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WHEN A (DUAL-)BAER MODULE IS A DIRECT SUM OF (CO-)PRIME MODULES

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Abstract. Since 2004, Baer modules have been considered by many authors as a generalization of the Baer rings. A module M_R is called Baer if every intersection of the kernels of endomorphisms on M_R is a direct summand of M_R. It is known that commutative Baer rings are reduced. We prove that if a Baer module M is a direct sum of prime modules, then every direct summand of M is retractable. The converse is true whenever the triangulating dimension of M is finite (e.g. if the uniform dimension of M is finite). Dually, if every direct summand of a dual-Baer module M is co-retractable, then it is a direct sum of co-prime modules and the converse is true whenever the sum is finite or M is a max-module. Among other applications, we show that if R is a commutative hereditary Noetherian ring then a finitely generated R-module is Baer iff it is projective or semisimple. Also, over a ring Morita equivalent to a perfect duo ring, all dual-Baer modules are semisimple.

Keywords: Baer module, co-prime module, co-retractable, prime module, dual-Baer, retractable module.

1. Introduction

Throughout the paper, all rings will have unit elements and all modules will be right unitary. A ring R is said to be *Baer* if for every non-empty subset X of R, the right annihilator of X in R is of the form eR for some $e = e^2 \in R$. Baer rings play an important role in the theory of rings of operators in functional analysis; see [15] and [5]. The concept of Baer ring was extended to modules by S.T. Rizvi and C. S. Roman in [20] and [22]. Baer modules and their generalizations have

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been studied among many other works. Every Baer module M is a D2-module (i.e., if M/A isomorphic to a direct summand of M then A is a direct summand of M), see [28], [11], [17], [18] and [7] for recent works on the subjects. It is easy to verify that " *if R is a commutative Baer ring then R is reduced*". For if $x^2 = 0$ then $x \in \mathrm{ann}_R(x) = eR$ for some $e = e^2 \in R$. Hence $x = ex = 0$. The following question naturally arises: " *Let R be commutative ring. When a finitely generated Baer R-module is a finite direct sum of prime modules?*". By a prime module M, we mean M is co-generated by each of its submodules [6]. We study Baer modules that are direct sums of primes in Theorems 4 and 5, and give an answer to the above question in Theorem 6. As an application, we show that if a ring R is Morita equivalent to a commutative hereditary Noetherian ring then a finitely generated R-module is Baer if and only if it is projective or semisimple (Corollary 2).
The dual notion of the Baer modules was introduced and studied in [25] where a module M_R is *dual-Baer* if for every $N \leq M_R$, the right ideal $\mathrm{Hom}_R(M, N)$ of $\mathrm{End}_R(M)$ is generated by an idempotent element. These modules are known to have the C2-property (i.e., every submodule in M isomorphic to a direct summand is a direct summand of M), see [1],[10], [14], [24] and [9] for some recent works on the dual-Baer modules and important generalizations of them. We shall dually investigate when a dual-Baer module is a direct sum of co-prime ones; see Theorem 7. This shows that if R is a right duo perfect ring, every dual-Baer module is semisimple. In order to extend the results to non-commutative case, we first give a categorical characterization for (dual-)Baer modules in Theorem 1. Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [3] and [16].

2. Main results

If K and L are two R-modules then $\mathrm{Tr}(K, L)$ means $\sum\{f(K) \mid f : K \to L\}$ and $\mathrm{Rej}(K, L)$ means $\bigcap\{\ker f \mid f : K \to L\}$. If $M = M_R$ is a module then the class of R-modules that are generated (resp. co-generated) by M is denoted by $\mathrm{Gen}(M_R)$ (resp. $\mathrm{Cog}(M_R)$). By $N \leq^{\oplus} M_R$, we mean N is a direct summand of M_R (i.e., there is a submodule K of M_R such that $M = N \oplus K$).

Theorem 1. *Every exact sequence $0 \to N \to M \to Y \to 0$ of R-modules with $Y \in$ Cog(M_R) splits if and only if M is a Baer R-module.*

Proof. ($\Rightarrow$). To show that the module M_R is Baer, Let $N = \mathrm{r}_M(X)$ for some non-empty $X \subseteq S$. Then $N = \bigcap_{f \in X} \ker f$. Hence, we can deduce that $M/N \in \mathrm{Cog}(M_R)$. Now the exact sequence $0 \to N \stackrel{\imath}{\to} M \stackrel{\pi}{\to} M/N \to 0$ splits by our assumption. This means $N \leq^{\oplus} M_R$, proving that M_R is Baer.
($\Leftarrow$). Suppose that $0 \to N \stackrel{f}{\to} M \to Y \to 0$ is an exact sequence of R-modules with $Y \in \mathrm{Cog}(M_R)$. Let $K = \mathrm{Im} f$. Since $M/K \simeq Y \in \mathrm{Cog}(M_R)$, there exists a one to one R-homomorphism $\theta : M/K \to M^{\Lambda}$ for some set Λ. For each $\lambda \in \Lambda$, let $f_\lambda = \pi_\lambda \theta$ where π_λ is the canonical projection on M^{Λ}. If now $X = \{f_\lambda \mid \lambda \in \Lambda\}$ then it is easily seen that $K = \mathrm{r}_M(X)$. Thus Im f is a direct summand of M_R by the Baer condition on M. It follows that the exact sequence splits, as desired. □

Lemma 1. *Let M_R be a nonzero module and $End_R(M) = S$. Then $N \in Gen(M_R)$ if and only if $N = IM$ for some right ideal I of S if and only if $Tr(M, N) = N$.*

Proof. By [27, Theorem 13.5]. □

Proposition 1. *Let M_R be a nonzero module with $End_R(M) = S$. Then the module M_R is dual-Baer if and only if $IM \leq^{\oplus} M_R$ for every $I \leq S_S$.*

Proof. (⇒) Let $I \leq S_S$ and $N = IM$. By our assumption, $\mathrm{Hom}_R(M, N)$ is a direct summand of S_S. Also, by Lemma 1, we have $N = \mathrm{Tr}(M, N) = \mathrm{Hom}_R(M, N)M$. It follows that $N \leq^{\oplus} M_R$.
(⇐) Let $N \leq M_R$ and $I = \mathrm{Hom}_R(M, N)$. By our assumption, $IM \leq^{\oplus} M_R$ and so $IM = eM$ for some $e = e^2 \in S$. It follows that $e(M) \subseteq N$ and $(1 - e)I = 0$. Hence, $e \in I$ and $I \subseteq eS$. Thus $I = eS$. □

Theorem 2. *Every exact sequence $0 \to N \to M \to Y \to 0$ of R-modules with $N \in Gen(M)$ splits if and only if M is a dual-Baer R-module.*

Proof. This obtained by Lemma 1 and Proposition 1. □

Theorem 3. *Being Baer (dual-Baer) module is a Morita invariant property.*

Proof. This is obtained by Theorems 1, 2 and the fact that the category equivalences preserve exact sequences and direct (co) products [3, Proposition 21.6(3) and (5)]. □

In [21, Propositions 2.12 and 2.14], it is investigated when a Baer module is a direct sum of indecomposable modules. In comparison, a dual-Baer module is always decomposed into the indecomposable modules [25, Corollary 2.6]. In the following, we investigate Baer modules that are decomposed into the *reduced modules* and give some applications. A module is said to be reduced if it has no non-trivial fully invariant direct summand. If X and Y are R-modules then it is well known that $\mathrm{Rej}(X, Y)$ is a fully invariant submodule of X and $X/\mathrm{Rej}(X, Y)$ lies in $\mathrm{Cog}(Y)$. Thus if M_R is a reduced Baer module then Theorem 1 shows that for any $0 \neq N \leq M$, $\mathrm{Hom}_R(M, N) = 0$ or $M \in \mathrm{Cog}(N)$. In [12], the *triangulating dimension (τ-dimension) τdim* was defined for a module M_R as follows: $\mathrm{Sup}\{k \in \mathbb{N} \mid M = \oplus_{i=1}^{k} M_i$ with $M_i \neq 0$ and $\mathrm{Hom}_R(M_i, M_j) = 0$ for any $i < j\}$. Theorem 2.5 of [12] states a series of necessary and sufficient conditions for $\tau dim(M_R)$ to be finite. In [8, Proposition 2.16] Baer rings with a generalized triangular matrix representation are studied. Note that *Noetherian condition ⇒ finite uniform dimension ⇒ ascending (descending) chain condition on direct summands ⇒ finite τ-dimension*; see [16, Chapter 3, page 208] for more information about the uniform dimension of a module. A module that is cogenerated by each of its nonzero submodule, is called *prime* in [6] and $*$-*prime* in [19]). It is easy to verify that *every prime module is reduced and retractable*. An R-module M_R is called *retractable* if $\mathrm{Hom}_R(M, N) \neq 0$ for every $0 \neq N \leq M_R$. The following theorem shows that *the study of Baer modules with finite τ-dimension reduces to the study of such modules when they are prime*. Recall that two R-modules X and Y are called *orthogonal to each other* (or orthogonal), if they do not contain nonzero isomorphic submodules.

We state a few lemmas and use them to prove Theorems 4 and 5.

Lemma 2. (a) *Let M_R be a nonzero retractable Baer module. Then either M_R is prime or there is a decomposition $M = N \oplus K$ such that N is a non-trivial fully invariant direct summand of M_R.*
(b) *If $M = M_1 \oplus M_2$ is a Baer R-module and each M_i is a prime R-module ($i = 1, 2$) such that $Hom_R(M_1, M_2) = 0$, then $Hom_R(M_2, M_1) = 0$.*
(c) *If M_1 and M_2 are prime R-modules such that $Hom_R(M_i, M_j) = 0$ ($i \neq j$) then M_1 and M_2 are orthogonal.*

Proof. (a) Let M_R be a non-prime module. Thus there exists a non-trivial submodule X of M such that $\text{Rej}(M, X) =: N$ is nonzero. Clearly N is a fully invariant direct summand of M such that $M/N \in \text{Cog}(M)$. Since M_R is retractable, $N \neq M$. Thus N is a non-trivial fully invariant submodule of M_R by Theorem 1.
(b) If $f : M_2 \to M_1$ is nonzero, we have $M/\ker f \in \text{Cog}(M)$. Then by the Baer condition on M (Theorem 1), we must have $M_2 \simeq \ker f \oplus \text{Im } f$. Now since M_1 is prime, it lies in $\text{Cog}(\text{Im } f)$. Hence $\text{Hom}_R(M_1, M_2) \neq 0$, contradiction.
(c) This is routine. □

Corollary 1. *A nonzero prime module is retractable and reduced. The converse is true when the module is Baer.*

Proof. If M_R is prime then for any nonzero $X, Y \leq M_R$, we have $\text{Hom}_R(X, Y) \neq 0$. It follows that M_R is reduced and retractable. The converse is obtained by Lemma 2(a). □

Lemma 3. *Let $M = \bigoplus_{i \in I} M_i$($I$ is an index set) such that $Hom_R(M_i, M_j) = 0$ ($i \neq j$). If $N \oplus L = M$ then $N = \bigoplus_{i \in I}(N \cap M_i)$.*

Proof. It is obtained by [22, Lemma 1.3.18], we give a proof for the reader's convenience. By hypothesis each M_i is a fully invariant submodule of M_R. Hence for each i we have $M_i = (N \cap M_i) \oplus (L \cap M_i)$. Thus $M = [\bigoplus_i (N \cap M_i)] \oplus [\bigoplus_i (L \cap M_i)] = N \oplus L$. It follows that $N = \bigoplus_i (N \cap M_i)$. □

Lemma 4. [23, Lemma 2.1] *Let a module $M = \bigoplus_{i \in I} M_i$($I$ is an index set) be a direct sum of submodules $M_i (i \in I)$ and let N be any nonzero submodule of M. Then there exists a subset J of I such that the canonical projection $\pi : M \to \bigoplus_{j \in J} M_j$ is injective on N and $\pi(N) \cap M_j \neq 0$.*

We say that X and Y are *sub-cog of each other* and write ($X \overset{sub-cog}{\simeq} Y$), if $X \in \text{Cog}(Y)$ and $Y \in \text{Cog}(X)$.

Lemma 5. *If $M = M_1 \oplus M_2$ is a Baer R-module and M_i is prime R-module ($i = 1, 2$) then either $Hom_R(M_1, M_2) = Hom_R(M_1, M_2) = 0$ or $M_1 \overset{sub-cog}{\simeq} M_2$*

Proof. Let $N = \text{Rej}(M_1, M_2)$. Then $M_1/N \in \text{Cog}(M_2)$. It follows that $M/N \in \text{Cog}(M)$ and so N is a direct summand of M by Theorem 1. Therefore N is a fully invariant direct summand of the Baer module M_1 by [20, Theorem 2.17]. Now by Corollary 1, we must have $N = 0$ or $N = M_1$. Consequently, $M_1 \in \text{Cog}(M_2)$ or $\text{Hom}_R(M_1, M_2) = 0$. Similarly $M_2 \in \text{Cog}(M_1)$ or $\text{Hom}_R(M_2, M_1) = 0$. The proof is now completed by Lemma 2(b). □

Theorem 4. *Let $M = \bigoplus_{i \in I} P_i$($I$ is an index set) be a Baer module such that each P_i is a prime R-module. Then we have:*

(a) $M=\bigoplus_{j\in J}M_j$ *(J is an index set) such that each* M_j *is a prime R-module and* $M_i's$ *are mutually orthogonal with* $Hom_R(M_i,M_j)=0$ $(i\neq j)$.
(b) *Every direct summand of* M_R *is retractable.*

Proof. (a) For every $j\in I$, let $M_j=\bigoplus\{P_i \mid P_i \overset{sub-cog}{\simeq} P_j\}$. Since every direct summand of a Baer module is Baer ([20, Theorem 2.17]), we have by Lemma 5, $\mathrm{Hom}_R(M_i,M_j)=0$ for every $(i\neq j)$. Thus it is enough to prove that each M_i is a prime module. Fix j and let $P=M_j=\bigoplus_{i\in A_j}P_i(A_j$ an index set). Suppose that $0\neq N\leq P$, we may suppose by Lemma 4 that $N\cap P_i\neq 0$ for all $i\in A_j$. Thus every P_i can be embedded into N because P_i is assumed to be prime. It follows that $P\in\mathrm{Cog}(N)$, as desired. The proof is now completed by 2(c).
(b) We use (a). Let $0\neq K\leq N\leq M=N\oplus N'$. We shall show that $\mathrm{Hom}_R(N,K)\neq 0$. By Lemma 3, we have $N=\bigoplus_{j\in J}(N\cap M_j)$. Again by Lemma 4, there is $t\in J$ such that K and $(N\cap M_t)$ contain nonzero isomorphic submodules. Thus the prime condition on M_t implies that $\mathrm{Hom}_R(N\cap M_t,K)\neq 0$, proving that $\mathrm{Hom}_R(N,K)\neq 0$. □

Theorem 5. *The following statements are equivalent for a non-zero Baer module* M.
(a) $\tau dim(M_R)<\infty$ *and every direct summand of* M_R *is retractable.*
(b) $M=\bigoplus_{i=1}^n M_i$ *such that each* M_i *is a prime R-module.*
(c) $M=\bigoplus_{i=1}^n M_i$ *such that each* M_i *is a prime R-module and* $M_i's$ *are mutually orthogonal with* $Hom_R(M_i,M_j)=0$ $(i\neq j)$.

Proof. (a) ⇒ (b). This obtained by [12, Theorem 2.5] and Corollary 1.
(b) ⇒ (c). By Proposition 4.
(c) ⇒ (a). By Proposition 4, every direct summand of M_R is retractable and by [12, Theorem 2.5] and Corollary 1, we have $\tau dim(M_R)<\infty$. □

A module M_R is called *compressible* if M can be embedded into every non-zero submodule of M_R. R-modules X and Y are said to be *sub-isomorphic* if X can be embedded into Y and vise versa; see for example [23].

Proposition 2. *Let* M_R *be a non-zero Baer module with ascending (descending) chain condition on direct summands. Then the following statements are equivalent.*
(a) *Every direct summand of* M_R *is retractable.*
(b) $M=\bigoplus_{i=1}^n M_i$ *such that* $Hom_R(M_i,M_j)=0$ $(i\neq j)$ *and each* M_i *is a finite direct sum of indecomposable compressible R-modules that are mutually sub-isomorphic.*

Proof. (a) ⇒ (b). By our assumption $\tau dim(M_R)$ is finite ([12, Theorem 2.5]). Hence by Theorem 5, $M=\bigoplus_{i=1}^n M_i$ is a direct sum of prime modules such that $\mathrm{Hom}_R(M_i,M_j)=0$ $(i\neq j)$. It follows that $\tau\dim(M_i)<\infty$ for all i by [12, Proposition 2.14]. Therefore, we may suppose that M_R is a prime module. Now by [3, proposition 10.14], $M=\bigoplus_{i=1}^m P_i$ is a finite direct sum of indecomposable submodules. Thus the Baer condition on $P_i\oplus P_j$ implies that every $f:P_i\to P_j$ is zero or one to one. On the other hand, since M_R is prime, $\mathrm{Hom}_R(X,Y)\neq 0$ for every non zero submodules X and Y of M_R. Consequently, all P_i are indecomposable compressible R-modules that are mutually sub-isomorphic, as desired.

(b) ⇒ (a). Let $P = \bigoplus_{i=1}^{m} P_i$ where $P_1, \cdots P_m$ are compressible and mutually subisomorphic R-modules. It is easy to show that P_R is a prime. Therefore, (a) is now obtained by Theorem 4. □

Lemma 6. *Let R be commutative ring, P and Q be prime ideals of R. Then for every finitely generated nonzero ideal A in R/P, $Hom_R(A, R/Q) = 0$ if and only if $P \nsubseteq Q$.*

Proof. If there exist a finitely generated ideal A in R/P and $0 \neq f \in \mathrm{Hom}_R(A, R/Q)$ then $P \subseteq \mathrm{ann}_R(f(A)) = Q$. Conversely, let $P \subseteq Q$ and A be a finitely generated ideal in R/P. Clearly R/P can be embedded into A_R. Hence, there is an R-homomorphism $f : A \to E$, where E is the injective hull of $(R/Q)_R$. It follows that $f(A) \cap R/Q \neq 0$. Since now $f(A)$ is finitely generated, it is retractable by [13, Theorem 2.7]. Thus $\mathrm{Hom}_R(f(A), (f(A) \cap R/Q))$ is nonzero, proving that $\mathrm{Hom}_R(A, R/Q) \neq 0$ □

Theorem 6. *Let R be a commutative ring. The following statements are equivalent for every finitely generated Baer R-module M.*
(a) $\tau dim(M_R) < \infty$.
(b) $M = \bigoplus_{i=1}^{n} M_i$ *such that each* M_i *is a prime R-module.*
(c) *There are mutually incomparable prime ideals* $P_1 \cdots P_n$ *such that* $M = \bigoplus_{i=1}^{n} M_i$ *and each* M_i *is a finite direct sum of right ideals of* R/P_i.

Proof. Let M be a finitely generated R-module. By [13, Theorem 2.7], every finitely generated R-module is retractable. Hence, every direct summand of M_R is retractable. Thus (a) ⇔ (b) by Theorem 5.
(c) ⇒ (b). This is clear.
(b) ⇒ (c). We have the decomposition $M = \bigoplus_{i=1}^{n} M_i$ of prime modules such that $\mathrm{Hom}_R(M_i, M_j) = 0$ $(i \neq j)$ by Theorem 4. Let $P_i = \mathrm{ann}_R(M_i)$. Since M_i is a prime R-module then P_i is a prime ideal of R and M_i is prime torsionfree as an R/P_i-module. Fix i, replace R with R/P_i and let $M = M_i$. Thus M is a torsionfree Baer module over a commutative domain R. If $M = R^{(n)}/K$ then K is a closed submodule of $R^{(n)}$. Also R being commutative domain has uniform dimension 1. Hence, the uniform dimension of M_R is finite by [16, Theorem 6.35]. Thus we can now apply Proposition 2 to deduce that M_R is a finite direct sum of (uniform) right ideals of R. The proof is completed by Lemma 6. □

The following result can be a generalization of [20, Proposition 2.19].

Corollary 2. *Let R be a ring Morita invariant to a commutative hereditary Noetherian ring and M_R is finitely generated. Then M_R is a Baer R-module M if and only if it is semisimple or projective.*

Proof. By Theorem 3, we can suppose that R is a commutative hereditary Noetherian ring. For the necessity, we apply Theorem 6 for M. If $P_j = 0$ for some j then $n = 1$ and M a finite direct sum of right ideals of R, because all P_i are mutually incomparable. Hence M_R is projective by the hereditary condition on R. On the other hand, every singular R-module has a nonzero socle by [**?**, Proposition 5.4.5]. Hence if M_R is singular, it must be semisimple by Proposition 2. For the sufficiency, let M_R be projective. To show that M_R is Baer, we apply Theorem 1. Let $0 \to N \to M \to Y \to 0$ be an exact sequence of R-modules with $Y \in \mathrm{Cog}(M)$. Since

M_R is projective, $Y \in \mathrm{Cog}(R)$. Thus Y_R can be embedded in a free R-module by [**?**, Proposition 3.4.3]. Our assumption on R implies that Y_R must be projective. Hence the exact sequence splits, as desired. □

We now consider the dual of Theorem 5. Following [26], a module M is called *co-prime* whenever $M \in \mathrm{Gen}(M/N)$ for every proper submodule $N < M_R$. Furthermore, we say that M_R is *co-compressible* if M_R is a homomorphic image of every nonzero factor of M_R. Clearly, co-compressible ⇒ co-prime ⇒ co-retractable. However, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is co-compressible, dual-Baer which is not Baer. For the dual notion of sub-isomorphic, by $X \overset{epi}{\simeq} Y$ the R-modules X and Y are called *epi-invariant* if X and Y are homomorphic image of each other.

Lemma 7. *(a) Let M_R be a nonzero co-retractable dual-Baer module. Then either M_R is co-prime or there is a decomposition $M = N \oplus K$ such that N is a non-trivial fully invariant direct summand of M_R.*
(b) If $M = M_1 \oplus M_2$ is a dual-Baer R-module and M_i is co-prime R-module ($i = 1, 2$) such that $Hom_R(M_1, M_2) = 0$, then $Hom_R(M_2, M_1) = 0$.

Proof. (a) and (b) are dual of Lemma 2 and obtained from the dual arguments. □

Corollary 3. *Every co-prime module is reduced and co-retractable. The converse is true if the module is dual-Baer.*

Proof. If M_R is co-prime then clearly it is co-retractable. Also if $M = N \oplus K$ such that N is a non-trivial fully invariant direct summand of M_R. Then M, hence K is generated by M/K. This follows that $\mathrm{Hom}_R(N, K) \neq 0$, a contradiction. The converse is obtained by Lemma 7(a). □

Lemma 8. *Let $M = \bigoplus_{i \in I} C_i$ such that for any $i \in I$, the R-module C_i is reduced. If V is a non-zero fully invariant direct summand of M_R then there is non-empty subset J of I such that $V = \bigoplus_{j \in J} C_j$ and $Hom_R(V, C_i) = 0$ for any $i \in I \setminus J$.*

Proof. Let $M = V \oplus V'$ where V is fully invariant submodule of M_R. Then we have $V = \bigoplus_{i \in I} V \cap C_i$ and $\mathrm{Hom}_R(V, V') = 0$. On the other hand, for every $i \in I$, it is easy to verify that $V \cap C_i$ is a fully invariant direct summand of C_i. Now let $J = \{i \in I \mid V \cap C_i \neq 0\}$. □

Lemma 9. *Every indecomposable co-retractable dual-Baer module is a co-compressible module.*

Proof. Suppose Y is a proper submodule of X and X is an indecomposable co-retractable dual-Baer module. Since X is co-retractable, there exits nonzero $f : X/Y \to X$. Now $\mathrm{Im}(f)$ lies in $\mathrm{Gen}(X)$ and so must be a direct summand of X by the dual-Baer condition (Theorem 2). It follows that f is surjective, proving that X is co-compressible. □

A module M is called *max* if every proper submodule of M lies in a maximal submodule.

Theorem 7. *Consider the following conditions for a non-zero dual-Baer module M_R.*
(a) *Every direct summand of M_R is co-retractable.*

(b) *M is a direct sum of co-compressible R-modules.*
(c) *M is a direct sum of co-prime R-modules.*
(d) *$M = \bigoplus_{i\in I} M_i$ is a direct sum of reduced R-modules with $Hom_R(M_i, M_j) = 0$ $(i \neq j)$ such that each M_i is a direct sum of indecomposable co-compressible modules that are mutually epi-invariant.*
Then (a)$\Rightarrow$(b) $\Leftrightarrow$ (c) and (c)$\Rightarrow$(d). All conditions (a) $\sim$ (d) are equivalent whenever I is a finite set or M_R is a max-module.

Proof. (a)$\Rightarrow$(b). By [25, Corollary 2.6], M is a direct sum of indecomposable modules. Thus (b) is obtained by our assumption and Lemma 9 and the fact that a direct summand of dual-Baer is dual-Baer ([25, Corollary 2.5]).
(b)$\Leftrightarrow$(c). One side is clear. For the other side, note that every direct summand of a co-prime module is co-prime (hence, co-retractable). Thus we are done by (a)$\Rightarrow$(b).
(b)$\Rightarrow$(d) We may suppose that $M = \bigoplus_{i\in I} C_i$ such that every C_i is indecomposable co-compressible R-module. Since M_R is dual-Baer, if $f \in \mathrm{Hom}_R(C_i, C_j)$ $(i \neq j)$ then $\mathrm{Im}(f)$ must be a direct summand of C_j (Theorem 2). If follows that $f = 0$ or f is epic. Thus for every $i \neq j$, either $\mathrm{Hom}_R(C_i, C_j) = \mathrm{Hom}_R(C_j, C_i) = 0$ or $C_i \overset{epi}{\simeq} C_j$ by Lemma 7(b). Now for any $i \in I$, let $M_i = \bigoplus_j \{C_j \mid C_i \overset{epi}{\simeq} C_j\}$. By Lemma 8, each M_i is reduced.
(d)$\Leftrightarrow$(a). Suppose that I is a finite set or M_R is a max-module. Then by [2, Propositions 2.6 and 2.7], each M_i is a co-retractable R-module. Now let $N \oplus L = M$ then by Lemma 3, $N = \bigoplus_{i\in I}(N \cap M_i)$. It is easily seen that each $(N \cap M_i)$ is a direct summand of M_i, and so is a co-retractable R-module. Therefore N_R is co-retractable by our assumption, as desired. □

Recall that the singular submodule $\mathrm{Z}(M_R)$ of an R-module M is defined by $\mathrm{Z}(M_R) = \{m \in M_R \mid mA = 0$ for some essential right ideal A of $R\}$. The module M_R is called *singular* (resp. *nonsingular*) if $\mathrm{Z}(M_R) = M$ (resp. $\mathrm{Z}(M_R) = 0$).

Corollary 4. *Let M be a non-zero dual-Baer R-module such that every direct summand of M_R is co-retractable. Then $M = M_1 \oplus M_2$ where M_1 is singular and M_2 is a nonsingular semisimple module.*

Proof. If C_R is co-compressible and N is a proper essential submodule of C, then C/N and hence $C \simeq C/N$ is singular. This follows that every co-compressible R-module is either singular or semisimple (as semisimple R-modules have no proper essential submodules). The proof is now completed by Theorem 7. □

Corollary 5. *A projective dual-Baer M_R is semisimple if and only if every direct summand of M_R is co-retractable.*

Proof. This follows from Corollary 5 and the fact that nonzero projective modules are not singular. □

A ring R is said to be *right duo* if every right ideal in R is a two sided ideal.

Lemma 10. *If R is a ring Morita invariant to a right duo perfect ring then every nonzero R-module is max and co-retractable.*

Proof. We may suppose that R is a right duo and a perfect ring. Let N be a proper submodule of a nonzero module M_R. Since R is right perfect then the nonzero module M/N has a maximal submodule K/N. On the other hand, by [4, Theorem 2.14], the simple R-module M/K can be embedded in M_R. It follows that $\mathrm{Hom}_R(M/N, M)$ is nonzero, as desired. □

Theorem 8. *Let R be a ring Morita invariant to a right duo perfect ring, then dual-Baer R-modules are precisely semisimple R-modules.*

Proof. Since a co-compressible module with a maximal submodule is simple, the result follows from Lemma 10 and Theorem 7. □

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