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LOCALLY FREE SUBGROUPS OF ONE-RELATOR GROUPS

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Abstract. Let $G_1 = \langle x_1, \dots x_s; [x_1, x_{n+1}][x_2, x_{n+2}] \dots [x_n, x_{2n}]S\rangle$, $G_2 = \langle a, x_1, \dots, x_s; [a, x_1][a, x_2] \dots [a, x_n]S\rangle$ be one-relator groups. We find conditions on S and n under which the normal closure of each $(n-1)$-generated subgroup of G_1 and of each 3-generated subgroup of G_2 is locally free.

Keywords: one-relator group, locally free group, n-free group.

1. Introduction

In this paper we find conditions when the normal closure of each n-generated subgroup of an one-relator group is a locally free group.

A group is said to be n-free if each of its n-generated subgroup is free. An example of such group [1, lemma 3] is the fundamental group

$$A_n = \langle x_1, x_2, \dots, x_{2n}; [x_1, x_2][x_3, x_4] \dots [x_{2n-1}, x_{2n}]\rangle$$

of genus n orientable surface. There are quite a lot of articles related to n-free groups, we will note only a few of them.

In [2] it is found the conditions under which every subgroup of infinite index of a fundamental group of a surface is free. In particular, this implies that all subgroups of infinite index of A_n are free and A_n is n-free.

The proof of lemma 3 [1] without changes is suitable for establishing the following result: the group

$$\langle x_1, x_2, \dots, x_{2n}, z_1, \dots, z_m; [x_1, x_2][x_3, x_4] \dots [x_{2n-1}, x_{2n}]S\rangle \ (n \geq 2),$$

where S is a word in the alphabet $z_1, \dots, z_m$, is n-free. A detailed proof of this fact is given in [3, proposition 2] and in [4, lemma 1.2.3].

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It follows from [5] that a free product with amalgamation $G = F *_C F'$, where the factors F and F' are free groups and the amalgamated subgroup C is a maximal cyclic subgroup of each factor, is 2-free. Corollary 4.2 (see [6]) says that this group G is in fact 3-free. In [6] it is defined a class C_k of groups which contains the class of all k-free groups, and it is proved if $G = K *_K F'$, where K and K' belongs to C_3 and the amalgamated subgroup C is a maximal cyclic subgroup of each factor, then $G \in C_3$.

In [7] it is found the conditions on the defining relations of a group under which it is 2-free. In [8] so-called fully residually free groups were studied. It is proved in [8], if such group is 2-free then it is 3-free. Interesting results about n-free groups can also be found in [9 – 13].

In this paper we generalize the notion of a n-free group. We consider a group

$$G_1 = \langle x_1, \dots x_s; [x_1, x_{n+1}][x_2, x_{n+2}] \dots [x_n, x_{2n}]R\rangle,$$

where $x_{n+1}, x_{n+2}, x_{n+3}, \dots, x_{2n}$ do not occur in R. We prove that if $t_1, \dots, t_{n-1} \in G_1$ then $G_1'\langle t_1, \dots, t_{n-1}\rangle^{G_1}$ is a free group. Note that such groups can also be found in [11, 14, 15].

Also we investigate a group G_2 which have the representation

$$G_2 = \langle a, x_1, \dots, x_s; P\rangle,$$

where $P = P(a, x_1, x_2, \dots, x_s) = [a, x_1][a, x_2] \dots [a, x_n]S$ $(n \geq 7)$ and $x_1, \dots, x_7$ do not occur in S. We show that if $t_1, t_2, t_3 \in G_2$ then $G_2'\langle t_1, t_2, t_3\rangle^{G_2}$ is a locally free group. Such group were also studied in [16] and for $R = 1$, $n = 2$ in [15].

2. Preliminary remarks

We recall some definitions and notation. As usual, $\langle S\rangle$ is a group generated by S; $\langle t_1, \dots, t_n\rangle^G$ is a normal closure of a subgroup $\langle t_1, \dots, t_n\rangle$ in G (i.e. a normal subgroup of G generated by $t_1, \dots, t_n$); G' is the commutant of G; $Z(G)$ is the center of G; $[x, y] = x^{-1}y^{-1}xy$; $\mathbf{Z}$ is a set of integers.

An embedding of a group A into B is any homomorphism $\varphi : A \to B$ which is an isomorphism of A onto A^φ. A group A is said to be embeddable in a group B if there exists an embedding $\varphi : A \to B$.

A group is locally free if each of its finitely generated subgroups is free.

We say that a set $\{f_1, \dots, f_r\}$ of numbers has the property (E), if it contains exactly one largest non-zero positive number or exactly one smallest negative non-zero number. If this set does not have the property (E), we write not(E).

Let $X = \{x_1, x_2, \dots\}$ be an alphabet, $F(X)$ be a free group freely generated by the set X, and α be an homomorphism of $F(X)$ onto some group G. If $\{P, Q, R, \dots\}$ is a set of defining relations for G in the alphabet X under the map α, then G can suitably be represented thus:

$$G = \langle X; P, Q, R, \dots\rangle. \tag{1}$$

In what follows we identify symbols x_i with their corresponding generating elements x_i^α. The elements of X are called generating symbols for G. We shall suppose that X contains fixed symbols $x_1, \dots, x_n$.

If T is a group word in the alphabet X then $\sigma_{x_i}(T)$ $(i = 1, \dots, n)$, or simply $\sigma_i(T)$, denotes the sum of exponents over x_i in T, and $[T]$ denotes the set elements from X occurring in T.

Let $t \in G$ and $T = T(x_1, \dots, x_n)$ be a group word in X for which $T^\alpha = t$; then, by definition, we put $\sigma_i(t) = \sigma_i(T)$. Since by assumption the sum of exponents over x_i in every defining relation of G is zero, it is readily checked that $\sigma_i(t)$ is independent of the choice of the word T.

Often we will write a_{ij} instead of $\sigma_j(t_i)$.

If $\widetilde{t} = (t_1, \dots, t_k)$ is an k-tuple of elements $t_1, \dots, t_k$ of G it is legitimate to consider the following matrix with integer entries:

$$M(G, \widetilde{t}) = M(G, t_1, \dots, t_k) = \begin{pmatrix} \sigma_1(t_1) & \dots & \sigma_n(t_1) \\ \dots & \dots & \dots \\ \sigma_1(t_k) & \dots & \sigma_n(t_k) \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}.$$

Let L be a subgroup of G generated by elements $t_1, \dots, t_k$. We consider the following elementary row transformations of the matrix $M(G, \widetilde{t})$:

1) multiplication of elements of the j-th row by -1,

2) replacement of the j-th row by the difference between the j-th and m-th rows, $m \neq j$.

Each of these transformations effects a transformation of the set $t_1, \dots, t_k$ to a new set of generators of the subgroup L; this new set may be derived from the old by corresponding transformations:

1′) replacement of t_j by t_j^{-1},

2′) replacement of t_j by $t_j t_m^{-1}$ for $m \neq j$.

It is easy to see that the matrix $M(G, \widetilde{t})$ with integer entries may be brought by transformations of the above two types to a trapezoidal form. We may thus assume that the generators $t_1, \dots, t_k$ of L have been so chosen that $M(G, \widetilde{t})$ is a trapezoidal matrix with only zeros below the principal diagonal (i.e. $a_{ij} = 0$ for all $i > j$) and positive elements on the principal diagonal (i.e. $a_{ii} \geq 0$ for all i). We will always assume that the matrix $M(G, \widetilde{t})$ has this form.

We will also consider

3) permutation of the columns of the matrix $M(G, \widetilde{t})$.

This transformation corresponds to

3′) renaming of generators of the group G.

Note that transformation 3′ changes the defining relations of the group G.

We recall the Reidemeister-Schreier rewriting process. Let G have representation

$$G = \langle x_1, x_2, \dots ; P, Q, R, \dots \rangle, \tag{2}$$

$H \leq G$, and N be the preimage of a subgroup H under the homomorphism α. Choose the Schreier system S of representatives of right cosets of the classes $F(X)$ modulo N. A representative of the coset Nu in S is denoted by $\overline{u}$. Consider $\tau : F(X) \to F(Y)$ which maps an element $u = x_{i_1}^{\epsilon_1} \dots x_{i_r}^{\epsilon_r}$ $(\epsilon_l = \pm 1)$ of $F(X)$ to an element $\tau(u) = y_{s_1, x_{i_1}}^{\epsilon_1} \dots y_{s_r, x_{i_r}}^{\epsilon_r}$ of $F(Y)$, where $s_j = \overline{x_{i_1}^{\epsilon_1} \dots x_{i_{j-1}}^{\epsilon_{j-1}}}$ if $\epsilon_j = 1$, and $s_j = \overline{x_{i_1}^{\epsilon_1} \dots x_{i_j}^{\epsilon_j}}$ if $\epsilon_j = -1$. Note that the restriction of τ to N is an homomorphism [17].

A representation of the group H relative to the mapping $y_{s,x} \to (sx\overline{sx}^{\ -1})^\alpha$ may be obtained using the subgroup representation theorem [17, theorem 2.4]. Namely, Y is taken to be the set of generators for H, and as the set of defining relations we take

$$\{\tau(sPs^{-1}), \tau(sQs^{-1}), \dots \mid s \in S\} \cup \{y_{s,x} \mid sx = \overline{sx} \text{ in } F(X)\}.$$

In particular, if $H = \langle x_2, x_3, \ldots\rangle^G$, and G/H is an infinite cyclic group, then we take the Schreier system of representatives of the right cosets of $F(X)$ modulo N to be the set $S = \{x_1^k \mid k \in \mathbf{Z}\}$. Then, by the subgroup representation theorem,

$$H = \langle \{y_{x_1^k, x_1}, y_{x_1^k, x_2}, \ldots\}_{k\in\mathbf{Z}}; \{y_{x_1^k, x_1}, \tau(x_1^k P x_1^{-k}), \ldots\}_{k\in\mathbf{Z}}\rangle.$$

In order to pass from the above-given representation to a new one, we use Tietze transformations, detailed information about which is contained in [17].

Later we shall also use the following corollary of the Magnus freedom theorem [17, corollary 4.10.2]: if

$$H = \langle x, c, \ldots, t; R(x^p, c, \ldots, t)\rangle,\ p \neq 0),$$

then the subgroup G of H generated by the elements $x^p, c, \ldots, t$, has the representation

$$G = \langle b, c, \ldots; R(b, c, \ldots, t)\rangle,$$

where b corresponds to the element x^p, c to the element c, $\ldots$, t to the element t of G.

3. Locally free subgroups

Let's fix the matrices:

$$A = \begin{pmatrix} 1 & a_{12} & p_2a_{13} & p_2p_3a_{14} & \ldots & p_2p_3p_4\ldots p_{k-2}a_{1,k-1} \\ 0 & 1 & a_{23} & p_3a_{24} & \ldots & p_3p_4\ldots p_{k-2}a_{2,k-1} \\ 0 & 0 & 1 & a_{34} & \ldots & p_4\ldots p_{k-2}a_{3,k-1} \\ & & & \ldots & & \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} p_2p_3p_4\ldots p_{k-2}p_{k-1}a_{1,k} \\ p_3p_4\ldots p_{k-2}p_{k-1}a_{2,k} \\ p_4\ldots p_{k-2}p_{k-1}a_{3,k} \\ \ldots \\ a_{k-1,k} \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_{k-1} \end{pmatrix},$$

where $p_2 \neq 0, \ldots, p_{k-1} \neq 0$.

Lemma 1. *The column $(q_1, \ldots, q_{k-1})^\top$ is a solution of the system of equations $AX = B$, where*

$$q_1 = (-1)^{k-2}\begin{vmatrix} a_{12} & a_{13} & a_{14} & \ldots & a_{1,k-1} & a_{1,k} \\ p_2 & a_{23} & a_{24} & \ldots & a_{2,k-1} & a_{2,k} \\ 0 & p_3 & a_{34} & \ldots & a_{3,k-1} & a_{3,k} \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & p_{k-1} & a_{k-1,k} \end{vmatrix},$$

$$q_2 = (-1)^{k-3}\begin{vmatrix} a_{23} & a_{24} & a_{25} & \ldots & a_{2,k-1} & a_{2,k} \\ p_3 & a_{34} & a_{35} & \ldots & a_{3,k-1} & a_{3,k} \\ 0 & p_4 & a_{45} & \ldots & a_{4,k-1} & a_{4,k} \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & p_{k-1} & a_{k-1,k} \end{vmatrix},$$

$\ldots$

$$q_{k-2} = -\begin{vmatrix} a_{k-2,k-1} & a_{k-2,k} \\ p_{k-1} & a_{k-1,k} \end{vmatrix},$$

$q_{k-1} = a_{k-1,k}$.

PROOF. First we find x_i by Cramer' rule. Then we put the i-th column in the place of the last one. Using the formula for the decomposition of the determinant by column it is easy to see that

$$q_i = (-1)^{k-i-1} \times$$
$$\begin{vmatrix} a_{i,i+1} & p_{i+1}a_{i,i+2} & p_{i+1}p_{i+2}a_{i,i+3} & \dots & p_{i+1}\dots p_{k-2}a_{i,k-1} & p_{i+1}\dots p_{k-1}a_{i,k} \\ 1 & a_{i+1,i+2} & p_{i+2}a_{i+1,i+3} & \dots & p_{i+2}\dots p_{k-2}a_{i+1,k-1} & p_{i+2}\dots p_{k-1}a_{i+1,k} \\ 0 & 1 & a_{i+2,i+3} & \dots & p_{i+3}\dots p_{k-2}a_{i+2,k-1} & p_{i+3}\dots p_{k-1}a_{i+2,k} \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & a_{k-1,k} \end{vmatrix}$$

Now we multiply the 2nd row by p_{i+1}, the 3rd row by $p_{i+1}p_{i+2}$, $\dots$, the last row by $p_{i+1}p_{i+2}\dots p_{k-1}$. Then we divide the 2nd column by p_{i+1}, the 3rd column by $p_{i+1}p_{i+2}$, $\dots$, column by $p_{i+1}p_{i+2}\dots p_{k-1}$. We obtain that

$$q_i = (-1)^{k-i-1} \begin{vmatrix} a_{i,i+1} & a_{i,i+2} & a_{i,i+3} & \dots & a_{i,k-1} & a_{i,k} \\ p_{i+1} & a_{i+1,i+2} & a_{i+1,i+3} & \dots & a_{i+1,k-1} & a_{i+1,k} \\ 0 & p_{i+2} & a_{i+2,i+3} & \dots & a_{i+2,k-1} & a_{i+2,k} \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & p_{k-1} & a_{k-1,k} \end{vmatrix}. \qquad \square$$

Now we present the construction of the embedding ψ of a group G into a suitable group C with the specially selected representation in which the matrix $M(C, t_1^\psi, \dots, t_k^\psi)$ has zero k-th column.

Lemma 2. *Let*

$$G = \langle a, x_1, \dots, x_s; R(a, x_1, \dots, x_s)\rangle$$

be a group, $t_1, \dots, t_k$ ($k \le s$) be elements of G and $\sigma_i(R) = 0$ ($i = 1, \dots, k$). Suppose that the matrix $M(G, \widetilde{t})$ has the form:

$$(3) \qquad M(G,\widetilde{t}) = \begin{pmatrix} p_1 & a_{12} & a_{13} & \dots & a_{1,k-1} & a_{1k} & \dots \\ 0 & p_2 & a_{23} & \dots & a_{2,k-1} & a_{2k} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & p_{k-1} & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots \end{pmatrix},$$

where $a_{ij} = \sigma_j(t_i)$ (note that $a_{k,k+1}$ is an element of matrix or is missing). Assume that the group C has the representation

$$(4) \qquad C = \langle a, x_1, \dots, x_{k-1}, x_k, \dots x_s; R_3\rangle,$$

where

$$R_3(a, x_1, \dots, x_{k-1}, x_k, \dots, x_s) = R(a, x_1x_k^{-q_1}, x_2x_k^{-q_2p_1}, x_3x_k^{-q_3p_1p_2}, \dots,$$
$$x_{k-1}x_k^{-q_{k-1}p_1p_2\dots p_{k-2}}, x_k^m, x_{k+1}, \dots, x_s),$$

$m = p_1 \dots p_{k-1}$ and $q_1, \dots, q_{k-1}$ be the same as in lemma 1. Then there exists the embedding $\psi : G \to C$ of G into C such that the matrix consisting of the first k columns of the matrix $M(C, t_1^\psi, \dots, t_k^\psi)$ is triangular with a zero angle under the principal diagonal and with a zero k-th column.

PROOF. Assume that the group C_1 has the representation $C_1 = \langle a, x_1, \dots, x_{k-1}, x_k, \dots x_s; R_1\rangle$, where $R_1 = R(a, x_1, \dots, x_{k-1}, x_k^m, x_{k+1}, \dots)$. By corollary of the Magnus freedom theorem formulated above, there exists the embedding ψ of G into C_1 such that $a^\psi = a, x_i^\psi = x_i (i = 1, 2, \dots, k-1, k+1, \dots)$, $x_k^\psi = x_k^m$. Clearly, in generators

$$a, y_1 = x_1 x_k^{q_1}, y_2 = x_2 x_k^{q_2 p_1}, \dots, y_{k-1} = x_{k-1} x_k^{q_{k-1} p_1 p_2 \dots p_{k-2}}, x_k, x_{k+1}, \dots, x_s$$

C_1 has the following representation: $C_1 = \langle a, y_1, \dots, y_{k-1}, x_k, \dots x_s; R_2 \rangle$, where

$$R_2(a, y_1, \dots, y_{k-1}, x_k, \dots, x_s) = R(a, y_1 x_k^{-q_1}, y_2 x_k^{-q_2 p_1}, y_3 x_k^{-q_3 p_1 p_2}, \dots,$$

$$y_{k-1} x_k^{-q_{k-1} p_1 p_2 \dots p_{k-2}}, x_k^m, x_{k+1}, \dots, x_s).$$

Rename generators $y_1, \dots, y_{k-1}$ by $x_1, \dots, x_{k-1}$ respectively, we see that C_1 in new generators has the representation

$$C_1 = \langle a, x_1, \dots, x_{k-1}, x_k, \dots x_s; R_3(a, x_1, \dots, x_s) \rangle,$$

i.e. C_1 coincides with C.

Express elements $t_1^\psi, \dots, t_k^\psi$ through these new generators. Suppose that t_i is a value of $T_i(a, x_1, \dots, x_s)$ on generators $a, x_1, \dots, x_s$ of G. We see

$$t_i^\psi = T_i(a, x_1 x_k^{-q_1}, x_2 x_k^{-q_2 p_1}, x_3 x_k^{-q_3 p_1 p_2}, x_{k-1} x_k^{-q_{k-1} p_1 p_2 \dots p_{k-2}}, x_k^m, x_{k+1}, \dots, x_s).$$

Therefore

$$\sigma_k(t_i^\psi) = -a_{i1} q_1 - a_{i2} p_1 q_2 - a_{i3} p_1 p_2 q_3 - \dots - a_{i,k-1} p_1 p_2 \dots p_{k-2} q_{k-1} + a_{i,k} m,$$

where $i = 1, \dots, k-1$. As $M(G, \widetilde{t})$ is a trapezoidal matrix we note that $a_{ij} = 0$ for all $j, j < i$. Hence

$$\sigma_k(t_i^\psi) = -p_1 \dots a_{ii} q_i - a_{i,i+1} p_1 \dots p_i q_{i+1} - \dots - a_{i,k-1} p_1 \dots p_{k-2} q_{k-1} + a_{i,k} m =$$

$$= -p_1 \dots p_i (q_i + a_{i,i+1} q_{i+1} + a_{i,i+2} p_{i+1} q_{i+2} + \dots + a_{i,k-1} p_{i+1} \dots q_{k-1} - a_{i,k} p_{i+1} \dots p_{k-1}).$$

The expression in parentheses coincides with the i-th row of the matrix $AX - B$ (if x_j replace by q_j) therefore it is equal to 0. Thus

$$\sigma_k(t_i^\psi) = 0 \text{ for } i = 1, \dots, k-1.$$

Since $a_{k1} = a_{k2} = \dots = a_{kk} = 0$ and $\sigma_k(t_k^\psi) = a_{k1} q_1 - a_{k2} p_1 q_2 - a_{k3} p_1 p_2 q_3 - \dots - a_{k,k-1} p_1 p_2 \dots p_{k-2} q_{k-1} + a_{k,k} m$, we obtain that $\sigma_k(t_k^\psi) = 0$.

So we found the embedding $\psi : G \to C$ of G into C with the representation (4), such that the matrix consisting of the first k columns of the matrix $M(C, t_1^\psi, \dots, t_k^\psi)$ is triangular with a zero angle under the principal diagonal and with a zero k-th column. $\square$

A similar construction also appeared in [18].

Lemma 3. *Let a group G have the representation*

$$G = \langle a, x_1, \dots, x_s; R(a, x_1, \dots, x_s) \rangle,$$

$t_1, \dots, t_k \in G$ $(k \le s)$ *and* $\sigma_i(R) = 0$ $(i = 1, \dots, k)$. *Suppose that the matrix $M(G, \widetilde{t})$ has the form (3). Let a group C have the representation (4), where $m = p_1 p_2 \dots p_{k-1}$, $(q_1, \dots, q_{k-1})^\top$ is a solution of a system of equations $AX = B$ from lemma 1. Then if $\langle a, x_1, \dots, x_{k-1}, x_{k+1}, x_{k+2}, \dots, x_s \rangle^C$ is a locally free (or free) group then $G' \langle t_1, \dots, t_k \rangle^G$ is also a locally free group (respectively, free group).*

PROOF. According to the above construction of the embedding ψ (lemma 2), the matrix $M(C, t_1^\psi, \dots, t_k^\psi)$ has zero k-th column. This means that $C'\langle t_1^\psi, \dots, t_k^\psi\rangle^C$ is contained in a subgroup $\langle a, x_1, \dots, x_{k-1}, x_{k+1}, x_{k+2}, \dots, x_s\rangle^C$, which by condition is locally free. Hence $C'\langle t_1^\psi, \dots, t_k^\psi\rangle^C$ is a locally free group. As $\psi : G \to C$ is an embedding and $(G')^\psi(\langle t_1, \dots, t_k\rangle^G)^\psi \subseteq C'\langle t_1^\psi, \dots, t_k^\psi\rangle^C$, then $G'\langle t_1, \dots, t_k\rangle^G$ is a locally free group. □

Lemma 4. *Let a group G have the representation*

$$G = \langle a, b, c, d, \dots, u, h, v, \dots, w, \dots, z; PR\rangle,$$

where $P = [a, bh^{-f_1}][a, ch^{-f_2}][a, dh^{-f_3}] \dots\dots [a, uh^{-f_{k-1}}][a, h^m][a, v] \dots [a, w]$. *Suppose that the set $\{f_1, f_2, \dots, f_{k-1}, -m\}$ of numbers satisfies the property (E). If*

(i) $h \notin [R]$ or

(ii) $a \notin [R]$ and $\sigma_h(R) = 0$,

then the normal closure $N = \langle a, b, c, d, \dots, u, v \dots, w, \dots, z\rangle^G$ generated by all generators except h is a locally free group.

PROOF. We proceed as Magnus [17] did when he proved the freedom theorem for one-relator groups.

Since G/N is an infinite cyclic group then N is generated by all elements

$$(5) \qquad a_s = h^s a h^{-s}, b_s = h^s b h^{-s}, \dots, w_s = h^s w h^{-s}, \dots, z_s = h^s z h^{-s} \ (s \in \mathbf{Z}).$$

By section 2 the group N in generators (5) is defined by the set of P_k ($k \in \mathbf{Z}$) defining relations, where $P_k = \tau(h^k PR h^{-k})$. We have

$$P_0 = P(a_0, \dots, u_{f_{k-1}}, \dots, z_0)R_0 =$$

$$= a_0^{-1} b_{f_1}^{-1} a_{f_1} b_{f_1} \dots a_0^{-1} u_{f_{k-1}}^{-1} a_{f_{k-1}} u_{f_{k-1}} a_0^{-1} a_{-m} [a_0, v_0] \dots [a_0, w_0] R_0,$$

where the word R_0 is obtained by rewriting the word R through generators (5). We note that from of a_l-symbols, only a_0 can be occurred in R_0.

Let the word P_i be defined as (R_i obtained from R_0 by adding to each index of the variable included in R_0 the number i)

$$P_i = P(a_i \dots u_{f_{k-1}+i}, \dots, z_i)R_i =$$

$$= a_i^{-1} b_{f_1+i}^{-1} a_{f_1+i} b_{f_1+i} \dots a_0^{-1} u_{f_{k-1}+i}^{-1} a_{f_{k-1}+i} u_{f_{k-1}+i} a_i^{-1} a_{-m+i} [a_i, v_i] \dots [a_i, w_i] R_i,$$

i.e. the number i added to each index of the variable included in P_0.

Let μ and M respectively, be smallest and largest indexes of a_l-symbols included in P_0.

For every $i = 0, 1, 2, \dots,$ we represent $N_{-i,i}$ by generators S_i and defining relations Σ_i as follows. As a generating set S_i, take the set of generators from (5), except for those a_k for which $k < \mu - i$ or $k > M + i$, and take the set $\{P_{-i}, \dots, P_i\}$ of defining words to be Σ_i. Thus

$$N_{-i,i} = \langle S_i; P_{-i}, \dots, P_i\rangle.$$

In [17, theorem 4.10, case 3] these groups $N_{-i,i}$ are introduced for any one-relator groups in which the sum of the exponents in our notation for h is 0. In [17] it is proved that we can assume that $N_{-i,i}$ is a subgroup of $N_{-i-1,i+1}$ generated by elements from (5) included in the list of generators of $N_{-i,i}$. In addition, in [17] it is noted that the union an ascending sequence of groups $N_{-i,i}$ coincides with N. We need to check that $N_{-i,i}$ is a free group. To do this, use the Tietze transformations.

First, we assume that the set $\{f_1, f_2, \dots, f_{k-1}, -m\}$ contains exactly one largest non-zero positive number. Deleting, via the Tietze transformations, a_{M+i} (i.e. a_s with largest index) from the set of generators of $N_{-i,i}$ and the defining relation P_i. Since a_{M+i} is not occurred in $P_{-i}, \dots, P_{i-1}$, these defining relations will not change. We continue this process of sequentially removing generators. Delete $a_{M+i-1}, a_{M+i-2}, \dots$ from the set of generators and, respectively, $P_{i-1}, P_{i-2}, \dots$ from the set of defining relations. As a result, we get that $N_{-i,i}$ can be defined by an empty set of defining relations. It means that $N_{-i,i}$ is a free group.

In the case when the set $\{f_1, f_2, \dots, f_{k-1}, -m\}$ contains exactly one smallest non-zero negative number, we will do the same, starting with the deletion of $a_{\mu-i}$.

Since the union of an increasing sequence of free groups is a locally free group, we see that N is a locally free group. $\square$

If $P = [a, h^m]$ $(m \neq 0)$ we have a special case of lemma 3. In fact, as shown in [1, Lemma 1], the following more general statement is true.

Lemma 5. *Let a group G have the representation*

$$G = \langle a, b, c, d, \dots, u, h, v, \dots, w, \dots, z; [a, h^m]R\rangle \ (m \neq 0).$$

If $a \notin [R]$ and $\sigma_h(R) = 0$, then the normal closure $N = \langle a, b, c, d, \dots, u, v \dots, w, \dots, z\rangle^G$ generated by all generators except h is a free group.

Theorem 1. *Let a group G have the representation*

$$G = \langle x_1, \dots x_s; [x_1, x_{n+1}][x_2, x_{n+2}] \dots [x_n, x_{2n}]R\rangle$$

where $x_{n+1}, x_{n+2}, x_{n+3}, \dots, x_{2n} \notin [R]$. If $t_1, \dots, t_{n-1} \in G$ then $G'\langle t_1, \dots, t_{n-1}\rangle^G$ is a free group.

PROOF. If there exists i $(i \leq n)$ such that i-th column of $M(G, \widetilde{t})$ is zero, we apply lemma 5 ($h = x_i$, $a = x_{i+n} \notin [R]$, $m = 1$). By lemma 5 $\langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_s\rangle^G$ is a free group. But $G'\langle t_1, \dots, t_{n-1}\rangle^G \subseteq \langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_s\rangle^G$ therefore $G'\langle t_1, \dots, t_{n-1}\rangle^G$ is a free group. We will assume that the first n columns of the matrix $M(G, \widetilde{t})$ are non-zero.

Let's apply the transformations 1), 2) defined before. By renaming (if necessary) generators $x_1, \dots, x_n$ of G, we can assume that the matrix $M(G, \widetilde{t})$ has the form

$$M(G, \widetilde{t}) = \begin{pmatrix} p_1 & a_{12} & a_{13} & \dots & a_{1,k-1} & a_{1k} & \dots \\ 0 & p_2 & a_{23} & \dots & a_{2,k-1} & a_{2k} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & p_{k-1} & a_{k-1,k} & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{6}$$

for some k $(k \leq n)$, where $p_1 > 0, \dots, p_{k-1} > 0$. In particular, at the intersection of the row with the number $> k$ and the column with the number $< n+1$, there is a zero.

By the construction of the embedding described earlier (lemma 2), there is an embedding $\psi : G \to C$ of G into C in which the k-th $(k \leq n)$ column of the matrix $M(G, \widetilde{t}^{\psi})$ is zero. We can assume that the defining relation of C has the form $[x_j, x_k^{p_1 \dots p_{k-1}}]T$ $(j > k,\ x_j \notin [T])$. Let's apply 5 ($h = x_k$, $a = x_j \notin [T]$, $\sigma_k(T) = 0$, $m = p_1 \dots p_{k-1}$). By lemma 5, $N = \langle x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_s\rangle^C$ is a free group. Since $(G'\langle t_1,\dots,t_{n-1}\rangle^G)^\psi \subseteq N$ then by lemma 3, $G'\langle t_1,\dots,t_{n-1}\rangle^G$ is a free group. $\square$

4. Normal closures of 3-generated subgroups

In this section, we will consider a group $G=\langle a,x_1,\dots,x_s;P\rangle$, where

$$P=P(a,x_1,x_2,\dots,x_s)=[a,x_1][a,x_2]\dots[a,x_n]S \ (n\geq 7)$$

and $x_1,\dots,x_7\notin[S]$. We will prove that a normal closure of every 3-generated subgroup of this group is a locally free group.

Let t_1,t_2,t_3 be any elements of G, $R=\langle t_1,t_2,t_3\rangle^G$ be a normal closure of the subgroup $\langle t_1,t_2,t_3\rangle$ in G.

First, we will assume (lemmas 6, 7) that the matrix $M(G,\widetilde{t})$ does not contain zero columns.

Lemma 6. *Let's suppose that there are three columns in $M(G,\widetilde{t})$ with numbers i,j,u such that*

$$\sigma_i(t_2)=\sigma_j(t_2)=\sigma_u(t_2)=0,\ \sigma_i(t_3)=\sigma_j(t_3)=\sigma_u(t_3)=0.$$

Then RG' is a locally free group.

Proof. Renaming generators $\{x_1,\dots,x_n\}$ of G, we can assume that $i=1,j=2,u=3$, i.e. the matrix $M(G,\widetilde{t})$ has the form $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ 0 & 0 & 0 & a_{24} & \dots \\ 0 & 0 & 0 & a_{34} & \dots \end{pmatrix}$, where $a_{11}\neq 0,a_{12}\neq 0,a_{13}\neq 0$. We can suppose that $a_{11}\neq -a_{12}$.

Note that when we rename generators, the defining relation is transformed into the relation denoted by $P_1(a,x_1,x_2,\dots,x_s)$.

We apply the construction of the embedding described earlier (lemma 2). According to it, there is the embedding ($k=2$, $q_1=a_{12}$, $m=a_{11}$) $\psi:G\to C$, where C has the representation

$$\langle a,x_1,x_2,\dots,x_{2n};P_1(a,x_1x_2^{-a_{12}},x_2^{a_{11}},x_3,\dots,x_s\rangle.$$

Since the 2nd column of the matrix $M(C,t_1^\psi,t_2^\psi,t_3^\psi)$ is zero, then $\langle t_1^\psi,t_2^\psi,t_3^\psi\rangle$ is contained in the normal closure $N=\langle a,x_1,x_3,x_4,\dots,x_s\rangle^C$. By lemma 4 (here $q_1=a_{12},m=a_{11},a_{12}\neq -a_{11}$) the subgroup N is locally free. As $(RG')^\psi\subseteq NC'=N$ then $(RG')^\psi$, and hence RG', are locally free groups. $\square$

Lemma 7. *Let the matrix $M(G,\widetilde{t})$ have the form*

$$M(G,\widetilde{t})=\begin{pmatrix} p_1 & a_{12} & a_{13} & a_{14} & \dots \\ 0 & p_2 & a_{23} & a_{24} & \dots \\ 0 & 0 & 0 & a_{34} & \dots \end{pmatrix},$$

where $p_1>0,p_2>0,a_{23}>0$. Then RG' is a locally free group.

Proof. We apply the construction of embedding described earlier (lemma 2). According to it, there is the embedding ($k=3$, $m=p_1p_2$) $\psi:G\to C$, where C has the representation

$$C=\langle a,x_1,x_2,\dots,x_s;P(a,x_1x_2^{-q_1},x_2^{-p_1q_2},x_3^{p_1p_2},\dots,x_s\rangle,$$

$(q_1,q_2)^\top$ is a solution of a system of equations

$x_1p_1+x_2p_1a_{12}=p_1p_2a_{13}$

$x_2p_1p_2 = p_1p_2a_{23}$,
i.e. $q_1 = -\begin{vmatrix} a_{12} & a_{13} \\ p_2 & a_{23} \end{vmatrix}$, $q_2 = a_{23}$.

Since the 3rd column of the matrix $M(C, t_1^\psi, t_2^\psi, t_3^\psi)$ is zero then $\langle t_1^\psi, t_2^\psi, t_3^\psi \rangle$ is contained in the normal closure $N = \langle a, x_1, x_2, x_4, \ldots, x_s \rangle^C$. Since numbers $q_2p_1, -p_1p_2$ have different signs then numbers $q_1, q_2p_1, -p_1p_2$ contain exactly one the non-zero largest or smallest number. By lemma 4 the subgroup N is locally free. Since $(RG')^\psi \subseteq NC' = N$ then $(RG')^\psi$, and hence RG', are locally free groups. $\square$

Let the matrix $M(G, \widetilde{t})$ have the form

$$M = M(G, \widetilde{t}) = \begin{pmatrix} p_1 & a_{12} & a_{13} & a_{14} & a_{15} & \ldots \\ 0 & p_2 & a_{23} & a_{24} & a_{25} & \ldots \\ 0 & 0 & p_3 & a_{34} & a_{35} & \ldots \end{pmatrix},$$

where $p_1 > 0, p_2 > 0, p_3 > 0, a_{34} > 0, a_{35} > 0$.

We apply the construction of embedding described earlier (lemma 2). According to it ($k = 4$, $m = p_1p_2p_3$), there is the embedding $\psi : G \to C_0$ of G into C_0 from construction, where C_0 has the representation

$$C_0 = \langle a, x_1, x_2, \ldots, x_n; P(a, x_1x_2^{-q_1}, x_2^{-p_1q_2}, x_3^{-p_1p_2q_3}, x_4^{p_1p_2p_3}, x_5, \ldots, x_s \rangle.$$

Here $(q_1, q_2, q_3)^\top$ is a solution of a system of equations
$x_1p_1 + x_2p_1a_{12} + x_3p_1p_2a_{13} = p_1p_2p_3a_{14}$
$x_2p_1p_2 + x_3p_1p_2a_{23} = p_1p_2p_3a_{24}$
$x_3p_1p_2p_3 = p_1p_2p_3a_{34}$,
in particular, $q_3 = a_{34} > 0$.

Rename generators of G twice. First, we denote x_3 by x_4, x_4 by x_5, x_5 by x_3, then we denote x_4 by x_5, x_5 by x_4. This corresponds to a permutation of columns of the matrix $M(G, \widetilde{t})$. The following matrix are obtained

$$\overline{M} = \begin{pmatrix} p_1 & a_{12} & a_{14} & a_{15} & \ldots \\ 0 & p_2 & a_{24} & a_{25} & \ldots \\ 0 & 0 & a_{34} & a_{35} & \ldots \end{pmatrix}, \quad \widetilde{M} = \begin{pmatrix} p_1 & a_{12} & a_{13} & a_{15} & a_{14} & \ldots \\ 0 & p_2 & a_{23} & a_{25} & a_{24} & \ldots \\ 0 & 0 & p_3 & a_{35} & a_{34} & \ldots \end{pmatrix}.$$

Again, in each of these cases, we use lemma 2. We have groups C_1, C_2 of the form

$$C_1 = \langle a, x_1, x_2, \ldots, x_n; P_1(a, x_1x_2^{-\overline{q}_1}, x_2^{-p_1\overline{q}_2}, x_3^{-p_1p_2\overline{q}_3}, x_4^{p_1p_2a_{34}}, x_5, \ldots, x_s \rangle,$$

$$C_2 = \langle a, x_1, x_2, \ldots, x_n; P_2(a, x_1x_2^{-\widetilde{q}_1}, x_2^{-p_1\widetilde{q}_2}, x_3^{-p_1p_2\widetilde{q}_3}, x_4^{p_1p_2p_3}, x_5, \ldots, x_s \rangle,$$

and embeddings $\overline{\psi} : G \to C_1$, $\widetilde{\psi} : G \to C_2$ (P_1, P_2 is obtained from P by renaming suitable generators).

For each matrix a corresponding system of equations arises. Its solutions for the matrix $\overline{M}$ are denoted by $\overline{q}_1$, $\overline{q}_2$, $\overline{q}_3$, for the matrix $\widetilde{M}$ by $\widetilde{q}_1$ $\widetilde{q}_2$, $\widetilde{q}_3$. By lemma 1

$$q_1 = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ p_2 & a_{23} & a_{24} \\ 0 & p_3 & a_{34} \end{vmatrix}, \quad q_2 = -\begin{vmatrix} a_{23} & a_{24} \\ p_3 & a_{34} \end{vmatrix}, \quad q_3 = a_{34}, \tag{7}$$

$$\overline{q}_1 = \begin{vmatrix} a_{12} & a_{14} & a_{15} \\ p_2 & a_{24} & a_{25} \\ 0 & a_{34} & a_{35} \end{vmatrix}, \quad \overline{q}_2 = -\begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix}, \quad \overline{q}_3 = a_{35}, \tag{8}$$

$$\widetilde{q}_1 = \begin{vmatrix} a_{12} & a_{13} & a_{15} \\ p_2 & a_{23} & a_{25} \\ 0 & p_3 & a_{35} \end{vmatrix}, \ \widetilde{q}_2 = - \begin{vmatrix} a_{23} & a_{25} \\ p_3 & a_{35} \end{vmatrix}, \ \widetilde{q}_3 = a_{35}. \tag{9}$$

We will assume that each of sets

$$q_1, \ q_2p_1, \ q_3p_1p_2, \ -p_1p_2p_3, \tag{10}$$

$$\overline{q}_1, \ \overline{q}_2p_1, \ \overline{q}_3p_1p_2, \ -p_1p_2a_{34}, \tag{11}$$

$$\widetilde{q}_1, \ \widetilde{q}_2p_1, \ \widetilde{q}_3p_1p_2, \ -p_1p_2p_3, \tag{12}$$

does not have the property (E) (i.e. it has the property not(E)).

Lemma 8.

$$\widetilde{q}_1 - \frac{p_3}{a_{34}}\overline{q}_1 = \frac{a_{35}}{a_{34}}q_1, \tag{13}$$

$$\widetilde{q}_2 - \frac{p_3}{a_{34}}\overline{q}_2 = \frac{a_{35}}{a_{34}}q_2. \tag{14}$$

Proof. We calculate

$$\widetilde{q}_1 - \frac{p_3}{a_{34}}\overline{q}_1 = \begin{vmatrix} a_{12} & a_{13} & a_{15} \\ p_2 & a_{23} & a_{25} \\ 0 & p_3 & a_{35} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14}\frac{p_3}{a_{34}} & a_{15} \\ p_2 & a_{24}\frac{p_3}{a_{34}} & a_{25} \\ 0 & a_{34}\frac{p_3}{a_{34}} & a_{35} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{12} & a_{13} - a_{14}\frac{p_3}{a_{34}} & a_{15} \\ p_2 & a_{23} - a_{24}\frac{p_3}{a_{34}} & a_{25} \\ 0 & 0 & a_{35} \end{vmatrix} = a_{35} \begin{vmatrix} a_{12} & a_{13} - a_{14}\frac{p_3}{a_{34}} \\ p_2 & a_{23} - a_{24}\frac{p_3}{a_{34}} \end{vmatrix} =$$

$$= a_{35}\left(\begin{vmatrix} a_{12} & a_{13} \\ p_2 & a_{23} \end{vmatrix} - \frac{p_3}{a_{34}} \begin{vmatrix} a_{12} & a_{14} \\ p_2 & a_{24} \end{vmatrix} \right).$$

Since

$$q_1 = a_{34} \begin{vmatrix} a_{12} & a_{13} \\ p_2 & a_{23} \end{vmatrix} - p_3 \begin{vmatrix} a_{12} & a_{14} \\ p_2 & a_{24} \end{vmatrix},$$

then the equality (13) is true. Also we have

$$\widetilde{q}_2 - \frac{p_3}{a_{34}}\overline{q}_2 = - \begin{vmatrix} a_{23} & a_{25} \\ p_3 & a_{35} \end{vmatrix} + \begin{vmatrix} a_{24}\frac{p_3}{a_{34}} & a_{25} \\ a_{34}\frac{p_3}{a_{34}} & a_{35} \end{vmatrix} =$$

$$= \begin{vmatrix} -a_{23} + a_{24}\frac{p_3}{a_{34}} & a_{25} \\ 0 & a_{35} \end{vmatrix} = \frac{a_{35}}{a_{34}}(-a_{23}a_{34} + a_{24}p_3) = \frac{a_{35}}{a_{34}}q_2.$$

□

Lemma 9. *From the equalities*

$$q_1 = -p_1p_2p_3, \ \overline{q}_1 = -p_1p_2a_{34}, \ \widetilde{q}_1 = -p_1p_2p_3$$

no more than one is true.

Proof. Suppose that $\overline{q}_1 = -p_1p_2a_{34}, \ \widetilde{q}_1 = -p_1p_2p_3$. By (13)

$$-p_1p_2p_3 + \frac{p_3}{a_{34}}p_1p_2a_{34} = \frac{a_{35}}{a_{34}}q_1.$$

This means that $q_1 = 0$. Now we see that the set $\{0, \ q_2p_1, \ a_{34}p_1p_2, \ -p_1p_2p_3\}$ has the property (E). We got a contradiction.

Assume that $q_1 = -p_1p_2p_3$, $\overline{q}_1 = -p_1p_2a_{34}$. By (13)

$$\widetilde{q}_1 + \frac{p_3}{a_{34}}p_1p_2a_{34} = -\frac{a_{35}}{a_{34}}p_1p_2p_3.$$

Hence $\widetilde{q}_1 = -p_1p_2p_3(1 + \frac{a_{35}}{a_{34}}) \neq -p_1p_2p_3$. We see that

$$\widetilde{q}_1 < 0,\ -p_1p_2p_3 < 0,\ \widetilde{q}_1 \neq -p_1p_2p_3,\ \widetilde{q}_3p_1p_2 = a_{35}p_1p_2 > 0,$$

then the set $\{\widetilde{q}_1,\ \widetilde{q}_2p_1,\ \widetilde{q}_3p_1p_2,\ -p_1p_2p_3\}$ has the property (E), what's wrong.

Let $q_1 = -p_1p_2p_3$, $\widetilde{q}_1 = -p_1p_2p_3$. Since $q_3p_1p_2 > 0$, $\widetilde{q}_3p_1p_2 > 0$, not(E) for sets $\{q_1,\ q_2p_1,\ q_3p_1p_2,\ -p_1p_2p_3\}$ and $\{\widetilde{q}_1,\ \widetilde{q}_2p_1,\ \widetilde{q}_3p_1p_2,\ -p_1p_2p_3\}$ bring us to equalities $q_2p_1 = q_3p_1p_2$, $\widetilde{q}_2p_1 = \widetilde{q}_3p_1p_2$. It means $q_2 = a_{34}p_2$ and $\widetilde{q}_2 = a_{35}p_2$. By (14)

$$a_{35}p_2 - \frac{p_3}{a_{34}}\overline{q}_2 = \frac{a_{35}}{a_{34}}a_{34}p_2.$$

Hence $\overline{q}_2 = 0$. We see that the set $\{\overline{q}_1,\ 0,\ a_{35}p_1p_2,\ -p_1p_2p_3\}$ has (E). We got a contradiction. □

Lemma 10. $\widetilde{q}_1 \neq -p_1p_2p_3$.

PROOF. Suppose $\widetilde{q}_1 = -p_1p_2p_3$. As $\widetilde{q}_3p_1p_2 > 0$, from not(E) for the set $\{\widetilde{q}_1,\ \widetilde{q}_2p_1,\ \widetilde{q}_3p_1p_2, -p_1p_2p_3\}$ it follows that $\widetilde{q}_2p_1 = \widetilde{q}_3p_1p_2$. Hence $\widetilde{q}_2 = a_{35}p_2$.

By lemma 9, $q_1 \neq -p_1p_2p_3$. Since $\{q_1,\ q_2p_1,\ q_3p_1p_2,\ -p_1p_2p_3\}$ has not(E) and $q_3p_1p_2 > 0$, $-p_1p_2p_3 < 0$, we have that numbers q_1, q_2p_1 are different signs. Hence, as not(E) and $q_2p_1 = -p_1p_2p_3$, then $q_2 = -p_2p_3$. Similarly, it is proved that $\overline{q}_2 = -p_2a_{34}$. By (14)

$$a_{35}p_2 + \frac{p_3}{a_{34}}p_2a_{34} = -\frac{a_{35}}{a_{34}}p_2p_3.$$

Thus $a_{35}p_2 < 0$, what's wrong. □

Lemma 11. $\overline{q}_1 \neq -p_1p_2a_{34}$.

PROOF. Assume $\overline{q}_1 = -p_1p_2a_{34}$. Since $\overline{q}_3p_1p_2 > 0$, from not(E) for the set $\{\overline{q}_1,\ \overline{q}_2p_1,\ \overline{q}_3p_1p_2, -p_1p_2a_{34}\}$ we have $\overline{q}_2p_1 = \overline{q}_3p_1p_2$, hence $\overline{q}_2 = a_{35}p_2$.

By lemma 9 $q_1 \neq -p_1p_2p_3$. As $\{q_1,\ q_2p_1,\ q_3p_1p_2,\ -p_1p_2p_3\}$ has not(E) and $q_3p_1p_2 > 0$, $-p_1p_2p_3 < 0$ then numbers q_1, q_2p_1 are different signs. If $q_1 < 0$ then this set has (E), hence $q_1 > 0$. Therefore, since not(E), $q_2p_1 = -p_1p_2p_3$, i.e. $q_2 = -p_2p_3$.

By lemma 10 $\widetilde{q}_1 \neq -p_1p_2p_3$. Similarly to the previous one, it is proved that $\widetilde{q}_2 = -p_2p_3$. By (14)

$$-p_2p_3 - \frac{p_3}{a_{34}}a_{35}p_2 = -\frac{a_{35}}{a_{34}}p_2p_3.$$

It means $-p_2p_3 = 0$, what's wrong. □

Lemma 12. $q_1 \neq -p_1p_2p_3$.

PROOF. Suppose $q_1 = -p_1p_2p_3$. As $q_3p_1p_2 > 0$, from not(E) for the set $\{q_1,\ q_2p_1,\ q_3p_1p_2, -p_1p_2p_3\}$ it follows that $q_2p_1 = q_3p_1p_2$. Hence $q_2 = a_{34}p_2$.

By lemmas 10, 11 $\overline{q}_1 \neq -p_1p_2a_{34}$, $\widetilde{q}_1 \neq -p_1p_2p_3$. As in lemma 11 we show that $\overline{q}_2 = -a_{34}p_2$, $\widetilde{q}_2 = -p_2p_3$. By (14)

$$-p_2p_3 + \frac{p_3}{a_{34}}a_{34}p_2 = -\frac{a_{35}}{a_{34}}a_{34}p_2.$$

It implies $p_2 = 0$. This is not true. □

Lemma 13. *At least one of the sets (10), (11), (12) has the property (E).*

PROOF. Let's assume that none of these sets has property (E). So far, we have been under this assumption. By lemmas 10, 11, 12, $\widetilde{q}_1 \neq -p_1p_2p_3$, $\overline{q}_1 \neq -p_1p_2a_{34}$, $q_1 \neq -p_1p_2p_3$. As in the proof of lemma 11, we find that $\widetilde{q}_2 = -p_2p_3$, $\overline{q}_2 = -a_{34}p_2$, $q_2 = -p_2p_3$. By (14)

$$-p_2p_3 + \frac{p_3}{a_{34}}a_{34}p_2 = -\frac{a_{35}}{a_{34}}p_2p_3.$$

Hence $\frac{a_{35}}{a_{34}}p_2p_3 = 0$. This is not true. □

Theorem 2. *Let a group G have the representation*

$$G = \langle a, x_1, \dots, x_s; P\rangle,$$

where $P = P(a, x_1, x_2, \dots, x_s) = [a, x_1][a, x_2]\dots[a, x_n]S$ $(n \geq 7)$ and $x_1, \dots, x_7 \notin [S]$. If $t_1, t_2, t_3 \in G$ then $G'\langle t_1, t_2, t_3\rangle^G$ is a locally free group.

PROOF. Let $R = \langle t_1, t_2, t_3\rangle^G$ be a normal closure of the subgroup $\langle t_1, t_2, t_3\rangle$ in G. If $\sigma_i(t_1) = \sigma_i(t_2) = \sigma_i(t_3) = 0$ for some i, then we get into the situation of lemma 4, assuming that $h = x_i$, $m = 1$, $f_1 = \dots = f_{k-1} = 0$. By lemma 4 N is a locally free group, where $N = \langle a, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_s\rangle^G$. In this case $t_1, t_2, t_3 \in N$, hence $RG' \subseteq N$, therefore RG' is a locally free group.

Note that if there are three columns in the matrix $M(G, \widetilde{t})$ with numbers i, j, u such that

$$\sigma_i(t_2) = \sigma_j(t_2) = \sigma_u(t_2) = 0,\ \sigma_i(t_3) = \sigma_j(t_3) = \sigma_u(t_3) = 0,$$

then by lemma 6 RG' is a locally free group.

We can suppose that the matrix $M(G, \overline{t})$ has the form

$$M(G, \overline{t}) = \begin{pmatrix} p_1 & a_{12} & a_{13} & a_{14} & a_{15} & \dots \\ 0 & p_2 & a_{23} & a_{24} & a_{25} & \dots \\ 0 & 0 & p_3 & a_{34} & a_{35} & \dots \end{pmatrix},$$

where $p_1 > 0, p_2 > 0, p_3 \neq 0, a_{34} \neq 0, a_{35} \neq 0, a_{36} \neq 0, a_{37} \neq 0$. If it is necessary to replace t_3 with t_3^{-1} and rename generators of G, we can assume (since $n \geq 7$) that $p_3 > 0, a_{34} > 0, a_{35} > 0$. By lemma 13, in this case at least one of the sets (10), (11), (12) has the property (E), and one can apply lemma 4. This set with the property (E) corresponds to one of matrices $M_0 = M(C_0, \widetilde{t})$, $M_1 = M(C_1, t_1^{\overline{\psi}}, t_2^{\overline{\psi}}, t_3^{\overline{\psi}})$, $M_2 = M(C_2, t_1^{\widetilde{\psi}}, t_2^{\widetilde{\psi}}, t_3^{\widetilde{\psi}})$. Let's say it's M_j. Let N be a normal closure in C_j of a subgroup generated by all generators of C_j except x_4. By lemma 4 N is a locally free group. Hence by lemma 3 RG' is a locally free group. □

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