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DIVISIBLE DESIGN GRAPHS WITH PARAMETERS $(4n, n+2, n-2, 2, 4, n)$ AND $(4n, 3n-2, 3n-6, 2n-2, 4, n)$

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Abstract. A k-regular graph is called a divisible design graph (DDG for short) if its vertex set can be partitioned into m classes of size n, such that two distinct vertices from the same class have exactly λ_1 common neighbors, and two vertices from different classes have exactly λ_2 common neighbors. A 4-by-n-lattice graph is the line graph of $K_{4,n}$. This graph is a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$. In the paper, we consider DDGs with these parameters. We prove that if n is odd, then such graph can only be a 4-by-n-lattice graph. If n is even, we characterise all DDGs with such parameters. Moreover, we characterise all DDGs with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$ that are related to 4-by-n-lattice graphs. Also, we prove that if Deza graph with parameters $(4n, n+2, n-2, 2)$ or $(4n, 3n-2, 3n-6, 2n-2)$ is not a DDG, then $n \leq 8$. All such Deza graphs were classified by computer search.

Keywords: divisible desing graph, divisible design, Deza graph, lattice graph.

1. Introduction

A k-regular graph on v vertices is called a *divisible design graph* (DDG for short) with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if its vertex set can be partitioned into m classes of size n, such that two distinct vertices from the same class have exactly λ_1 common neighbors, and two vertices from different classes have exactly λ_2 common neighbors. A DDG with $m = 1$, $n = 1$, or $\lambda_1 = \lambda_2$ is called improper; in the opposite case, it is called proper. Divisible design graphs were first studied by

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M.A. Meulenberg in his master's thesis [9], and later the studies were developed in three papers by D. Crnkovic, W.H. Haemers, H. Kharaghani and M.A. Meulenberg [2, 3, 5] in 2011.

Every graph G can be interpreted as a design, by taking the vertices of G as points, and the neighborhoods of the vertices as blocks. In other words, the adjacency matrix of G is interpreted as the incidence matrix of a design. We refer to this design as the *neighborhood design* of G. An incidence structure with constant block size k is a *divisible design* if the point set can be partitioned into m classes of size n, such that two points from one class occur together in λ_1 blocks, and two points from different classes occur together in exactly λ_2 blocks. A divisible design D is called symmetric (or is said to have the dual property) if the dual of D (that is, a design with a transposed incidence matrix) is again a divisible design with the same parameters as D. The definition of a DDG implies that its neighborhood design is a symmetric divisible design.

A *Deza graph* with parameters (v, k, b, a) is a k-regular graph with v vertices, such that every two distinct vertices have b or a common neighbors, where $b \geqslant a$. The definition of a DDG implies that DDG is a Deza graph with $\{b, a\} = \{\lambda_1, \lambda_2\}$.

An $m{\times}n$-*lattice graph* is a graph with the vertex set $\{1, \dots, m\} \times \{1, \dots, n\}$. It is well known that two vertices are adjacent if they share their first or second coordinate. An $m{\times}n$-lattice graph is a DDG when $m = 4$ (or $n = 4$). In this paper, we characterise DDGs with parameters that are similar to the parameters of the $4{\times}n$-lattice graph. These graphs have parameters $(4n, n+2, n-2, 2, 4, n)$. We also characterise all DDGs with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$ that are related to $4{\times}n$-lattice graphs.

In [5, Construction 4.8], the construction of two series of DDGs with parameters $(4n, n+2, n-2, 2, 4, n)$ and $(4n, 3n-2, 3n-6, 2n-2, 4, n)$ from Hadamard matrices was presented. The first one corresponds to $4{\times}n$-lattice graphs and the second one can be obtained from $4{\times}n$-lattice graphs by switching all edges between two pairs of classes

The main result of this article is a characterisation of DDGs with parameters $(4n, n+2, n-2, 2, 4, n)$ and $(4n, 3n-2, 3n-6, 2n-2, 4, n)$.

The paper is organised as follows. In Section 2, we provide some definitions, notations, and preliminaries about DDGs and Deza graphs. In Section 3, we consider Deza graphs with parameters $(4n, n+2, n-2, 2)$ and $(4n, 3n-2, 3n-6, 2n-2)$ that are not DDGs, in particular, we prove that in this case $n \leq 8$. In Section 4, we characterise DDGs with parameters $(4n, n+2, n-2, 2, 4, n)$. In Section 5, we characterise DDGs with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$.

2. Preliminaries

We denote the neighborhood of the vertex x by $N(x)$ and the set of common neighbors of the vertices x and y by $N(x, y)$.

2.1. Properties of DDGs.

Proposition 1. [5, Lemma 2.1] *The eigenvalues of the adjacency matrix of a DDG with parameters* $(v, k, \lambda_1, \lambda_2, m, n)$ *are*

$$\{k, \sqrt{k-\lambda_1}, -\sqrt{k-\lambda_1}, \sqrt{k^2-\lambda_2 v}, -\sqrt{k^2-\lambda_2 v}\},$$

with the multiplicities $1, f_1, f_2, g_1, g_2$*, respectively. Moreover,* $f_1 + f_2 = m(n-1)$ *and* $g_1 + g_2 = m - 1$*.*

The equation [5, Equation 2] states that the trace of adjacency matrix A of a DDG is:

$$\mathrm{trace}(A) = 0 = k + (f_1 - f_2)\sqrt{k - \lambda_1} + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v} \tag{1}$$

Let $V_1 \cup V_2 \cup \ldots \cup V_t$ be the partition of the vertex set of a graph Γ with the property that every vertex of V_i has exactly r_{ij} neighbors in V_j. Then $V_1 \cup V_2 \cup \ldots \cup V_t$ will be an *equitable t-partition* of Γ. Matrix $R = (r_{ij})_{t\times t}$ is called the *quotient matrix* of the equitable partition.

Proposition 2. [5, Theorem 3.1] *The vertex partition from the definition of a DDG (the canonical partition) is equitable and the quotient matrix* R *satisfies the following equation*

$$R^2 = RR^T = (k^2 - \lambda_2 v)I_m + \lambda_2 n J_m$$

.

Further, by a factor matrix of DDG we mean the matrix corresponding to the canonical partition of DDG.

Proposition 3. [5, Proposition 3.2] *The quotient matrix* R *of DDG satisfies the following conditions:*
$\sum\limits_i (R)_{i,j} = k$*, for all* $j = 1, 2, \ldots, m$*,*
$0 \le \mathrm{trace}(R) = k + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v} \le m(n-1)$*.*

2.2. **Construction that arises from Hadamard matrices.** An $m \times m$ matrix H is a *Hadamard matrix* if every entry is 1 or -1, and $HH^T = mI$. A Hadamard matrix H is called *graphical*, if H is symmetric with a constant diagonal, and regular, if all row and column sums are equal (for example, l). Without loss of generality we assume that a graphical Hadamard matrix has diagonal entries -1. Consider a regular graphical Hadamard matrix H.

The next construction is based on [5, Contruction 4.8].

Construction 1. *Consider the smallest regular graphical Hadamard matrices*

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{bmatrix}.$$

We replace every entry with value -1 *by* $J_n - I_n$ *and each* $+1$ *by* I_n*, and obtain the adjacency matrix of a DDG with parameters* $(4n, n+2, n-2, 2, 4, n)$ *and* $(4n, 3n-2, 3n-6, 2n-2, 4, n)$ *respectively.*

The second graph can be obtained from the first one by switching edges between two pairs of classes of the canonical partition.

2.3. **Switching constructions.** *Switching* a set of vertices in a graph means reversing the adjacencies of each pair of vertices, where one belongs to the set and the other does not: thus, the edge set is changed such that an adjacent pair becomes nonadjacent and vice versa. The edges, both of whose endpoints either belong to the set, or do not belong, are not changed. Graphs are *switching equivalent* if one of them can be obtained from the other one by switching. Switching was introduced by van Lint and Seidel (see [11]) and developed by Seidel.

An involutive automorphism of a graph is called a *Seidel automorphism* if it only interchanges nonadjacent vertices.

Construction 2 (Dual Seidel switching; [4, Theorem 3.1]). *Let G be a strongly regular graph with parameters (v, k, λ, μ), where $k \neq \mu$, $\lambda \neq \mu$. Suppose that M is the adjacency matrix of G, and P is a non-identity permutation matrix of a similar size. Then PM is the adjacency matrix of a Deza graph Γ if and only if P represents a Seidel automorphism. Moreover, Γ is a Deza graph if and only if $\lambda \neq 0$, $\mu \neq 0$.*

Construction 3 (Generalised dual Seidel switching 2; [6, Theorem 6]). *Let G be a Deza graph with the adjacency matrix $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, and H be its induced subgraph with the adjacency matrix M_{11}. If there exists a Seidel automorphism of H with the permutation matrix P_{11} such that $P_{11}M_{12}M_{22} = M_{12}M_{22}$, then the matrix*

$$N = \begin{pmatrix} P_{11}M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is the adjacency matrix of a Deza graph.

Remark 1. *The combinatorial meaning of the matrix condition is as follows. Condition $P_{11}M_{11}M_{12} = M_{11}M_{12}$ from Theorem 3 means that for every $v \in V(G) \setminus V(H)$ and for every $x, y \in V(H)$ such that $\varphi(x) = y$, the number of common neighbors for v and x in H is equal to the number of common neighbors for v and y in H.*

Remark 2. *Note that in* [6], *Construction 3 was considered only for Deza graphs with strongly regular children but the proof does not use this property, therefore, this construction can be applied to every Deza graph.*

3. Deza graphs which are not DDGs

Proposition 4. [7, Theorem 1] *If G is a Deza graph with parameters (v, k, b, a) and $a < 2b - k$, then G is a DDG.*

Lemma 1. *If G is a Deza graph with parameters $(4n, n+2, n-2, 2)$ or $(4n, 3n-2, 3n-6, 2n-2)$ and G is not a DDG, then $n \leq 8$.*

Proof. (1) Let G have parameters $(4n, n+2, n-2, 2)$. By Lemma 4, if G is not a DDG, then $a \geq 2b - k$. Thus, $2 \geq 2(n-2) - (n+2)$, and $n \leq 8$.
(2) Let G have parameters $(4n, 3n-2, 3n-6, 2n-2)$. By Lemma 4, if G is not a DDG, then $a \geq 2b - k$. Thus, $2n - 2 \geq 2(3n-6) - (3n-2)$, and $n \leq 8$. □

Deza graphs with parameters $(4n, n+2, n-2, 2)$ and $(4n, 3n-2, 3n-6, 2n-2)$ in the case when $n \leq 8$ were determined completely by computer search. For $n = 6$, we found 48 non-isomorphic Deza graphs with parameters $(4n, n+2, n-2, 2)$ and

10 non-isomorphic Deza graphs with parameters $(4n, 3n-2, 3n-6, 2n-2)$ that are not DDGs. For the remaining values of n (i.e., 2, 3, 4, 5, 7, and 8), we found only DDGs.

Adjacency matrices of all Deza graphs with such parameters are available on the web pages `http://alg.imm.uran.ru/dezagraphs/deza.php?v=24&k=8&b=4&a=2&form=None` and `http://alg.imm.uran.ru/dezagraphs/deza.php?v=24&k=16&b=12&a=10&form=None`.

4. DDGs with parameters $(4n, n+2, n-2, 2, 4, n)$

The main result of this section is the following theorem.

Theorem 1. *Let G be a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$. Then one of the following cases hold.*

1. *If n is odd, then G is isomorphic to a $4\times n$-lattice graph,*
2. *If n is even, then the quotient matrix R of G equals with respect to the canonical partition to one of matrices (4), (5) or (6)*
   - (a) *if R equals matrix (4), then G is isomorphic to the $4\times n$-lattice graph,*
   - (b) *if R equals matrix (5), then G is isomorphic to the graph G' from Construction 4 (see below),*
   - (c) *if R equals matrix (6), then G is isomorphic to one of the graphs obtained from Construction 5 (see below).*

For $n \leq 8$, we verified the assertion of Theorem 1 by computer enumeration, and it is true. Therefore, we will assume that $n > 8$ if necessary.

By Proposition 1, we have $g_1 + g_2 = m - 1 = 3$. We can calculate all possibilities for g_1, g_2 and $\mathrm{trace}(R)$ using Proposition 3 and Equation (1).

g_1	g_2	$\mathrm{trace}(R)$
3	0	$4n-4$
2	1	$2n$
1	2	4
0	3	$8-2n$

Таблица 1

If $g_1 = 0$ and $n > 8$, then $\mathrm{trace}(R) < 0$, which is impossible.

If $g_1 = 3$, then G has exactly four eigenvalues $\{k, \pm 2, n-2\}$. The classification of graphs with the smallest eigenvalue -2 (see [1, Section 3.12]) implies that G is isomorphic to the $4\times n$-lattice graph.

4.1. Quotient matrices.

Lemma 2. *Let G be a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ and suppose that n is odd. If $R = [r_{ij}]$ is the quotient matrix of G, then r_{ii} is even for all $i = 1, \ldots, m$.*

Proof. Since r_{ii} is the valency of the subgraph induced by the vertices of the i-th class of canonical partition, r_{ii} is even for an odd n. □

Proposition 5. *Let G be a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$ where $n > 8$ and suppose that (a, b, c, d) is a row of the quotient matrix R. Then $\{a, b, c, d\} = \{n-1, 1, 1, 1\}$.*

Proof. By Proposition 3, we have the equality

$$a + b + c + d = n + 2 \tag{2}$$

and by Proposition 2, we have

$$a^2 + b^2 + c^2 + d^2 = n^2 - 2n + 4. \tag{3}$$

First, note that if there is $x \in \{a, b, c, d\}$ such that $x \geq n$, then equation (3) does not hold. Now denote by x the largest element in $\{a, b, c, d\}$ and by y the second largest element. In the case $x \leq n-2$, the sum $a^2 + b^2 + c^2 + d^2$ is maximal if $x = n-2$, $y = 4$ and two other elements equal 0. Then we have $a^2+b^2+c^2+d^2 \leq (n-2)^2+16$. By (3), we have $n^2 - 2n + 4 \leq (n-2)^2 + 16$. Hence, $n \leq 8$. Since $n > 8$, we have $x = n-1$ and we need to check three possibilities for $\{a, b, c, d\}$: $\{n-1, 3, 0, 0\}$, $\{n-1, 2, 1, 0\}$, and $\{n-1, 1, 1, 1\}$. It is easy to see that only the third of them satisfies (3). □

Corollary 1. *Let G be a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$. Then there are exactly three following possibilities for quotient matrix R of G with respect to the numbering of classes.*

$$\begin{bmatrix} n-1 & 1 & 1 & 1 \\ 1 & n-1 & 1 & 1 \\ 1 & 1 & n-1 & 1 \\ 1 & 1 & 1 & n-1 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} 1 & n-1 & 1 & 1 \\ n-1 & 1 & 1 & 1 \\ 1 & 1 & n-1 & 1 \\ 1 & 1 & 1 & n-1 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} 1 & n-1 & 1 & 1 \\ n-1 & 1 & 1 & 1 \\ 1 & 1 & 1 & n-1 \\ 1 & 1 & n-1 & 1 \end{bmatrix} \tag{6}$$

Remark 3. *It is easy to see that these matrices correspond to rows of table 1. Therefore, if G has a quotient matrix (4), then G is isomorphic to a $4\times n$-lattice graph.*

Statement 1 of Theorem 1 immediately follows from Lemma 2 and Remark 3.

Further we will assume that n is even. We need to consider graphs with quotient matrices (5) and (6).

4.2. **Graphs with quotient matrix (5).**

Construction 4. *The $4\times n$-lattice graph G has an induced subgraph H such that H is isomorphic to the $2\times n$-lattice graph. By Remark 1, Construction 3 is applicable for these G and H. For every even $n \geqslant 6$, there is a Seidel automorphism φ of H, such that for every $i \in \{1,2\}$ and for every $j \in \{1,\dots,n\}$ we have that $\varphi((i,j)) = (3-i, n+1-j)$. The subgraph H satisfies the condition of Construction 3 since the combinatorial condition of Remark 1 holds. Indeed, for every $v \in V(G)\setminus V(H)$ and for every $x, y \in V(H)$ such that $\varphi(x) = y$, there is exactly one common neighbor in H for v and x, and exactly one for v and y. Hence, by Theorem 3, for $n \geqslant 6$ there is a Deza graph with parameters $(4n, n+2, n-2, 2)$. Since automorphism φ interchanges two blocks of the canonical partition of the $4\times n$-lattice graph, the obtained Deza graph is a DDG. It is clear that it has quotient matrix (5). We denote this graph by $G'(n)$.*

In this section, we prove the statement $2b$ of Theorem 1. If G is a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$ and quotient matrix (5), then G is isomorphic to $G'(n)$.

Proof. We denote the blocks of the canonical partition of G by V_1, V_2, V_3, and V_4 with respect to the quotient matrix (5). According to the quotient matrix (5), the vertices of V_1 and V_2 induce cliques in G. There is a perfect matching between them. We denote the vertices of the first block by $(1,1), (1,2), \dots, (1,n)$, and the vertices of the second block by $(2,1), (2,2), \dots, (2,n)$, such that vertices $(1,i)$ and $(2,i)$ are adjacent for every i.

Consider all pairs of vertices from blocks V_1 and V_2. Recall that in G, vertices from the same block have $n-2$ common neighbors, and vertices from different blocks have 2 common neighbors. Every two vertices from V_1 have $n-2$ common neighbors in V_1 and no common neighbors in other blocks. That is also true for every two vertices from V_2. Vertices $(1,i)$ and $(2,j)$, where $i \neq j$, have two common neighbors ($(1,j)$ and $(2,i)$) in $V_1 \cup V_2$, and no common neighbors in V_3 and V_4. Vertices $(1,i)$ and $(2,i)$ have no common neighbors in $V_1 \cup V_2$, and 2 common neighbors in $V_3 \cup V_4$. But $(1,i)$ has only one neighbor in V_3 and one neighbor in V_4. We denote these two vertices by $(3,i)$ in V_3 and $(4,i)$ in V_4. Then $(3,i)$ and $(4,i)$ are adjacent with the vertex $(2,i)$ and have no other neighbors in $V_1 \cup V_2$.

Now consider common neighbors of vertices $(1,i)$ and $(3,i)$. In this case, $N((1,i)) \cap N((3,i)) = \{(2,i),(4,i)\}$ because $N((1,i)) \cap (V_3 \cup V_4) = \{(3,i),(4,i)\}$, $N((3,i)) \cap (V_1 \cup V_2) = \{(1,i),(2,i)\}$ and vertices $(1,i)$, $(3,i)$ have 2 common neighbors in the graph G. Similarly, $N((1,i)) \cap N((4,i)) = \{(2,i),(3,i)\}$, hence, vertices $\{(1,i),(2,i),(3,i),(4,i)\}$ induce a clique in graph G.

From the quotient matrix (5) we know that the subgraph of G induced by V_3 is isomorphic to $n/2$ copies of K_2. Without loss of generality we assume that every vertex $(3,i)$ is adjacent to the vertex $(3, n-i+1)$ for all $i = 1, 2, \dots, n/2$. Now consider the vertices $(4,i)$ and $(4, n-i+1)$. Since vertices $(1,i)$ and $(3, n-i+1)$ have 2 common neighbors in G and $(1, n-i+1), (3,i) \in N((1,i),(3,n-i+1))$, it follows that every vertex $(3, n-i+1)$ is not adjacent to the vertex $(4,i)$. Similarly, every vertex $(3,i)$ is not adjacent to the vertex $(4, n-i+1)$. Therefore, the vertices $(1,i)$ and $(4, n-i+1)$ have only one common neighbor in $V_1 \cup V_2 \cup V_3$. But every vertex $(1,i)$ has one neighbor (the vertex $(4,i)$) in V_4. Hence, vertices $(4,i)$ and $(4, n-i+1)$

are adjacent for every i. Moreover, according to the quotient matrix (5), each vertex $(3,i)$ is adjacent with all vertices $(4,j)$ with the exception of $(4,n-i+1)$.

We described all edges of the graph G. Hence, there is a unique DDG with such parameters and quotient matrix (5). Thus, this graph is isomorphic to $G'(n)$. □

4.3. **Graphs with quotient matrix (6).** Let G be a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$ and the quotient matrix (6). Further, suppose that $A = [A_{ij}]$ is the adjacency matrix of G with blocks A_{ij} corresponding to the canonical partition of G. Now consider an auxiliary graph G^* with the following adjacency matrix:

$$A^* = \begin{bmatrix} A_{11} & J_n - A_{12} & A_{13} & A_{14} \\ J_n - A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & J_n - A_{34} \\ A_{41} & A_{42} & J_n - A_{43} & A_{44} \end{bmatrix}. \tag{7}$$

This matrix can be obtained by the operation that corresponds to switching of all edges between V_1 and V_2, and between V_3 and V_4. The partition (V_1, V_2, V_3, V_4) is equitable for the graph $G**$ with quotient matrix J_4.

Note that graph G^* is regular with valency 4 and it can be disconnected. In the following lemma, we calculate the number of common neighbors of all pairs of vertices in the graph G^*. We denote the set of common neighbors of vertices x and y in the graph G^* by $N_{G^*}(x,y)$. Each vertex $x \in V_i$ in the graphs G and G^* has only one neighbor in its own class V_i; we denote this neighbor by x'.

Lemma 3. *Let G^*, V_1, V_2, V_3, and V_4 be as described above, and suppose that x and y are distinct vertices of G^*. Without loss of generality we can assume that $x \in V_1$. Consider all possible cases of the location of the vertex y:*

(1) *If $y \in V_1$, then $|N_{G^*}(x,y)| = 0$,*

(2) *If $y \in V_2$, and*
   - (a) *if $|N(x,y) \cap (V_1 \cup V_2)| = 2$, then $|N_{G^*}(x,y)| = 0$,*
   - (b) *if $|N(x,y) \cap (V_1 \cup V_2)| = 1$, then $|N_{G^*}(x,y)| = 2$,*
   - (c) *if $|N(x,y) \cap (V_1 \cup V_2)| = 0$, then $|N_{G^*}(x,y)| = 4$,*

(3) *If $y \in V_i$, where $i \in \{3,4\}$, and*
   - (a) *if $|N(x,y) \cap (V_1 \cup V_i)| = 2$, then $|N_{G^*}(x,y)| = 4$,*
   - (b) *if $|N(x,y) \cap (V_1 \cup V_i)| = 1$, then $|N_{G^*}(x,y)| = 2$,*
   - (c) *if $|N(x,y) \cap (V_1 \cup V_i)| = 0$, then $|N_{G^*}(x,y)| = 0$,*

Proof. (1) Since G has quotient matrix 6, all common neighbors of x and y in G lie in V_2. Since the edges between V_1 and V_2 in G^* form a perfect matching, we have that x and y have no common neighbors in G^*,

(2) (a) The vertex x is adjacent to y' and the vertex y is adjacent to x' in G, but in G^* these adjacencies are removed. Thus, x and y have no common neighbors in G^*.
   - (b) In G, either x is adjacent to y' or y (but not both of them) is adjacent to x' and they have one more common neighbor in $V_3 \cup V_4$. In G^*, edges between V_1 and V_2 are switched; hence, x and y have one common neighbor in $V_1 \cup V_2$ and one common neighbor in $V_3 \cup V_4$.
   - (c) In G, neither x is adjacent to y', nor y is adjacent to x'. Hence, x is adjacent to y' and y is adjacent to x' in G^*. Moreover, x and y have two common neighbors in $V_3 \cup V_4$.

(3) Let $y \in V_3$. If $y \in V_4$, then the proof is the same as if $y \in V_3$.
   (a) In G, vertices x and y have two common neighbors in $V_1 \cup V_3$ and have no common neighbors in $V_2 \cup V_4$. Hence, in G^*, they have two common neighbors in $V_1 \cup V_3$ and two common neighbors in $V_2 \cup V_4$. New common neighbors are the unique neighbor of y in V_2 and the unique neighbor of x in V_4.
   (b) In G, vertices x and y have one common neighbor in $V_1 \cup V_3$ and one common neighbor in $V_2 \cup V_4$. Therefore, in G^*, they have the same common neighbor in $V_1 \cup V_3$ and the other common neighbor in $V_2 \cup V_4$.
   (c) In G, vertices x and y have no common neighbors in $V_1 \cup V_3$ and two common neighbors in $V_2 \cup V_4$. Hence, in G^*, they have no common neighbors in $V_1 \cup V_3$ and in $V_2 \cup V_4$.

□

Consider a connected component of G^*. We denote it by D and suppose that $D_1 = D \cap V_1$, $D_2 = D \cap V_2$, $D_3 = D \cap V_3$, $D_4 = D \cap V_4$. Since G^* has the equitable partition with parts V_1, V_2, V_3, V_4 and with quotient matrix J_4, it follows that the set $D_i \cup D_j$ induces a subgraph of valency 2, which is the union of cycles, for every i and j $(i \neq j)$.

Lemma 4. *The size of D is divisible by* 8.

Proof. Consider the largest independent set S in D_1. For each vertex $x \in S$, we have one neighbor x' in D_1. Also for each vertex x we have one neighbor in D_2, one neighbor in D_3 and one in D_4. That is also true for the vertex x'. Since every other vertex must have one neighbor in D_1 which is not adjacent to vertices from S, then there are no other vertices in D. Finally, we have $|D| = 8|S|$ and $|D|$ is divisible by 8. □

In the next two lemmas, we consider the case when D contains vertices x and y with 4 common neighbors.

Lemma 5. *There exist vertices $x, y \in D$ of type* 2*c from Lemma 3 if and only if there are four vertices in $D_1 \cup D_2$ (or in $D_3 \cup D_4$) that induce a cycle. In this case, the induced subgraphs on $D_1 \cup D_2$ and $D_3 \cup D_4$ are isomorphic to sC_4 for some s. Moreover, D is isomorphic to $C_s[\overline{K_2}]$ where every copy of $\overline{K_2}$ is a pair of vertices of type* 2*c from Lemma 3. Alternately, two of such pairs are from $V_1 \cup V_2$ and two pairs are from $V_3 \cup V_4$. We denote a cycle of length s by C_s and a pair of nonadjacent vertices by $\overline{K_2}$.*

Proof. First, consider vertices $x, y \in D$ of type $2c$ from Lemma 3. They have two common neighbors (x' and y') in $D_1 \cup D_2$ and two common neighbors (say, z and t) in $D_3 \cup D_4$. Hence, the pairs x', y' and z, t are of type $2c$. Then z', t' and two common neighbors x', y' in $D_3 \cup D_4$ are of type $2c$. Then the common neighbors of z', t' are also of type $2c$. We continue this process until the cycle of pairs of vertices is closed. Thus, we have a cycle of pairs of type $2c$, where two pairs x, y and x', y' are taken from $D_1 \cup D_2$, the next two pairs are from $D_3 \cup D_4$ and further alternately, because we can start with any of these pairs.

Conversely, if we have the cycle (x, x', y, y') as the induced subgraph in $D_1 \cup D_2$ (or $D_3 \cup D_4$), then the pairs of vertices x, y and x', y' are of type $2c$, and the lemma is proved. □

Lemma 6. *There exist vertices $x, y \in D$ of type $3a$ from Lemma 3 if and only if there are four vertices in $D_1 \cup D_3$ (or in $D_2 \cup D_4$) that induce the cycle. In this case, the induced subgraphs on $D_1 \cup D_2$ and $D_3 \cup D_4$ are isomorphic to sC_4 for some s. Moreover, D is isomorphic to $C_s[\overline{K_2}]$, where each copy of $\overline{K_2}$ is a pair of vertices of type $2c$ from Lemma 3. Alternately, two such pairs are from the set $V_1 \cup V_3$ and two pairs are from $V_2 \cup V_4$.*

Proof. First, consider vertices $x, y \in D$ of type $3a$ from Lemma 3. They have two common neighbors (x' and y') in $D_1 \cup D_3$ and two common neighbors (say, z and t) in $D_2 \cup D_4$. Then the pairs x', y' and z, t are of type $3a$. Therefore, z', t' and two common neighbors x', y' in $D_2 \cup D_4$ are of type $3a$. Then the common neighbors of z', t' are also of type $3a$. We continue this process until the cycle of pairs of vertices is closed. Thus, we have a cycle of pairs of type $3a$, where two pairs x, y and x', y' are taken from $D_1 \cup D_3$, the next two pairs are from $D_2 \cup D_4$ and further alternately, because we can start with any of these pairs.

Conversely, if we have a cycle (x, x', y, y') as the induced subgraph in $D_1 \cup D_3$ (or $D_2 \cup D_4$), then the pairs of vertices x, y and x', y' are of type $3a$, and the lemma is proved. □

The subgraph D in Lemma 5 and Lemma 6 is the same one, but it is embedded in a different way in the graph G^*. In the first case, we alternate two pairs of vertices from V_1 and V_2 with two pairs of vertices from V_3 and V_4; in the second one, we alternate two pairs of vertices from V_1 and V_3 with two pairs of vertices from V_2 and V_4. And we switch only edges between V_1 and V_2 and between V_3 and V_4. Hence, corresponding subgraphs in the graph G are not isomorphic, excluding the case when D is isomorphic to $C_4[\overline{K_2}]$, because in this case for D both lemmas hold.

A $(0, 2)$-*graph* is a connected graph such that every two vertices have 0 or 2 common neighbors.

Lemma 7. *If the connected component D has no pairs of vertices with four common neighbors, then D is isomorphic to a 4-cube.*

Proof. If the connected component D has no pairs of vertices with four common neighbors, then D is a $(0, 2)$-graph of valency 4. In [8], Mulder proved that if a k-regular $(0, 2)$-graph on v vertices has a diameter d, then $v \leq 2^k$ and $d \leq k$. In both cases, equality is true only for a k-cube with parameters $(2^k, k, 2, 0)$. By Lemma 4, the size of D is divisible by 8, so D is isomorphic to a 4-cube or is a Deza graph with parameters $(8, 4, 2, 0)$. But in the last case, $D_1 \cup D_2$ induces the cycle C_4, so by Lemma 5 we have a contradiction. □

Lemma 8. *There exist three non-isomorphic equitable partitions of the 4-cube with quotient matrix J_4.*

Proof. The inner edges of parts of the partition form a perfect matching in the 4-cube. Moreover, two edges from the same part are antipodal in the 4-cube. Now we can use the result that was obtained in paper [10], that states that there are eight equivalence classes of perfect matchings in a 4-cube and only three of them satisfy the necessary condition. It is enough to test these perfect matchings to prove that there are three antipodal perfect matchings and three corresponding equitable partitions. □

Construction 5. *Consider some copies of the graph $C_s[\overline{K_2}]$ with equitable partitions as in Lemmas 5 or 6 and some copies of the 4-cube with one of three equitable partitions. Then we can switch edges between V_1 and V_2 and also between V_3 and V_4. This switching gives us a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$ and quotient matrix (6).*

To complete the proof of Theorem 1, we prove statement 2*c*.

Proof of statement 2c of Theorem 1. By Lemmas 5, 6, and 7, the connected component D of the graph G^* is isomorphic to $C_s[\overline{K_2}]$ or a 4-cube. The graph $C_s[\overline{K_2}]$ has two non-isomorphic embeddings into the canonical partition of G^*. The 4-cube has three non-isomorphic embeddings into the canonical partition of G^*. Now we can obtain G^* only from such connected components. Hence, a divisible design graph G can be obtained from G^* by the reverse switching. Thus, Theorem 1 is proved.

5. DDGs with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$

The main result of this section is the following theorem.

Theorem 2. *Let G be a DDG with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$. Then G can be obtained from a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$ by switching between the first two classes and the last two classes of the canonical partition of DDG. Classes are numbered according to the quotient matrix from Corollary 1.*

For $n \leq 8$, we verified the assertion of Theorem 2 by computer enumeration, and it is true. Therefore, we will assume that $n > 8$ if necessary.

5.1. **Quotient matrices.** By Proposition 1, we have $g_1 + g_2 = m - 1 = 3$. We can calculate all possibilities for g_1, g_2, f_1, f_2 and trace(R) using Proposition 3 and Equation (1).

g_1	g_2	trace(R)
3	0	$6n-8$
2	1	$4n-4$
1	2	$2n$
0	3	4

Таблица 2

If $g_1 = 3$ and $n > 8$, then trace$(R) > 4n-4$, which is impossible by Proposition 3.

Proposition 6. *Let G be a DDG with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$, where $n > 8$ and suppose that (a, b, c, d) is a row of the quotient matrix R. Then $\{a, b, c, d\} = \{n-1, n-1, n-1, 1\}$.*

Proof. By Proposition 3, we have

$$a + b + c + d = 3n - 2 \tag{8}$$

and by Proposition 2, we have

$$a^2 + b^2 + c^2 + d^2 = 3n^2 - 6n + 4 \tag{9}$$

Assume that d is the smallest element in $\{a, b, c, d\}$. First, note that $a^2+b^2+c^2+d^2$ is maximal if a, b and c are maximal. If $d \geq 2$, then $a^2+b^2+c^2$ is maximal when $a = n$, $b = n$, $c = n-4$, and $d = 2$. But in this case, $a^2+b^2+c^2+d^2 = 3n^2-8n+20$. If $3n^2-8n+20 \geq 3n^2-6n+4$, then $n \leq 8$. Since $n > 8$, we can assume that $d = 1$. Then $\{a, b, c\}$ is equal to $\{n, n, n-3\}$ or $\{n, n-1, n-2\}$ or $\{n-1, n-1, n-1\}$. Only the last case gives us the equality $a^2+b^2+c^2+1 = 3n^2+6n+4$. If $d = 0$, then $\{a, b, c\}$ is equal to $\{n, n, n-2\}$ or $\{n, n-1, n-1\}$. Both cases give us the inequality $a^2+b^2+c^2 \neq 3n^2+6n+4$. □

Corollary 2. *Let G be a DDG with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$. Then there exist exactly three following possibilities for quotient matrix R of G with respect to the numbering of classes.*

$$\begin{bmatrix} n-1 & 1 & n-1 & n-1 \\ 1 & n-1 & n-1 & n-1 \\ n-1 & n-1 & n-1 & 1 \\ n-1 & n-1 & 1 & n-1 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} n-1 & 1 & n-1 & n-1 \\ 1 & n-1 & n-1 & n-1 \\ n-1 & n-1 & 1 & n-1 \\ n-1 & n-1 & n-1 & 1 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} 1 & n-1 & n-1 & n-1 \\ n-1 & 1 & n-1 & n-1 \\ n-1 & n-1 & 1 & n-1 \\ n-1 & n-1 & n-1 & 1 \end{bmatrix} \tag{12}$$

Remark 4. *These matrices correspond to the last three rows of table 2. If we switch edges between the first two and the second two classes of the canonical partition of the graph with quotient matrix (10), then we obtain a graph with corresponding equitable partition with quotient matrix (4). Therefore, if G has quotient matrix (10), then G is isomorphic to the second graph from Construction 1.*

If we switch edges between the first two and the second two classes of the canonical partition of graphs with quotient matrices (11) and (12), then we obtain graphs with corresponding equitable partition with quotient matrices (5) and (6) respectively. To prove Theorem 2, it suffices to check that the resulting graphs are DDGs with parameters $(4n, n+2, n-2, 2, 4, n)$.

5.2. **Proof of Theorem 2.** The proof is carried out by a simple check of the numbers of common neighbors of the pairs of vertices in all possible cases.

We denote the blocks of the canonical partition of G by V_1, V_2, V_3 and V_4 with respect to the quotient matrix.

Lemma 9. *Let G be a DDG with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$ with quotient matrix (11). Suppose that G' is the graph obtained from G by switching between the first two and the last two classes of the canonical partition. Then G' is a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$ and with quotient matrix (5).*

Proof. We need to consider all pairs of vertices in G and show that two vertices from the same class have $n-2$ common neighbors in G' and two vertices from the different classes have 2 common neighbors in G'. Consider all possibilities for the number of common neighbors of vertices $x, y \in G$.

- Consider $x, y \in V_i$, where $i = 1$ or 2. In this case, x and y have $n-2$ common neighbors in V_i, $n-2$ common neighbors in V_3 and $n-2$ common neighbors in V_4, since each of them has $n-1$ neighbors in V_i, V_3 and V_4. Then each vertex in $V_3 \cup V_4$ is adjacent to x or y. Hence, the vertices x and y have only $n-2$ common neighbors in V_i in G'.
- Consider $x \in V_1$, $y \in V_2$, where $x \sim y$. In this case, x and y have no common neighbors in $V_1 \cup V_2$. Hence, x and y have $n-1$ common neighbors in V_3 and $n-1$ common neighbors in V_4. Then there are two vertices in $V_3 \cup V_4$ that are nonadjacent to both x and y. Hence, x and y have 2 common neighbors in $V_3 \cup V_4$ in G'.
- Consider $x \in V_1$, $y \in V_2$, where $x \not\sim y$. In this case, x and y have 2 common neighbors in $V_1 \cup V_2$. Hence, x and y have $n-2$ common neighbors in V_3 and $n-2$ common neighbors in V_4. Then each vertex in $V_3 \cup V_4$ is adjacent with x or y. Hence vertices x and y have 2 common neighbors in $V_1 \cup V_2$ in G'.
- Consider $x, y \in V_i$, where $i = 3$ or 4. In this case, x and y have $n-2$ common neighbors in V_{7-i}, $n-2$ common neighbors in V_1 and $n-2$ common neighbors in V_2. Then each vertex in $V_1 \cup V_2$ is adjacent to x or y. Hence, x and y have only $n-2$ common neighbors in V_{7-i} in G'.
- Consider $x \in V_3$, $y \in V_4$, where $|N(x, y) \cap (V_3 \cup V_4)| = 2$. In this case, x and y have $2n-4$ common neighbors in $V_1 \cup V_2$. Hence, x and y have $n-2$ common neighbors in V_1 and $n-2$ common neighbors in V_2. Then each vertex in $V_1 \cup V_2$ is adjacent to x or y. Hence, x and y have no common neighbors in $V_1 \cup V_2$ and 2 common neighbors in $V_3 \cup V_4$ in G'.
- Consider $x \in V_3$, $y \in V_4$, where $|N(x, y) \cap (V_3 \cup V_4)| = 1$. In this case, x and y have $2n-3$ common neighbors in $V_1 \cup V_2$. Hence, x and y have $n-1$ common neighbors in V_1 and $n-2$ common neighbors in V_2 or vice versa. Then there are one vertex in $V_1 \cup V_2$ that is nonadjacent with x and y. Hence, vertices x and y have one common neighbor in $V_1 \cup V_2$ and one common neighbor in $V_3 \cup V_4$ in G'.
- Consider $x \in V_3$, $y \in V_4$ where $|N(x, y) \cap (V_3 \cup V_4)| = 0$. In this case, x and y have $2n-2$ common neighbors in $V_1 \cup V_2$. Hence, x and y have $n-1$ common neighbors in V_1 and $n-1$ common neighbors in V_2. Then there are two vertices in $V_1 \cup V_2$ that are nonadjacent with x and y. Hence, vertices x and y have two common neighbors in $V_1 \cup V_2$ and no common neighbors in $V_3 \cup V_4$ in G'.
- Consider $x \in V_i$ where $i \in \{1, 2\}$, and $y \in V_j$ where $j \in \{3, 4\}$, where $x \sim y$. In this case, x and y have $n-2$ common neighbors in V_i. There are two possibilities for the number of common neighbors of x and y in V_{7-j}. If x and y have $n-1$ common neighbors in V_{7-j}, then they have 1 common neighbor in $V_{3-i} \cup V_j$. In this case, x and y have 1 common neighbor in $V_{3-i} \cup V_j$ and 1 common neighbor in V_i in graph G'. If x and y have $n-2$ common neighbors in V_{7-j}, then they have 2 common neighbors in $V_{3-i} \cup V_j$. In this case, x and y have 1 common neighbor in V_{7-j}, 1 common neighbor in V_i and no common neighbors in other classes in graph G'.

- Consider $x \in V_i$ where $i \in \{1,2\}$, and $y \in V_j$ where $j \in \{3,4\}$, where $x \not\sim y$. In this case, x and y have $n-1$ common neighbors in V_i. There are two possibilities for the number of common neighbors of x and y in V_{7-j}. If x and y have $n-1$ common neighbors in V_{7-j}, then they have 0 common neighbors in $V_{3-i} \cup V_j$. In this case, x and y have 2 common neighbors in $V_{3-i} \cup V_j$ and no common neighbors in other classes in graph G'. If x and y have $n-2$ common neighbors in V_{7-j}, then they have 1 common neighbor in $V_{3-i} \cup V_j$. In this case, x and y have 1 common neighbor in $V_{3-i} \cup V_j$ and 1 common neighbor in V_{7-j} in graph G'.

Thus, in all cases, two vertices from the same class of canonical partition have $n-2$ common neighbors in G' and two vertices from different classes have 2 common neighbors. Hence, G' is a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$. □

Lemma 10. *Let G be a DDG with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$ and with quotient matrix (12). Suppose that G' is a graph obtained from G by switching between the first two and the last two classes of the canonical partition. Then G' is a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$ and with quotient matrix (6).*

Proof. The proof is similar to the proof of Lemma 9. We need to consider all pairs of vertices in G and show that two vertices from the same class have $n-2$ common neighbors in G' and two vertices from the different blocks have 2 common neighbors in G'. Consider all possibilities for the number of common neighbors of vertices $x, y \in G$. Without loss of generality we can assume that $x \in V_1$.

- Consider $y \in V_1$. In this case, x and y have $n-2$ common neighbors in V_2, $n-2$ common neighbors in V_3 and $n-2$ common neighbors in V_4. Then each vertex in $V_3 \cup V_4$ adjacent with x or y. Hence, x and y have only $n-2$ common neighbors in V_2 in G'.
- Consider $y \in V_i$, where $i \in \{2,3,4\}$. Let $\{j, s\} = \{2,3,4\} \setminus \{i\}$.
  - If $|N(x,y) \cap (V_1 \cup V_i)| = 2$. Then x and y have $2n-4$ common neighbors in $V_j \cup V_s$. Hence, x and y have $n-2$ common neighbors in V_j and $n-2$ common neighbors in V_s. Then each vertex in $V_j \cup V_s$ is adjacent to x or y. Hence, x and y have only two common neighbors in $V_1 \cup V_i$ in G'.
  - If $|N(x,y) \cap (V_1 \cup V_i)| = 1$. Then x and y have $2n-3$ common neighbors in $V_j \cup V_s$. Hence, x and y have $n-1$ common neighbors in V_j and $n-2$ common neighbors in V_s or vice versa. Then there is one vertex in $V_j \cup V_s$ that is nonadjacent with x and y. Hence, x and y have one common neighbor in $V_1 \cup V_i$ and one common neighbor in $V_j \cup V_s$ in G'.
  - If $|N(x,y) \cap (V_1 \cup V_i)| = 0$. Then x and y have $2n-2$ common neighbors in $V_j \cup V_s$. Hence, x and y have $n-1$ common neighbors in V_j and $n-1$ common neighbors in V_s. Then there are two vertices in $V_j \cup V_s$ that are nonadjacent to x and y. Hence, x and y have only two common neighbors in $V_j \cup V_s$ in G'.

Thus, in all cases, two vertices from the same class of canonical partition have $n-2$ common neighbors in G' and two vertices from different classes have 2 common neighbors. Hence, G' is a DDG with parameters $(4n, n+2, n-2, 2, 4, n)$. □

This completes the proof of Theorem 2.

Remark 5. *Graphs with quotient matrix (12) have four eigenvalues* $\{3n-2, \pm 2, -(n-2)\}$. *Hence, these graphs are walk regular (see* [3]*).*

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