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THE SUM OF ORDERS OF ELEMENTS IN NONABELIAN GROUPS OF ODD ORDER

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Abstract. Denote by $\psi(G)$ the sum of the orders of the elements of a finite group G. We obtain an exact upper bound for $\psi(G)$ on the set of nonabelian groups of given odd order n in terms of the minimal prime divisor of n. We also describe the finite groups on which this bound is achieved.

Keywords: orders of elements, solvable groups.

1. Introduction

Denote by $\psi(G)$ the sum of the orders of the elements of a finite group G. In [1] the authors show that the maximum of $\psi(G)$ on the set of groups of the given order n is attained at a cyclic group. This maximum is strict, that is, if H is a non-cyclic group of order n, then $\psi(H) < \psi(G)$. Using the function ψ, it is possible to formulate sufficient conditions for a finite group to be cyclic, abelian, nilpotent, solvable, or supersolvable [2]. The paper [2], in particular, gives an upper bound for the values of $\psi(G)$ on the set of nonabelian groups G of even order. The paper [3] contains a similar result for the non-cyclic q^*-groups. A finite group G is a q^*-group if q is the smallest prime divisor of the order of G. The main result of [3] states that if G is a finite non-cyclic q^*-group of order n and C_n is a cyclic group of order n, then

$$\psi(G) \leqslant \frac{(q^3 - q + 1)(q + 1)}{q^5 + 1}\psi(C_n).$$

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For a positive integer m and a prime q, put

$$M_{q^{m+1}} = \langle a, b | a^{q^m} = b^q = 1, a^b = a^{1+q^{m-1}} \rangle. \quad (1)$$

Here we prove the following statement.

Theorem 1. *Let q be an odd prime and let G be a nonabelian q^*-group of order n. Then*

$$\psi(G) \leqslant \frac{q^6 + q^3 - q^2 + 1}{q^7 + 1} \psi(C_n)$$

with the equality if and only if G is a direct product of M_{q^3} and a cyclic group of order coprime to q.

2. Preliminaries

The following lemma contains some useful facts about the function ψ.

Lemma 1. (1) [4, Lemma 2.2] *If A, B are finite groups of coprime orders, then $\psi(A \times B) = \psi(A)\psi(B)$.*

(2) [1, Corollary B] *If P is a normal cyclic Sylow subgroup of a finite group G, then $\psi(G) \leqslant \psi(P)\psi(G/P)$, with equality if and only if P is central in G.*

(3) [4, Lemma 2.9] *If n is a natural number, p and q are the largest and smallest prime divisors of n, then*

$$\psi(C_n) > \frac{q}{p+1} n^2.$$

(4) [4, Lemma 2.9] *If p is a prime and m is a non-negative integer, then*

$$\psi(C_{p^m}) = \frac{p^{2m+1} + 1}{p+1}.$$

(5) [4, Lemma 2.2] *Let p be a prime. Assume that G is a semidirect product of a normal cyclic p-subgroup C and a nontrivial subgroup F whose order is coprime to p. Put $Z = C_F(C)$. Then*

$$\psi(G) = \psi(C)\psi(Z) + |C| \left(\psi(F) - \psi(Z)\right).$$

Observe that [4, Lemma 2.9] states that $\psi(C_n) \geqslant 2/(p+1)$, but the authors prove strict inequality. Also it easy to see that the number 2 in the numerator can be replaced with q.

The following statement easily follows from Item (4) of the previous lemma.

Corollary 1. *Let p be a prime and m be a positive integer. Then the following statements hold.*

(1)

$$\frac{p^m}{\psi(C_{p^m})} < \frac{p+1}{p^2}.$$

(2) *If H is a proper subgroup of a cyclic group C and p is a prime divisor of the index $|C : H|$, then*

$$\frac{\psi(H)}{\psi(C)} \leqslant \frac{1}{p^2 - p + 1}.$$

Lemma 2. *If H is a normal abelian Hall subgroup of a finite group G, then $\psi(G) \leqslant \psi(H)\psi(G/H)$.*

Proof. By the Schur–Zassenhaus theorem, $G \simeq H \rtimes (G/H)$. It follows from Item (1) of Lemma 1 that it is sufficient to prove that $\psi(H \rtimes (G/H)) \leqslant \psi(H \times (G/H))$. Choose $h \in H$ and $g \in G/H$ and denote by gh and $g \cdot h$ their products as elements of semidirect and direct products respectively. Since H is a Hall subgroup, $|g \cdot h| = |g||h|$. Now

$$(gh)^{|g|} = h^{g^{|g|-1}} h^{g^{|g|-2}} \dots h^g h.$$

So $|gh|$ divides $|g \cdot h|$, and the lemma is proved. $\square$

Lemma 3. [4, Proposition 2.5] *Let G be a finite group and suppose that there exists $x \in G$ such that $|G : \langle x \rangle| < 2p$, where p is the maximal prime divisor of $|G|$. Then one of the following holds:*

(1) *G has a normal cyclic Sylow p-subgroup;*
(2) *G is solvable and $\langle x \rangle$ is a maximal subgroup of G of index either p or $p+1$.*

Lemma 4. [5, Theorem 4.2] *If q is an odd prime and m is a positive integer, then*

$$\psi(M_{q^{m+1}}) = \psi(C_{q^m} \times C_q) = \frac{q^{2m+2} + q^3 - q^2 + 1}{q+1}.$$

In particular,

$$\psi(M_{q^3}) = \frac{q^6 + q^3 - q^2 + 1}{q^7 + 1} \psi(C_{q^3}).$$

3. Proof of Theorem

Let q be an odd prime and let G be a nonabelian q^*-group of order n. Suppose that

$$\psi(G) \geqslant \frac{q^6 + q^3 - q^2 + 1}{q^7 + 1} \psi(C_n). \tag{2}$$

Let p be the greatest prime divisor of $|G|$. Since $\psi(C_n) > \frac{q}{p+1} n^2$ by Item (3) of Lemma 1,

$$\psi(G) > \frac{(q^6 + q^3 - q^2 + 1)q}{(q^7 + 1)(p+1)} n^2.$$

So G contains an element x such that

$$|x| \geqslant \frac{\psi(G)}{n} > \frac{(q^6 + q^3 - q^2 + 1)q}{(q^7 + 1)(p+1)} n.$$

Then

$$|G : \langle x \rangle| < \frac{(q^7 + 1)(p+1)}{(q^6 + q^3 - q^2 + 1)q}.$$

We proceed by induction on the number of prime divisors of the order of G. Assume that $p = q$. Then

$$|G : \langle x \rangle| < \frac{(q^7 + 1)(q+1)}{(q^6 + q^3 - q^2 + 1)q}.$$

Since G is nonabelian and

$$\frac{(q^7 + 1)(q+1)}{(q^6 + q^3 - q^2 + 1)q} < q^2,$$

the index $|G : \langle x\rangle|$ is q. It follows that G is isomorphic to $M_{q^{m+1}}$ for $m \geqslant 2$ (see, for example, [6, Theorem 1.2]). By Lemma 4 and Item (4) of Lemma 1, the inequality (2) leads to

$$\psi(G) = \frac{q^{2m+2}+q^3-q^2+1}{q+1} \geqslant \frac{q^6+q^3-q^2+1}{q^7+1}\psi(C_n) = \\ = \frac{q^6+q^3-q^2+1}{q^7+1}\cdot\frac{q^{2m+3}+1}{q+1},$$

or equivalently

$$(q^{2m+2}+q^3-q^2+1)(q^7+1)-(q^6+q^3-q^2+1)(q^{2m+3}+1) = \\ = -q^{2m+6}+q^{2m+5}-q^{2m+3}+q^{2m+2}+q^{10}-q^9+q^7-q^6 = \\ = (-q^{2m+2}+q^6)(q^3+1)(q-1) \geqslant 0.$$

This inequality holds and becomes equality only if $m = 2$ as stated.

Suppose that $p > q$. Then

$$|G : \langle x\rangle| < \frac{(q^7+1)(p+1)}{(q^6+q^3-q^2+1)q} < p+1.$$

It follows from Lemma 3 that either G contains a normal cyclic Sylow p-subgroup or $|G : \langle x\rangle| = p$.

Assume that G contains a normal cyclic Sylow p-subgroup P. By Item (2) of Lemma 1, we have $\psi(G) \leqslant \psi(P)\psi(G/P)$ with the equality if and only if P is central in G.

If G/P is a nonabelian group, then

$$\psi(G/P) \leqslant \frac{q^6+q^3-q^2+1}{q^7+1}\psi(C_{|G/P|})$$

by the inductive hypothesis. Hence

$$\psi(G) \leqslant \frac{q^6+q^3-q^2+1}{q^7+1}\psi(P)\psi(C_{|G/P|}) = \frac{q^6+q^3-q^2+1}{q^7+1}\psi(C_n).$$

Since we assume (2), the latter inequality must be equality. It is possible only if P is central. Then G has the stated structure, that is, it is a direct product of M_{q^3} and a cyclic group of order coprime to q.

Let G/P be an abelian group. Denote by H a p-complement of G. We write Z for $C_H(P)$. By Item (5) of Lemma 1,

$$\psi(G) < \psi(C_{p^m})\psi(H)\left(\frac{\psi(Z)}{\psi(H)} + \frac{p^m}{\psi(C_{p^m})}\right).$$

If H is cyclic, then

$$\psi(G) < \left(\frac{\psi(Z)}{\psi(H)} + \frac{p^m}{\psi(C_{p^m})}\right)\psi(C_n).$$

Let us bound the right-hand side from above. Consider the fraction $\psi(Z)/\psi(H)$. If $Z = H$, then $G = P \times H$ and G is abelian, contradicting the assumption. So Z is a proper subgroup of H. It follows from Item (2) of Corollary 1 that

$$\frac{\psi(Z)}{\psi(H)} \leqslant \frac{1}{r^2-r+1} \tag{3}$$

for a prime divisor r of $|H : Z|$. Since $G/C_G(P) \leqslant Aut(C_{p^m})$, we have $H/C_H(P) \leqslant Aut(C_{p^m})$. Therefore, $|H/C_H(P)|$ divides $p-1$. In particular, r divides $p-1$ and so $p \geqslant 2r+1$. Since $r \geqslant q$, it follows that $p \geqslant 2q+1$. Now it follows from (3) and Item (1) of Corollary 1 that

$$\left(\frac{\psi(Z)}{\psi(H)} + \frac{p^m}{\psi(C_{p^m})}\right)\psi(C_n) \leqslant \left(\frac{1}{(2q+1)^2} + \frac{1}{2q+1} + \frac{1}{q^2-q+1}\right)\psi(C_n).$$

Since

$$\frac{1}{(2q+1)^2} + \frac{1}{2q+1} + \frac{1}{q^2-q+1} < \frac{q^6+q^3-q^2+1}{q^7+1}$$

holds for all $q > 2$, we have a contradiction.

If H is a non-cyclic subgroup of G, then by Item (5) of Lemma 1, we have

$$\psi(G) = |P|\psi(H) + (\psi(P) - |P|)\psi(Z).$$

Dividing the both sides by $\psi(C_n)$, we get

$$\frac{\psi(G)}{\psi(C_n)} = \frac{|P|}{\psi(P)}\frac{\psi(H)}{\psi(C_{|H|})} + \left(1 - \frac{|P|}{\psi(P)}\right)\frac{\psi(Z)}{\psi(C_{|Z|})}\frac{\psi(C_{|Z|})}{\psi(C_{|H|})}.$$

Since $\psi(Z) \leqslant \psi(C_{|Z|})$,

$$\frac{\psi(G)}{\psi(C_n)} \leqslant \frac{|P|}{\psi(P)}\frac{\psi(H)}{\psi(C_{|H|})} + \left(1 - \frac{|P|}{\psi(P)}\right)\frac{\psi(C_{|Z|})}{\psi(C_{|H|})} =$$

$$= \frac{|P|}{\psi(P)}\left(\frac{\psi(H)}{\psi(C_{|H|})} - \frac{\psi(C_{|Z|})}{\psi(C_{|H|})}\right) + \frac{\psi(C_{|Z|})}{\psi(C_{|H|})}.$$

Since H is a non-cyclic q^*-group, according to [3, Theorem 4]

$$\frac{\psi(H)}{\psi(C_{|H|})} \leqslant \frac{(q^3-q+1)(q+1)}{q^5+1}.$$

Thus

$$\frac{\psi(G)}{\psi(C_n)} \leqslant \frac{|P|}{\psi(P)}\left(\frac{(q^3-q+1)(q+1)}{q^5+1} - \frac{\psi(C_{|Z|})}{\psi(C_{|H|})}\right) + \frac{\psi(C_{|Z|})}{\psi(C_{|H|})}.$$

Since $\psi(C_{|Z|})/\psi(C_{|H|}) > 0$, Corollary 1 implies that

$$\frac{\psi(G)}{\psi(C_n)} < \frac{(q+1)}{q^2}\frac{(q^3-q+1)(q+1)}{q^5+1} + \frac{1}{q^2-q+1}.$$

The inequality

$$\frac{(q+1)}{q^2}\frac{(q^3-q+1)(q+1)}{q^5+1} + \frac{1}{q^2-q+1} < \frac{q^6+q^3-q^2+1}{q^7+1}$$

is equivalent to

$$q^{15} - 3q^{14} + 2q^{12} - 3q^{11} + q^{10} - q^9 - q^7 - 3q^6 + 3q^5 - q^4 - 3q^3 + q^2 - 1 > 0$$

which holds for all $q > 2$ (indeed, $q^{15} \geqslant 3q^{14}$, $2q^{12} > 3q^{11}$, and so on); that is a contradiction and we finished the case when G contains a normal cyclic Sylow p-subgroup.

Assume that $|G : \langle x\rangle| = p$. Let $P \in Syl_p(G)$ and $|P| = p^{m+1}$. There are three options: P is cyclic (this situation has been considered already), $P \simeq C_{p^m} \times C_p$, or $P \simeq M_{p^{m+1}}$.

Let us show that P is normal in G. If $Q \in Syl_q(G)$ then Q lies in $\langle x\rangle$. So Q is cyclic and G contains a normal q-complement N (see, for example, [7, Theorem 5.14]). By the inductive hypothesis, P is normal in N and, therefore, in G. So $G = P \rtimes H$ where H is a cyclic subgroup of G.

If P is isomorphic to $C_{p^m} \times C_p$, then $\psi(G) \leqslant \psi(P)\psi(H)$ by Lemma 2. It follows from Lemma 4 that

$$\frac{\psi(G)}{\psi(C_n)} \leqslant \frac{\psi(P)}{\psi(C_{p^{m+1}})} = \frac{p^{2m+2}+p^3-p^2+1}{p^{2m+3}+1} \leqslant \frac{q^6+q^3-q^2+1}{q^7+1}.$$

The last inequality holds and is strict for all $m \geqslant 2$. If $m = 0$, then P is cyclic. Assume that $m = 1$. Since H acts nontrivially on the cyclic group $P/C_P(H)$ of order p, its order has a non-identity common divisor with $p-1$. So $p \geqslant 2q+1$. The inequality

$$\frac{(2q+1)^4+(2q+1)^3-(2q+1)^2+1}{(2q+1)^5+1} < \frac{q^6+q^3-q^2+1}{q^7+1}$$

is equivalent to

$$8q^9+20q^8+24q^7+31q^6+28q^5+q^4-4q^3+17q^2+16q+3>0,$$

which holds for all $q \geqslant 1$; that is a contradiction.

Finally, consider the case $P \simeq M_{p^{m+1}}$. Let h be an arbitrary element of the group H and φ be the automorphism of P induced by conjugation by h. Let a, b be generators for P such that

$$a^{p^m} = b^p = 1, a^b = a^{1+p^{m-1}}.$$

Obviously, $\varphi(g) = g$ for every $g \in \langle a\rangle$. If $\varphi(b) = b^\gamma a^\alpha$ for $0 \leqslant \gamma \leqslant p-1$ and $0 \leqslant \alpha \leqslant p^m - 1$, then

$$a^{(1+p^{m-1})} = \varphi(a^b) = a^{\varphi(b)} = a^{b^\gamma a^\alpha} = a^{b^\gamma} = a^{(1+p^{m-1})^\gamma}.$$

Hence

$$(1+p^{m-1})^\gamma \equiv 1+p^{m-1} \pmod{p^m},$$

or equivalently

$$(1+p^{m-1})((1+p^{m-1})^{\gamma-1}-1) \equiv 0 \pmod{p^m}.$$

So $\gamma - 1 \equiv 0 \pmod{p}$, i.e. $\gamma = 1$.

Since h is an arbitrary element of H, the latter centralizes the normal series

$$1 \leqslant \langle a\rangle \leqslant P.$$

Since H is a p'-group, we have $G = P \times H$.

By Item (1) of Lemma 1

$$\frac{\psi(G)}{\psi(C_n)} = \frac{\psi(P)\psi(H)}{\psi(C_{|P|})\psi(C_{|H|})}.$$

Recall that H is a cyclic subgroup, and therefore

$$\frac{\psi(G)}{\psi(C_n)} = \frac{\psi(P)}{\psi(C_{|P|})} \leqslant \frac{p^6+p^3-p^2+1}{p^7+1} < \frac{q^6+q^3-q^2+1}{q^7+1};$$

that is a contradiction, and the proof is complete.

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