0 \ge 0$, then the zero solution to (1.1) is asymptotically stable; moreover, the stabilization rate is determined by the function

$$\exp\left(-\frac{1}{2}\int_{t_0}^t \gamma(\xi)d\xi\right).$$

Corollary 3. *Let the conditions of Theorem 1 hold. If*

$$\int_0^t \gamma(s)ds \ge \gamma_1 t + \gamma_2, \quad \gamma_1 > 0,$$

then the zero solution to (1.1) is exponentially stable; moreover, for a solution $y(t)$ to (2.6), we have the estimate

$$\|y(t)\| \le \sqrt{\frac{V(0,\varphi)}{h_0}} \exp\left(-\frac{\gamma_1 t}{2} - \frac{\gamma_2}{2}\right), \quad t > 0.$$

3. Estimates for solutions to nonlinear systems

Consider the initial value problem for (1.1)

$$\begin{aligned}
\frac{d}{dt}y(t) &= A(t)y(t) + B(t)y(t-\tau(t)) + \int\limits_{t-\tau(t)}^{t} D(t,t-s)y(s)\,ds \\
&\quad + F\left(t, y(t), y(t-\tau(t)), \int\limits_{t-\tau(t)}^{t} D(t,t-s)y(s)\,ds\right), \qquad t>0, \qquad (3.1)\\
y(t) &= \varphi(t), \quad t \in [-\tau_2, 0], \\
y(+0) &= \varphi(0),
\end{aligned}$$

where $\varphi(t) \in C([-\tau_2, 0])$ is a given real-valued vector-function. Below we establish some estimates for solutions to (3.1).

Theorem 2. *Let the conditions of Theorem 1 hold and $\omega = 0$ in (1.3). Then, for a solution $y(t)$ to (3.1), the following estimate holds*

$$\|y(t)\| \le \sqrt{\frac{V(0,\varphi)}{h(t)}} \exp\left(-\frac{1}{2}\int_0^t \gamma_q(\xi) d\xi\right), \quad t>0, \tag{3.2}$$

where

$$\gamma_q(t) = \min\left\{\left(p(t) - \frac{2q\|H(t)\|}{h(t)}\right), k(t)\right\}. \tag{3.3}$$

Proof. Let $y(t)$ be a solution to (3.1). Consider the Lyapunov–Krasovskii functional (2.11) on this solution. As in the proof of Theorem 1, differentiating $V(t,y)$, we obtain

$$\frac{d}{dt}V(t,y) = -\frac{1}{\tau(t)} \int\limits_{t-\tau(t)}^{t} \left\langle Q(t,t-s) \begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ y(s) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ y(s) \end{pmatrix} \right\rangle ds + W(t), \tag{3.4}$$

where $Q(t,s)$ is defined in (2.4), (2.5),

$$\begin{aligned}
W(t) &= \left\langle H(t)F\left(t, y(t), y(t-\tau(t)), \int\limits_{t-\tau(t)}^{t} D(t,t-s)y(s)\,ds\right), y(t)\right\rangle \\
&\quad + \left\langle H(t)y(t), F\left(t, y(t), y(t-\tau(t)), \int\limits_{t-\tau(t)}^{t} D(t,t-s)y(s)\,ds\right)\right\rangle. \qquad (3.5)
\end{aligned}$$

By (1.3) for $\omega = 0$, we have

$$\|W(t)\| \le 2q\|H(t)\|\|y(t)\|^2.$$

By (2.7), we arrive at the inequality

$$\frac{d}{dt}V(t,y) \le -p(t)\langle H(t)y(t), y(t)\rangle$$

$$-k(t)\int\limits_{t-\tau(t)}^{t}\langle K(t,t-s)y(s),y(s)\rangle ds+2q\|H(t)\|\|y(t)\|^2.$$

Using (2.12), we obtain

$$\frac{d}{dt}V(t,y)\le-\left(p(t)-\frac{2q\|H(t)\|}{h(t)}\right)\langle H(t)y(t),y(t)\rangle$$

$$-k(t)\int\limits_{t-\tau(t)}^{t}\langle K(t,t-s)y(s),y(s)\rangle ds.$$

According to the definition of (2.11), we have

$$\frac{d}{dt}V(t,y)\le-\gamma_q(t)V(t,y),$$

where $\gamma_q(t)$ is given in (3.3). Repeating similar reasoning as in the proof of Theorem 1, we obtain (3.2).

Theorem 2 is proven.

□

Theorem 3. *Let the conditions of Theorem 1 hold and $\omega>0$ in (1.3). Suppose that the integral*

$$\int\limits_0^\infty \omega q\|H(\xi)\|(h(\xi))^{-1-\omega/2}\exp\left(-\frac{\omega}{2}\int_0^\xi\gamma(s)ds\right)d\xi$$

converges. Then the estimate holds

$$\|y(t)\|\le\sqrt{\frac{V(0,\varphi)}{h(t)}}\exp\left(-\frac{1}{2}\int_0^t\gamma(\xi)d\xi\right)\left(1-V^{\omega/2}(0,\varphi)R\right)^{-1/\omega},\quad t>0,\quad (3.6)$$

for a solution y(t) to (3.1) with the initial function $\varphi(t)$ such that

$$V(0,\varphi)<R^{-2/\omega},\qquad(3.7)$$

where

$$R=\int\limits_0^\infty \omega q\|H(\xi)\|(h(\xi))^{-1-\omega/2}\exp\left(-\frac{\omega}{2}\int_0^\xi\gamma(s)ds\right)d\xi.\qquad(3.8)$$

Proof. Let $y(t)$ be a solution to (3.1). Consider the Lyapunov–Krasovskii functional (2.11) on this solution. As in the proof of Theorem 2, differentiating $V(t,y)$, we obtain (3.4), where $W(t)$ is defined in (3.5). By (1.3) for $\omega>0$, we have

$$\|W(t)\|\le2q\|H(t)\|\|y(t)\|^{2+\omega}.$$

By (2.7), we arrive at the inequality

$$\frac{d}{dt}V(t,y)\le-p(t)\langle H(t)y(t),y(t)\rangle$$

$$-k(t)\int\limits_{t-\tau(t)}^{t}\langle K(t,t-s)y(s),y(s)\rangle ds+2q\|H(t)\|\|y(t)\|^{2+\omega}.$$

According to the definition of (2.11), we have

$$\frac{d}{dt}V(t,y) \le -\gamma(t)V(t,y) + +2q\|H(t)\|\|y(t)\|^{2+\omega},$$

where $\gamma(t)$ is given in (2.10). Taking into account the inequality

$$\|y(t)\|^2 \le \frac{V(t,y)}{h(t)},$$

we obtain

$$\frac{d}{dt}V(t,y) \le -\gamma(t)V(t,y) + \beta(t)(V(t,y))^{1+\omega/2},$$

where

$$\beta(t) = 2q\|H(t)\|(h(t))^{-1-\omega/2}.$$

Hence, by Gronwall's inequality (see, for example, [8]), we arrive at the estimate

$$V(t,y) \le V(0,\varphi)\exp\left(-\int_0^t \gamma(\xi)d\xi\right)$$

$$\times\left(1-\frac{\omega}{2}V^{\omega/2}(0,\varphi)\int\limits_0^t \beta(\xi)e^{-\frac{\omega}{2}\int\limits_0^{\xi}\gamma(s)\,ds}d\xi\right)^{-2/\omega}, \quad t>0, \tag{3.9}$$

where $V(0,\varphi)$ is defined in (2.9). We now estimate the function in parentheses

$$U(t) = 1-\frac{\omega}{2}V^{\omega/2}(0,\varphi)\int\limits_0^t \beta(\xi)e^{-\frac{\omega}{2}\int\limits_0^{\xi}\gamma(s)\,ds}d\xi$$

$$\ge 1-\frac{\omega}{2}V^{\omega/2}(0,\varphi)\int\limits_0^\infty \beta(\xi)e^{-\frac{\omega}{2}\int\limits_0^{\xi}\gamma(s)\,ds}d\xi \ge 1-V^{\omega/2}(0,\varphi)R,$$

where R is defined in (3.8). If $\varphi(t)$ is such that (3.7) is valid, then $U(t) > 0$. Consequently, it follows from (3.9) that

$$V(t,y) \le V(0,\varphi)\exp\left(-\int_0^t \gamma(\xi)d\xi\right)\left(1-V^{\omega/2}(0,\varphi)R\right)^{-2/\omega}.$$

By the definition of (2.11), we obtain (3.6).

The theorem is proven.

□

Remark 1. If $q = 0$, then Theorems 2 and 3 turn into Theorem 1.

Remark 2. Let the conditions of Corollaries 2 and 3 in Section 1 be valid, which guarantee the asymptotic or exponential stability of the zero solution to the linear systems. Then Theorem 3 gives us estimates for attraction domains and estimates characterizing stabilization rates of solutions to (1.1) as $t \to \infty$.

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