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AN APPLICATION OF THE CHEBYSHEV COLLOCATION METHOD FOR THE CALCULATION OF A MASS FLUX IN A LONG CONCENTRIC ANNULAR CHANNEL

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Abstract. A rarefied gas flow through a long concentric annular channel due to pressure gradient is studied on the basis of the linearized BGK model of the Boltzmann kinetic equation using a Chebyshev collocation method. The method is based on the approximation by the truncated Chebyshev series. The linearized BGK model kinetic equation and boundary conditions are transformed into a matrix equation, which corresponds to a system of linear algebraic equations with the values of the unknown function at the Chebyshev collocation points. The mass flux is calculated as a function of the rarefaction parameter. The accuracy of the results is validated in several ways, including the recovery of the analytical solutions at the hydrodynamic and free molecular limits.

Keywords: linearized BGK model kinetic equation, model of diffuse reflection, collocation method, Chebyshev polynomials.

1. Introduction

Due to the development of micro- and nanotechnologies, in the last decade the interest has grown to the research of flows of a rarefied gas in channels with complex forms of cross sections. Examples on the use of channels in micro- and nanodevices are provided in [1]. A correct description of the flow of a rarefied gas in the channels of such devices can be obtained based on the Boltzmann kinetic equation. However, to get a numeric solution to the Boltzmann kinetic equation, one requires significant computational resources, while substituting in the kinetic equation the collision integral with its model that maintains fundamental properties of the integral,

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allows to reduce the computational expenses significantly. The methods for solving model kinetic equations known at the present time, such as the discrete velocities and ordinates method [2]-[5], the conservative method [6], require a considerable amount of detailed meshes in the configurational space and the velocity space even in the framework of the linear transport theory. To achieve significant savings in computer resources, as shown in [7]-[9], polynomial approximation can be used. The collocation method using Chebyshev's polynomials allows to obtain acceptable results given a comparatively small number of knots in every direction of the phase space [7]-[9]. It is also necessary to emphasize that the limits of applicability of the linearized model kinetic equations for describing the gas flow in the channels are much wider [10] than it was assumed before, hence, the problem of searching for the solutions of those is still relevant.

In this paper, we propose a modification of the approach [7]-[9] with the application of the properties of Chebyshev polynomial sums [11] and Hadamard's and Kronecker's matrix products. The proposed modification allows to perform a rational construction of the matrix of a system of linear non-homogeneous equations taking into account the boundary conditions for a more complex form of a cross section of the channel, which represents a ring. The numerical solution of the problem on calculation of the mass flow in a long concentric annular channel is obtained using the Bhatnagar–Gross–Krook (BGK) model of the Boltzmann kinetic equation [12] in the framework of diffuse boundary conditions on the walls of the channel.

2. Statement of the problem. The kinetic equaiton

Consider the flow of a rarefied gas in a long concentric annular channel formed by two cylinders with the radii R_1' and R_2' ($R_1' < R_2'$), under the action of a given gradient of pressure directed along the axis of the channel z'. The walls of the channel are kept at a constant temperature. We will consider the gas flow in the medium part of the channel. The condition of the rarefied gas at the point $\mathbf{r}'$ is defined by the function of distribution of the gas molecules $f'(\mathbf{r}', \mathbf{v})$, where $\mathbf{v}$ is a molecular gas velocity. The macroscopic parameters of the gas in the channel, such as concentration n', temperature T', pressure p', mass velocity u, are expressed via the function of distribution $f'(\mathbf{r}', \mathbf{v})$ in the form of integrals over the velocity space. As the scales of length, velocity, consent ration, temperature, distribution function, we choose respectively the values R_2', $\beta^{-1/2}$, n_0', T_0', $n_0'\beta^{3/2}$, where $\beta = m'/(2k_BT_0')$, k_B is the Boltzmann constant, m' is the mass of gas molecules, n_0', T_0' is the concentration, temperature of the gas at some point taken as the origin; $p' = n'k_BT'$. Then for dimensionless quantities we have the following relations:

$$\mathbf{r} = \frac{\mathbf{r}'}{R_2'},\ R_1 = \frac{R_1'}{R_2'},\ R_2 = 1,\ f = \frac{f'}{n_0'\beta^{3/2}},$$

$$\mathbf{C} = \beta^{1/2}\mathbf{v},\ \mathbf{u} = \beta^{1/2}\mathbf{u}',\ n = \frac{n'}{n_0'},\ T = \frac{T'}{T_0'}.$$

We assume that the length of the channel $L' \gg R_2'$, and the nondimentional pressure gradient is small in absolute value, that is,

$$G_p = \frac{dp}{dz},\ |G_p| \ll 1. \tag{1}$$

Taking into account the asymmetrical character of a gas flow in the channel, we introduce the cylindrical coordinates $\mathbf{r} = (\rho, r_\varphi, r_z)$ in the configurational space and $\mathbf{C} = (C_\perp, C_\psi, C_z)$ in the velocity space. Designating $\zeta = \cos\psi$, in the linear approximation we have

$$p(z) = 1+G_p z,\ f(\mathbf{r},\mathbf{C}) = f_0(C)\left(1 + G_p(z + h(\rho,\mathbf{C}))\right),\ f_0(C) = \pi^{-3/2}\exp\left(-C^2\right),$$

$$u_z = G_p U_z,\ U_z(\rho) = \frac{2}{\pi}\int\limits_0^{+\infty}\exp(-C_\perp^2)C_\perp\int\limits_{-1}^{1}\frac{1}{\sqrt{1-\zeta^2}}Z(\rho,C_\perp,\zeta)d\zeta dC_\perp, \tag{2}$$

$$Z(\rho,C_\perp,\zeta) = \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}\exp(-C_z^2)C_z h(\rho,\mathbf{C})dC_z.$$

The main calculated value is the reduced gas mass flow

$$J_M = \frac{4}{(1-R_1^2)}\int\limits_{R_1}^{1}U_z(\rho)\rho d\rho. \tag{3}$$

To obtain U_z, we use the linearized BGK equation [7]

$$\left(\frac{\partial Z}{\partial\rho}\zeta + \frac{\partial Z}{\partial\zeta}\frac{(1-\zeta^2)}{\rho}\right)C_\perp + \delta Z(\rho,C_\perp,\zeta) + \frac{1}{2} = \delta U_z(\rho), \tag{4}$$

and the diffuse boundary conditions on the channel walls

$$Z(1,C_\perp,\zeta) = 0,\ \zeta < 0, \quad Z(R_1,C_\perp,\zeta) = 0, \quad \zeta > 0, \tag{5}$$

where $\delta = \mathrm{Kn}^{-1}$ is a parameter of gas rarefaction, Kn is a Knudsen number.

3. Solution of the boundary problem

We decompose the unknown function $Z(\rho,C_\perp,\zeta)$, where $\rho\in[0,\,1]$, $C_\perp\in[0,+\infty)$ and $\zeta\in[-1,\,1]$, into a series of Chebyshev polynomials of the first kind T_{k_i}, and restricting ourselves in this series to the members with the numbers $k_i \le n_i$ ($i = \overline{1,3}$), we obtain

$$Z(\rho,C_\perp,\zeta) = \mathbf{T_1}(x_1)\otimes\mathbf{T_2}(x_2)\otimes\mathbf{T_3}(x_3)\mathbf{A}, \tag{6}$$

where $x_1 = (2\rho - 1 - R_1)/(1-R_1)$, $x_2 = (C_\perp - 1)/(C_\perp + 1)$, $x_3 = \zeta$, $\mathbf{T_i}$ is a matrix of the size $1\times n_i'$ ($n_i' = n_i + 1$, $i = \overline{1,3}$):

$$\mathbf{T}_i(x_i) = \left(T_0(x_i)\,T_1(x_i)\dots T_{n_i-1}(x_i)\,T_{n_i}(x_i)\right), \tag{7}$$

$\mathbf{A}$ is a matrix of the size $n_1'n_2'n_3'\times 1$:

$$\mathbf{A} = \left(a_{000}\,a_{001}\dots a_{n_1n_2n_3-1}\,a_{n_1n_2n_3}\right)^T, \tag{8}$$

by $\mathbf{T_1}(x_1)\otimes\mathbf{T_2}(x_2)$ a Kronecker product of matrices $\mathbf{T_1}(x_1)$ and $\mathbf{T_2}(x_2)$ is denoted.

Substituting (6) into (4) and (5), we obtain

$$\mathbf{B}(x_1,x_2,x_3)\mathbf{A} = -\frac{1}{2}, \tag{9}$$

$$\mathbf{T_1}(-1)\otimes\mathbf{T_2}(x_2)\otimes\mathbf{T_3}(x_3)\mathbf{A} = 0, \quad x_2 > 0, \tag{10}$$

$$\mathbf{T_1}(1)\otimes\mathbf{T_2}(x_2)\otimes\mathbf{T_3}(x_3)\mathbf{A} = 0, \quad x_2 < 0, \tag{11}$$

where

$$\mathbf{B}(x_1,x_2,x_3)=\frac{2}{1-R_1}\left(\frac{d\mathbf{T_1}(x_1)}{dx_1}\otimes\mathbf{T_2}(x_2)x_2+ \right. \\ \left. +\mathbf{T_1}(x_1)\otimes\frac{d\mathbf{T_2}(x_2)}{dx_2}\frac{(1-x_2^2)}{x_1+(1+R_1)/(1-R_1)}\right)\otimes \\ \otimes\mathbf{T_3}(x_3)\frac{1+x_3}{1-x_3}+\delta\mathbf{T_1}(x_1)\otimes(\mathbf{T_2}(x_2)\otimes\mathbf{T_3}(x_3)-\mathbf{P_2}\otimes\mathbf{P_3}), \tag{12}$$

$$\mathbf{P_2}=\frac{2}{\pi}\int\limits_{-1}^{1}\frac{\mathbf{T_2}(x_2)}{\sqrt{1-x_2^2}}dx_2=(2\,0\,\ldots 0\,0), \tag{13}$$

$$\mathbf{P_3}=2\int\limits_{-1}^{1}\frac{1+x_3}{(1-x_3)^3}\mathbf{T_3}(x_3)\exp\left(-\frac{(1+x_3)^2}{(1-x_3)^2}\right)dx_3. \tag{14}$$

We choose as collocation points in (9) for x_1 the extremum points of the polynomial $T_{n_1}(x_1)$ on the segment $[-1;1]$, for x_2 and for x_3 zeroes $T_{n'_2}(x_2)$ and $T_{n'_3}(x_3)$ on this segment:

$$x_{1,k_1}=\cos\left(\frac{\pi(n_1-k_1)}{n_1}\right),\quad k_1=\overline{0,n_1}, \tag{15}$$

$$x_{i,k_i}=\cos\left(\frac{\pi(2n_i-2k_i+1)}{2(n_i+1)}\right),\quad k_i=\overline{0,n_i},\ i=2,3. \tag{16}$$

To obtain the values of Chebyshev polynomials at the points (15) and (16) and their derivatives, we use the geometric definition $T_{j_i}(x_i)=\cos(j_i\arccos x_i)$, where $x_i\in[-1,1]$ [11]. Then

$$T_{j_1}(x_{1,k_1})=\cos\left(\frac{\pi j_1(n_1-k_1)}{n_1}\right),\quad j_1,k_1=\overline{0,n_1},$$

$$T_{j_i}(x_{i,k_i})=\cos\left(\frac{\pi j_i(2n_i-2k_i+1)}{2(n_i+1)}\right),\quad j_i,k_i=\overline{0,n_i},\ i=2,3.$$

$$\frac{dT_{j_i}(x_i)}{dx_i}=\frac{j_i\sin(j_i\arccos x_i)}{\sqrt{1-x_i^2}},\quad j_i=\overline{0,n_i},\ i=1,2. \tag{17}$$

Calculating the values of $\dfrac{dT_j(x_i)}{dx_i}$ $(i=1,2)$ respectively at the points (15) and (16), we have

$$\frac{dT_{j_1}(x_{1,0})}{dx_1}=(-1)^{j_1+1}j_1^2,\ \ \frac{dT_{j_1}(x_{1,n_1})}{dx_1}=j_1^2,$$

$$\frac{dT_{j_1}(x_{1,k_1})}{dx_1}=\frac{j_1\sin\left(\dfrac{\pi j_1(n_1-k_1)}{n_1}\right)}{\left|\sin\left(\dfrac{\pi(n_1-k_1)}{n_1}\right)\right|},\quad j_1=\overline{0,n_1},\ k_1=\overline{1,n_1-1}, \tag{18}$$

(19)
$$\frac{dT_{j_2}(x_{2,k_2})}{dx_2} = \frac{j_2 \sin\left(\dfrac{\pi j_2(2n_2-2k_2+1)}{2(n_2+1)}\right)}{\left|\sin\left(\dfrac{\pi(2n_2-2k_2+1)}{2(n_2+1)}\right)\right|}, \quad j_2,k_2=\overline{0,n_2}.$$

Substituting (15) and (16) into (9), we arrive at the system of linear $n'_1n'_2n'_3$-equations, where we substitute the equations with $x_{1,0}$, x_{2,k_2} $(k_2=\overline{n'_2/2,n_2})$ by the equations that follow from boundary condition (10):

(20)
$$\mathbf{T_1}(-1)\otimes\mathbf{T_2}(x_{2,k_2})\otimes\mathbf{T_3}(x_{3,k_3})\mathbf{A}=0, \quad k_3=\overline{0,n_3},$$

with x_{1,n_1}, x_{2,k_2} $(k_2=\overline{0,n'_2/2-1})$ by the equations that follow from boundary condition (11):

(21)
$$\mathbf{T_1}(1)\otimes\mathbf{T_2}(x_{2,k_2})\otimes\mathbf{T_3}(x_{3,k_3})\mathbf{A}=0, \quad k_3=\overline{0,n_3}.$$

Here and after we consider n_2 an odd number. As a result, the system of linear $n'_1n'_2n'_3$-equations is transformed into the form

(22)
$$\mathbf{CA}=\mathbf{F},$$

(23)
$$\mathbf{C}=\frac{2}{1-R_1}\mathbf{E_{JH}}\circ\left(\sum_{k=2,3}\mathbf{J_k}\otimes\mathbf{H_k}\right)\otimes\mathbf{G_2}+\delta\mathbf{J_1}\otimes\mathbf{H_1}\otimes\mathbf{G_1}-$$
$$-\delta\mathbf{E_{JH}}\circ(\mathbf{J_1}\otimes\mathbf{H_4}\otimes\mathbf{P_2})\otimes(\mathbf{G_3}\otimes\mathbf{P_3}), \quad \mathbf{F}=-\frac{1}{2}\mathbf{E_{JH,1}}\otimes\mathbf{G_3},$$

by $\circ$ Hadamard's matrix product is denoted. The elements of the square matrices $\mathbf{J_l}=(\mathbf{J_l})_{n'_1\times n'_1}$, $\mathbf{H_l}=(\mathbf{H_l})_{n'_2\times n'_2}$ $(l=\overline{1,3})$ and $\mathbf{G_l}=(\mathbf{G_l})_{n'_3\times n'_3}$ $(l=1,2)$ and column matrices $\mathbf{H_4}=(\mathbf{H_4})_{n'_2\times 1}$ and $\mathbf{G_3}=(\mathbf{G_3})_{n'_3\times 1}$ are defined the following way:

(24)
$$J_{1,i_1,j_1}=T_{1,j_1}(x_{1,i_1}),\ J_{2,i_1,j_1}=\frac{dT_{1,j_1}(x_{1,i_1})}{dx_1},$$
$$J_{3,i_1,j_1}=\frac{T_{1,j_1}(x_{1,i_1})}{x_{1,i_1}+(1+R_1)/(1-R_1)},\ (i_1,j_1=\overline{0,n_1});$$

(25)
$$H_{1,i_2,j_2}=T_{2,j_2}(x_{2,i_2}),\ H_{2,i_2j_2}=x_{2,i_2}T_2(x_{2,i_2}),$$
$$H_{3,i_2,j_2}=\frac{dT_{2,j_2}(x_{2,i_2})}{dx_2}(1-x^2_{2,i_2}),\ H_{4,i_2,1}=1,\ (i_2,j_2=\overline{0,n_2});$$

(26)
$$G_{1,i_3,j_3}=T_{3,j_3}(x_{3,i_3}),\ G_{2,i_3,j_3}=\frac{1+x_{3,i_3}}{1-x_{3,i_3}}T_{3,j_3}(x_{3,i_3}),$$
$$G_{3,i_3,1}=1,\ (i_3,j_3=\overline{0,n_3});$$

$\mathbf{E_{JH}}$ is a square matrix of the size $n'_1n'_2\times n'_1n'_2$:

(27)
$$\mathbf{E_{JH}}=\mathbf{E}-\sum_{k=1,2}\mathbf{E_{J,k}}\otimes\mathbf{E_{H,k}},$$

$\mathbf{E_{JH,1}}$ is the first column $\mathbf{E_{JH}}$, the matrix $\mathbf{E}=(\mathbf{E})_{n'_1n'_2\times n'_1n'_2}$ consists of elements equal to 1, and the nonzero elements of $\mathbf{E_{J,k}}_{n'_1\times n'_1}$ and $\mathbf{E_{H,k}}_{n'_2\times n'_2}$ $(k=\overline{1,2})$ are $E_{J,1,0,j_1}=1$, $E_{J,2,n_1,j_1}=1$, $E_{H,1,i_2,j_2}=1$ $(i_2=\overline{n'_2/2+1,n_1})$, $E_{H,2,i_2,j_1}=1$, $(i_2=\overline{0,n'_2/2})$, $(j_k=\overline{0,n_k})$ $(k=\overline{1,2})$.

Taking into account the fact that at the points (15) we have that [11]

$$\frac{2}{n_1}\sum_{k_1=0}^{n_1}{}'' T_{j_1}(x_{1,k_1})T_{l_1}(x_{1,k_1}) = \gamma_{j_1}\delta_{j_1,l_1}, \tag{28}$$

where by $\sum\limits_{k_1=0}^{n_1}{}''$ we mean the finite sum in which the first and the last summands are multiplied by 1/2,

$$\gamma_{j_i} = \begin{cases} 2, & j_i = 0 \vee (i = 1 \wedge j_i = n_i) \\ 1 & j_i > 0 \wedge (i = 1 \wedge j_i < n_i), \end{cases} \qquad \delta_{j_i,l_i} = \begin{cases} 1, & j_i = l_i, \\ 0, & j_i \neq l_i, \end{cases} \quad i = \overline{1,3}, \tag{29}$$

and at the points (16) we have that [11]

$$\frac{2}{n_i+1}\sum_{k_i=0}^{n_i} T_{j_l}(x_{i,k_i})T_{l_l}(x_{i,k_i}) = \gamma_{j_i}\delta_{j_i,l_i}, \ \ i = 2,3, \tag{30}$$

$$\frac{2}{n_i+1}\sum_{k_i=0}^{n_i}{}' T_{k_i}(x_{i,j_1})T_{k_i}(x_{i,l_i}) = \delta_{j_1,l_1}, \ \ i = 2,3, \tag{31}$$

where by $\sum\limits_{k_i=0}^{n_i}{}'$ we mean the finite sum in which the first summand is multiplied by 1/2, then from the system $\mathbf{J_1} \otimes \mathbf{H_1} \otimes \mathbf{G_1}\mathbf{A} = \mathbf{Z}$, in which

$$\mathbf{Z} = (Z_{000},\, Z_{001} \ldots Z_{n_1n_2n_3-1}\, Z_{n_1n_2n_3})^T, \ Z_{k_1k_2k_3} = Z(\rho_{k_1}, C_{\perp,k_2}, \zeta_{k_3}), \tag{32}$$

we obtain

$$\mathbf{A} = \varrho\mathbf{J_1''} \otimes \mathbf{H_1'} \otimes \mathbf{G_1'}\mathbf{Z}, \quad \varrho = \frac{8}{n_1n_2'n_3'}, \tag{33}$$

where $\mathbf{J_1''}$ is a matrix in which $J''_{1,i_1,i_2} = J^T_{1,i_1,i_2}/4$ $(i_1, i_2 = 0, n_1)$, $J''_{1,i_1,i_2} = J^T_{1,i_1,i_2}/2$ $((i_1 = 0, n_1 \wedge i_2 \neq 0, n_1) \vee (i_2 = 0, n_1 \wedge i_1 \neq 0, n_1))$, and the rest of the elements are equal to the corresponding elements of the matrix $\mathbf{J_1}^T$; $\mathbf{H_1'}$ and $\mathbf{G_1'}$ are matrices in which the first rows coincide with the corresponding rows of the matrices $\mathbf{H_1}^T/2$ and $\mathbf{G_1}^T/2$, and the rest of the rows are respectively the rows $\mathbf{H_1}^T$ and $\mathbf{G_1}^T$. For example, given $n_1' = 4$, the matrix $\mathbf{J_1''}$ has the form

$$\mathbf{J_1''} = \begin{pmatrix} \frac{1}{4}T_0(x_{1,0}) & \frac{1}{2}T_0(x_{1,1}) & \frac{1}{2}T_0(x_{1,2}) & \frac{1}{2}T_0(x_{1,3}) & \frac{1}{4}T_0(x_{1,4}) \\ \frac{1}{2}T_1(x_{1,0}) & T_1(x_{1,1}) & T_1(x_{1,2}) & T_1(x_{1,3}) & \frac{1}{2}T_1(x_{1,4}) \\ \frac{1}{2}T_2(x_{1,0}) & T_2(x_{1,1}) & T_2(x_{1,2}) & T_2(x_{1,3}) & \frac{1}{2}T_2(x_{1,4}) \\ \frac{1}{2}T_3(x_{1,0}) & T_3(x_{1,1}) & T_3(x_{1,2}) & T_3(x_{1,3}) & \frac{1}{2}T_3(x_{1,4}) \\ \frac{1}{4}T_4(x_{1,0}) & \frac{1}{2}T_4(x_{1,1}) & \frac{1}{2}T_4(x_{1,2}) & \frac{1}{2}T_4(x_{1,3}) & \frac{1}{4}T_4(x_{1,4}) \end{pmatrix}.$$

Substituting (33) into (22), we arrive at the system of linear $n_1'n_2'n_3'$-equations with respect to $\mathbf{Z}$

$$\mathbf{L}\mathbf{Z} = \mathbf{F}, \tag{34}$$

$$\mathbf{L} = \varrho\mathbf{C}(\mathbf{J_1''} \otimes \mathbf{H_1'} \otimes \mathbf{G_1'}) = \frac{2}{1-R_1}\mathbf{B_1} + \delta\mathbf{I} - \delta\mathbf{B_2}, \tag{35}$$

$$\mathbf{B_1} = \mathbf{E_{JH}} \circ \left(\frac{2}{n_1}\mathbf{J_2}\mathbf{J_1''} \otimes \mathbf{I_{d_2}} + \frac{2}{n_2'}\mathbf{I_{d_1}} \otimes (\mathbf{H_3}\mathbf{H_1'})\right) \otimes \mathbf{I_{d_3}}, \tag{36}$$

$$\mathbf{B_2} = \frac{4}{n_2' n_3'} \mathbf{E_{JH}} \circ (\mathbf{I_1} \otimes \mathbf{E_H}) \otimes \mathbf{E_G}, \tag{37}$$

where $\mathbf{I_1}$ and $\mathbf{I}$ are identity matrices of the sizes $n_1' \times n_1'$ and $n_1' n_2' n_3' \times n_1' n_2' n_3'$, respectively; $\mathbf{I_{d_1}} = \mathbf{J_3 J_1''}$, $\mathbf{I_{d_2}} = \mathbf{H_2 H_1'}$ and $\mathbf{I_{d_3}} = \mathbf{G_2 G_1'}$ are diagonal matrices:

$$I_{d_1,i_1,i_1} = \frac{1}{x_{1,i_1} + (1+R_1)/(1-R_1)}, \; I_{d_2,i_2,i_2} = x_{2,i_2}, \; I_{d_3,i_3,i_3} = \frac{1 + x_{3,i_3}}{1 - x_{3,i_3}}; \tag{38}$$

$\mathbf{E_H} = (\mathbf{E_H})_{n_2' \times n_2'} = \mathbf{H_4} \otimes \mathbf{P_2 H_1'}$ is a matrix, all elements of which equal 1, and the matrix

$$\mathbf{E_G} = \mathbf{G_3} \otimes \mathbf{P_3 G_1'} = \mathbf{G_3} \otimes (\mathbf{P_3 G_1'}), \tag{39}$$

has equal rows $\mathbf{P_3 G_1'}$. We obtain the solution of equation (34) using the LU-method. Based on the obtained elements of the matrix $\mathbf{Z}$, we reconstruct $U_z(\rho)$:

$$U_z(\rho) = \varrho \mathbf{T_1} \left(\frac{2\rho - 1 - R_1}{1 - R_1} \right) \mathbf{J_1''} \otimes \mathbf{E_{H,r}} \otimes \mathbf{E_{G,r}} \mathbf{Z}, \tag{40}$$

where $\mathbf{E_{H,r}}$ and $\mathbf{E_{G,r}}$ are the first rows of the matrices $\mathbf{E_H}$ and $\mathbf{E_G}$.

Substituting (40) into (3), we obtain

$$J_M = \frac{\varrho}{1 + R_1} \mathbf{P_1 J_1''} \otimes \mathbf{E_{H,r}} \otimes \mathbf{E_{G,r}} \mathbf{Z}, \tag{41}$$

$$\mathbf{P_1} = \int_{-1}^{1} \mathbf{T_1}(x_1) \; (x_1(1 - R_1) + 1 + R_1) \, dx_1, \tag{42}$$

where to obtain the integrals of Chebyshev polynomials, we use the following formulae [11]:

$$x_1 T_i(x_1) = T_{i+1}(x_1) + T_{|i-1|}(x_1), \qquad \int_{-1}^{1} T_i(x_1) dx_1 = \begin{cases} \dfrac{2}{1 - i^2}, & i \text{ is even}, \\ 0, & i \text{ is odd} \end{cases} \tag{43}$$

4. Analysis of the obtained results

In the free molecular mode of the flow ($\delta = 0$), equation (4) with boundary conditions (5) transforms into the Boltzmann equation for a collisionless gas and can be obtained analytically [13]. In this case, we have

$$Z(\rho, C_\perp, \zeta) = -\frac{\rho\zeta + \sqrt{1 - \rho^2(1 - \zeta^2)}}{2C_\perp}, \quad -1 \le \zeta \le \frac{\sqrt{R_1^2 - \rho^2}}{\rho}, \tag{44}$$

$$Z(\rho, C_\perp, \zeta) = -\frac{\rho\zeta - \sqrt{R_1^2 - \rho^2(1 - \zeta^2)}}{2C_\perp}, \quad \frac{\sqrt{R_1^2 - \rho^2}}{\rho} < \zeta \le 1. \tag{45}$$

Substituting (44) and (45) into (2), we come to the following expression for $U_z(\rho)$:

$$U_z(\rho) = \frac{1}{2\sqrt{\pi}}\left(\int\limits_{\frac{\sqrt{R_1^2-\rho^2}}{\rho}}^{1} \frac{\sqrt{R_1^2-\rho^2(1-\zeta^2)}}{\sqrt{1-\zeta^2}}d\zeta - \right.$$
$$\left. - \int\limits_{-1}^{\frac{\sqrt{R_1^2-\rho^2}}{\rho}} \frac{\sqrt{1-\rho^2(1-\zeta^2)}}{\sqrt{1-\zeta^2}}d\zeta\right). \tag{46}$$

In the hydrodynamic mode of the flow ($\delta^{-1} \ll 1$), from the Navier–Stokes equation with a boundary condition of complete adhesion on the cylinders, following [14], we obtain

$$U_z(\rho) = \frac{\delta\left(\rho^2-1\right)}{4}\left(1-\frac{\ln\rho}{\ln R_1}\right). \tag{47}$$

In this case, substituting (47) into (3), we have that

$$J_M = \frac{\delta\left(R_1^2(1-\ln R_1)-1-\ln R_1\right)}{4\ln R_1}. \tag{48}$$

In the mode of the flow with sliding, the tangential mass velocity of the gas is proportional to its normal gradient close to the channel walls [15] and [16]. Following [16], we write the boundary conditions of sliding in the form

$$U_z(R_i) = \frac{\sigma_p}{\delta}\frac{\partial U_z}{\partial \mathbf{n}_{R_i}}(R_i),\ i=1,2, \tag{49}$$

where σ_p is a dimentionless coefficient of viscous sliding, $\mathbf{n}_{R_i}$ is a unit normal vector to the cylinder with the radius R_i ($i=1,2$), directed towards the gas. For a diffuse scattering, in the framework of the BGK model the coefficient σ_p equals 1.016 [15]. We will search for the solution of the differential equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dU_z(\rho)}{d\rho}\right) = \delta, \quad R_1 \le \rho \le R_2, \tag{50}$$

with boundary conditions (49) in the form

$$U_z(\rho) = U_{H,z}(\rho) + \sigma_p U_{s,z}(\rho). \tag{51}$$

Here $U_{H,z}(\rho)$ is a hydrodynamic solution with the boundary conditions of adhesion (47), and $U_{s,z}(\rho)$ is a small complement of order δ^0, conditioned by the sliding of the gas on the boundary. As a result, to determine the correction for sliding $U_{s,z}(\rho)$, we obtain a Dirichlet problem for the Laplace's equation in the ring

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dU_{s,z}(\rho)}{d\rho}\right) = 0, \tag{52}$$

$$U_{s,z}(R_i) = \frac{1}{\delta}\frac{\partial U_{H,z}}{\partial \mathbf{n}_{R_i}}(R_i),\ i=1,2. \tag{53}$$

Substituting (47) into (53), we obtain

$$U_{s,z}(R_i) = \frac{(-1)^i \delta}{4} \frac{R_1^2 - 2R_i^2 \ln R_1 - 1}{R_i \ln R_1}, \; i = 1, 2. \tag{54}$$

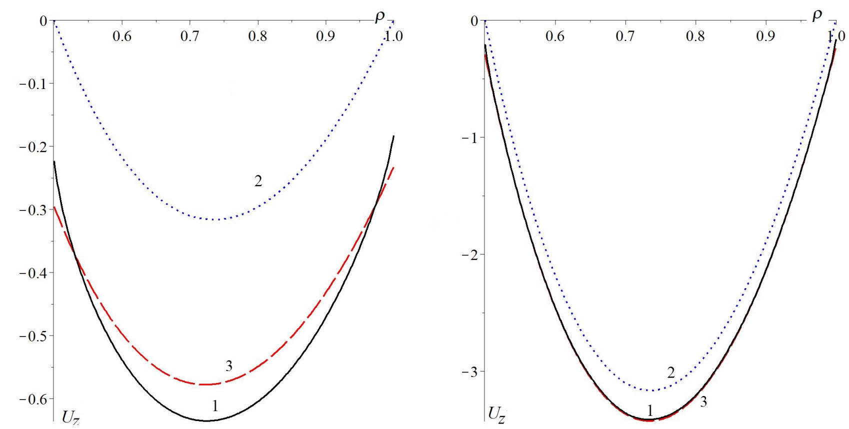
As a result, solving boundary problems (52) and (54), we get

$$U_{s,z}(\rho) = \frac{(1 + 2R_1^2 \ln R_1 - R_1^3 + 2R_1 \ln R_1 - R_1^2 + R_1) \ln \rho}{4R_1 \ln^2 R_1} + \\ + \frac{1}{4} \frac{R_1^2 - 2 \ln R_1 - 1}{\ln R_1}. \tag{55}$$

In the mode of the flow with sliding, the solution of equation (4) does not depend on $C_\perp$ and ζ and is determined by the condition that $Z(\rho, C_\perp, \zeta) = U_z(\rho)$, that is, it represents a composition of functions (47) in (55). Substituting (51) into (3) and taking into account (47) in (55), we obtain

$$J_M = J_{M,H} + \sigma_p J_{M,s} = \frac{\delta \left(R_1^2(1 - \ln R_1) - 1 - \ln R_1\right)}{4 \ln R_1} + \\ + \sigma_p \frac{4R_1^3 \ln^2 R_1 - 4R_1^2 \ln^2 R_1 - 4R_1^3 \ln R_1 + R_1^4 + 4R_1 \ln^2 R_1 + 4R_1 \ln R_1 - 2R_1^2 + 1}{4R_1(R_1 - 1) \ln^2 R_1}. \tag{56}$$

For $R_1 = 0.5$, Figures 1 (a) and 1 (b) show by curves 1, 2, and 3 the distributions of mass velocity U_z of the gas in the channel obtained by formulae (40), (47), and (51), respectively. It can be seen on Figures 1 (a) and (b) that given $\delta \geq 10$, the profile of the value U_z approaches the solution of the NavierвЂ“Stokes equation with boundary conditions of sliding (49). On Figure 2, the curves 1, 2, and 3 represent the distributions of mass velocity of gas in the channel with $R_1 = 0.5$, obtained by formula (40) given $\delta = 0.1$, 1, and 10. The dashed line 4 demonstrates the distribution U_z in a free molecular mode, calculated with the help of (46). It follows from Figure 2 that as the values of δ increase from 0 to 1, the values of U_z decrease in their absolute value to the minimum, that takes place near $\delta = 1$, which is confirmed in [4] and [13]. The results of the calculations of the reduced mass flow by formula (40) depending on δ for $R_1 = 0.5$ are shown on Figures 3 and 4 by the curve 1. The curve 2 on Figure 4 is constructed using cubic spline interpolation of values of the mass flow from [4]. It can be seen that the results of our work agree well with the results of work [4]. The difference increases as the values of δ decrease, achieving its maximum 2% given $\delta = 0.01$. Starting from $\delta = 20$, the curve 1 on Figure 3 almost coincides with the curve 3, constructed by formula (56). Moreover, the use of the boundary condition of adhesion is not satisfactory for the values of $\delta < 100$ (curve 2 on Figure 3). The point M on Figure 4 shows the free molecular limit that equals $J_M = -0.8658$, obtained by formula (3) with substituting in it the expression (46). To compare the values of the reduced mass flow (41) given distinct relations $R_1 = R_1'/R_2'$ with the results from [4], it is necessary to divide the values obtained by (40) by $2(1 - R_1)$, which is due to the introduction in [4] of a hydrodynamic diameter of the channel $2(R_2' - R_1')$ as a distinctive linear size. The results of test estimations show that the modified collocational method with the use of the properties of Chebyshev polynomial sums [11] and Hadamard's and Kronecker's matrix products, proposed in this paper, provides a good accuracy of calculations for different values of relations of cylinder radii $R_1 = R_1'/R_2'$ in a wide spectrum of values of the rarefaction parameter δ. When choosing the number of knots to be $n_i = 11, 21$ $(i = 1,2)$, and $n_3 = 11$, in equation (34) the calculated values of the reduced mass flow by formula (40) do not differ more than 2% for $0.1 \le \delta \le 100$.



(a) (b)

Figure 1. Distribution of mass velocity U_z of the gas in the channel for $R_1 = 0.5$ given $\delta = 10$ (a) and $\delta = 100$ (b)

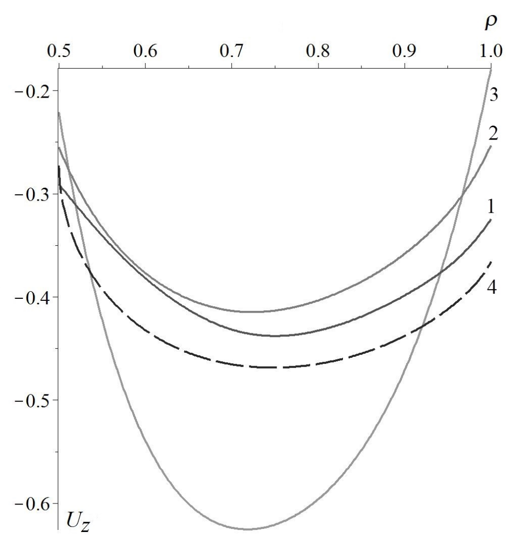


Figure 2. Distribution of mass velocity U_z of the gas in the channel for $R_1 = 0.5$ given different values of δ

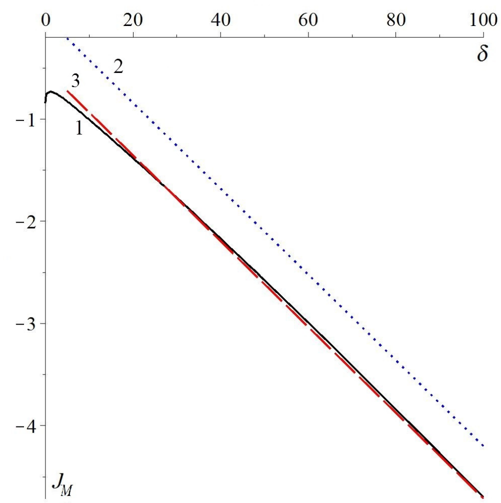


Figure 3. Dependence of the gas mass flow J_M of $0 < \delta \leq 100$ for $R_1 = 0.5$

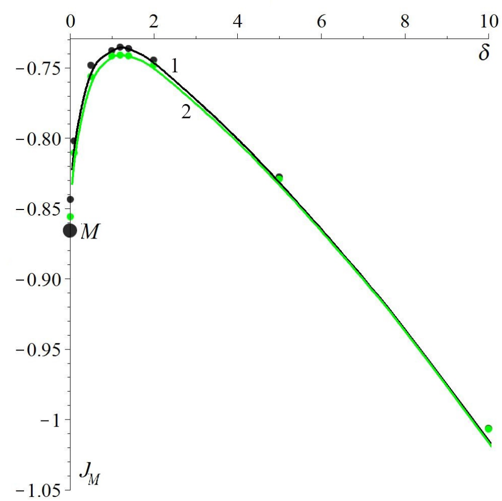


Figure 4. Dependence of the gas mass flow J_M of $0 < \delta \leq 10$ for $R_1 = 0.5$

5. Conclusion

The problem of calculation of the mass flow with the use of the linearized BGK equation given isotermic flow of a rarefied gas in a long concentric annular channel under the action of a constant gradient of pressure is solved with the help of the Chebyshev collocation method. The realization of the collocation method is performed using the properties of sums of Chebyshev polynomials and Hadamard's and Kronecker's matrix products aiming at minimization of influence of rounding errors when calculating the values of the required function, that satisfies the solution of the linearized kinetic equation and the boundary condition at the collocation points. We have obtained the values of the gas mass flow for the flow modes from the free molecular to the hydrodynamic one. It was shown that the application of Chebyshev polynomials when studying the gas flows provides a possibility to effectively obtain the integral characteristics of these flows with the subsequent analysis of those.

References

[1] P. Taheri, H. Struchtrup, *Poiseuille flow of moderately rarefied gases in annular channels*, Int. J. Heat Mass Transfer, **55**:4 (2012), 1291–1303. Zbl 1262.80057

[2] Y. Sone, *Molecular gas dynamics. Theory, techniques, and applications.* Birkhauser, Boston, 2007. Zbl 1144.76001

[3] F. Sharipov, *Rarefied gas dynamics: Fundamentals for research and practice.* Wiley-VCH, Weinheim, 2016.

[4] G. Breyiannis, S. Varoutis, D. Valougeorgis *Rarefied gas flow in concentric annular tube: Estimation of the Poiseuille number and the exact hydraulic diameter*, Eur. J. Mech. B, Fluids, **27**:5 (2008), 609–622. Zbl 1147.76052

[5] M. Hadj-Nacer, D. Maharjan, M. Ho, S.K. Stefanov, I. Graur, M. Greiner, *Continuum and kinetic simulations of heat transfer trough rarefied gas in annular and planar geometries in the slip regime.* J. Heat Transfer **139**:4 (2017), No 042002.

[6] V.A. Titarev, Implicit high-order method for calculating rarefied gas flow in a planar microchannel, J. Comput. Phys., **231**:1 (2012), 109–134. Zbl 1426.76432

[7] O.V. Germider, V.N. Popov, *An application of the Chebyshev polynomials for the calculation of a rarefied gas flow in the cylindrical geometry of the channels*, Sib. Électron Mat. Izv., **16** (2019), 1947–1959. Zbl 1433.35212

[8] O.V. Germider, V.N. Popov, *Nonisothermal rarefied gas flow through a long cylindrical channel under arbitrary pressure and temperature drops*, Fluid Dynamics. **55**:3 (2020), 407–422.

[9] O.V. Germider, V.N. Popov, *A collocation method for linear ellipsoidal model kinetic equation in a circular tube*, J. Phys.: Conf. Ser., **1560** (2020), 1560 012019.

[10] D. Valougeorgis, N. Vasileiadis, V. Titarev, *Validity range of linear kinetic modeling in rarefied pressure driven single gas flows through circular capillaries*, Eur. J. Mech., B, Fluids, **64** (2017), 2–7. Zbl 1408.76464

[11] J. Mason, D. Handscomb, *Chebyshev polynomials*, CRC Press, Boca Raton, 2003. Zbl 1015.33001

[12] P.L. Bhatnagar, E.P. Gross, M.A. Krook, *A model for collision processes in gases. I: Small amplitude processes in charged and neutral one-component systems*, Phys. Rev., II. Ser., **94** (1954), 511–525. Zbl 0055.23609

[13] E.M. Shakhov, *Rarefied gas flow between coaxial cylinders*, Comput. Math. Math. Phys. **43**:7 (2003), 1059–1068. Zbl 1095.76051

[14] L.D. Landau, E.M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford etc., 1987. Zbl 0655.76001

[15] F.M. Sharipov, V.D. Seleznev, *Motion of Rarefied Gases in Channels and Microchannels*, UrO RAN, Yekaterinburg, 2008.

[16] I. Graur, F. Sharipov, *Gas flow through an elliptical tube over the whole range of the gas rarefaction*, Eur. J. Mech., P', Fluids, **27**:3 (2008), 335–345. Zbl 1154.76376

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