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MODERATE DEVIATIONS PRINCIPLE FOR INDEPENDENT RANDOM VARIABLES UNDER SUBLINEAR EXPECTATIONS

Q.Q. ZHOU, A.V. LOGACHOV

Abstract. In this paper, we obtain the moderate deviations principle for a sums of weak independent random variables under sublinear expectations. Unlike known results, we will not require that random variables have the identical distribution.

Keywords: large deviations principle, moderate deviations principle, weak independence, sublinear expectation.

1. Introduction

With the development of economy, people's risk awareness is constantly increasing, and they are more concerned about the events with small probabilities. Large deviations principle (LDP) is a powerful tool to study the rare events with exponential distributions and it is the precision of law of large numbers. Since then LDP is widely used in mathematical statistics, statistical mechanics, quantum mechanics, risk management and so on. For more details of LDP one can refer to Dembo and Zeitouni [5] and, Deuschel and Stroock [6].

However, the classical LDP is confined to the linear case, such as linear expectation and linear probability. Motivated by the risk measure in finance and the problem of model uncertainties in statistics, Peng [10, 13, 14] put forward G-expectation, G-normal distribution and other concepts related to G-expectation, and then established the theory of nonlinear expectation and nonlinear probability. Thus it is necessary to investigate the LDP in nonlinear expectation space. The first result about LDP under sublinear expectation is from Hu [9] who obtained the upper

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bound of Cramér's theorem for capacities. After that, Gao and Xu [8] obtained the LDP for independent and identically distributed random variables under sublinear expectations. Chen and Feng [2] established the LDP for a sequence of negatively dependent random variables under sublinear expectations. Other works about LDP under sublinear expectations one can refer to Cao [1], Chen and Xiong [4], Gao and Jiang [7], Tan and Zong [15] and so on.

By the subadditive method, a moderate deviations principle (MDP) associated with Peng's CLT (introduced in Peng [11]) is obtained in Gao and Xu [8]. Chen and Feng [2] established the upper bound of MDP for a sequence of negatively dependent random variables under sublinear expectations. In this paper, we are going to investigate the MDP for a sequence of weak independent but not identically distributed random variables under sublinear expectations.

The rest of the paper is organized as follows. In section 2, we recall the necessary definitions and formulate the main result. And at the end of this section, we give an example. Section 3 is devoted to proving the main result of this paper. Auxiliary results are given in Section 4.

2. Main result, preliminaries

In this section, we first review the definition, related symbols and properties of sublinear expectation. One can refer to Peng [10, 12, 13, 14] for more details of sublinear expectations. Here, the relevant symbols and concepts are from Peng [14].

Let Ω be a given set, denote by $\mathcal{B}(\Omega)$ the $\sigma-$algebra of subsets of the set Ω. We denote by $\mathcal{H}$ a linear space of real−valued measurable functions defined on Ω.

We will assume that any constant $c \in \mathcal{H}$; $\mathbf{I}_A \in \mathcal{H}$ for any $A \in \mathcal{B}(\Omega)$, where $\mathbf{I}_A$ is the indicator of the set A; and if $X \in \mathcal{H}$, then $|X| \in \mathcal{H}$. The space $\mathcal{H}$ can be considered as the space of random variables.

Definition 1. *A functional $\mathbb{E} : \mathcal{H} \mapsto \mathbb{R}$ is called a sublinear expectation, if it satisfies the following properties: for all $X, Y \in \mathcal{H}$*

(i) monotonicity: $\mathbb{E}X \geq \mathbb{E}Y$, if $X \geq Y$;

(ii) constant preserving: $\mathbb{E}c = c$, for all constants $c \in \mathbb{R}$;

(iii) sub–additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}X + \mathbb{E}Y$;

(iv) positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}X$, for all $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a *sublinear expectation space.*

Given a sublinear expectation $\mathbb{E}$, let us denote the *conjugate expectation* $\mathcal{E}$ of sublinear expectation $\mathbb{E}$ by

$$\mathcal{E}X := -\mathbb{E}[-X] \text{ for all } X \in \mathcal{H}.$$

Obviously, that $\mathcal{E}X \leq \mathbb{E}X$ for all $X \in \mathcal{H}$.

In this paper, we use the following definition of independence of random variables under sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, which is weaker than that in Peng [13, 14].

Definition 2. *(***Independence***) (see* [2, Definition 2.2]*) Let $X_1, \dots, X_n$ be random variables in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, X_n is said to be independent from $(X_1, \dots, X_{n-1})$ under $\mathbb{E}$, if for every nonnegative measurable functions $\varphi_i(\cdot)$*

on $\mathbb{R}$ *with* $\mathbb{E}\varphi_i(X_i) < \infty$, $i = 1, \dots, n$, *we have*

$$\mathbb{E}\Big[\prod_{i=1}^{n} \varphi_i(X_i)\Big] = \mathbb{E}\Big[\prod_{i=1}^{n-1} \varphi_i(X_i)\Big]\mathbb{E}\varphi_n(X_n). \tag{1}$$

$\{X_n\}_{n=1}^{\infty}$ *is said to be a sequence of independent random variables, if for each* $n \in \mathbb{N}$, X_n *is independent from* $(X_1, X_2, \dots, X_{n-1})$.

We will consider a sequence of independent but not identically distributed random variables $X_1, X_2, \dots$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ with

$$\mathbb{E}X_i = \mathcal{E}X_i = 0 \text{ for all } i \in \mathbb{N} \text{ and } \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \sigma_i^2 = \sigma^2 > 0, \tag{2}$$

where $\sigma_i^2 := \mathbb{E}X_i^2$.

In addition, the following Cramér's uniform condition is imposed throughout this paper, i.e.

$$\sup_{i\in\mathbb{N}} \mathbb{E}e^{q|X_i|} < \infty, \text{ for some } q > 0. \tag{3}$$

Suppose that $x(n)$ is a sequence of positive real numbers satisfying

$$\lim_{n\to\infty} \frac{x(n)}{\sqrt{n}} = \infty, \ \lim_{n\to\infty} \frac{x(n)}{n} = 0. \tag{4}$$

We will be interested in the LDP for a sequence of random variables

$$s_n := \frac{S_n}{x(n)}, \text{ where } S_n = \sum_{i=1}^{n} X_i, \ n \in \mathbb{N}. \tag{5}$$

Define an *upper probability* (a *capacity*)

$$\mathbb{V}(A) := \mathbb{E}\mathbf{I}_A, \text{ for } A \in \mathcal{B}(\Omega).$$

Recall the definition of LDP under sublinear expectations with an upper probability $\mathbb{V}$. Denote by $\mathcal{B}(\mathbb{R})$ the Borel $\sigma-$algebra of subsets of the set $\mathbb{R}$.

Definition 3. *A function* $I : \mathbb{R} \to [0, \infty]$ *is called a rate function, if for all* $l \geq 0$ *the set* $\{x : I(x) \leq l\}$ *is a compact subset of* $\mathbb{R}$.

Definition 4. *Let* s_n *be a sequence of random variables in the sublinear expectation space,* $\psi(n)$ *be a sequence of positive real numbers such that* $\lim_{n\to\infty} \psi(n) = \infty$ *and* $I(\cdot)$ *be a rate function on* $\mathbb{R}$*. The sequence* s_n *is said to satisfy LDP in* $\mathbb{R}$ *with rate function* $I(\cdot)$ *and speed* $1/\psi(n)$ *if*
(a) For any closed set $F \in \mathcal{B}(\mathbb{R})$

$$\limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbb{V}(s_n \in F) \leq -\inf_{x\in F} I(x); \tag{6}$$

(b) for any open set $G \in \mathcal{B}(\mathbb{R})$

$$\liminf_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbb{V}(s_n \in G) \geq -\inf_{x\in G} I(x), \tag{7}$$

here we put $I(\varnothing) = \infty$.

Here (6) is referred to as upper bound of large deviations (ULD) and (7) is referred to as lower bound of large deviations (LLD). If (6) and (7) are satisfied with $\psi(n) = x^2(n)/n$ and $s_n = S_n/x(n)$, we say that s_n satisfies the LDP with rate function $I(\cdot)$ and speed $n/x^2(n)$, say also that s_n satisfies MDP.

The following theorem is the main result of this paper.

Theorem 1. *Let conditions (2)–(4) be met. Then the sequence s_n defined in (5) satisfies the MDP in $\mathbb{R}$ with speed $\frac{n}{x^2(n)}$ and rate function*

$$I(y) := \sup_{x\in\mathbb{R}} \left\{ xy - \frac{1}{2}\overline{\Lambda}(x) \right\} = \frac{y^2}{2\sigma^2},\ y \in \mathbb{R}, \tag{8}$$

where $\overline{\Lambda}(x) = x^2 \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=1}^{n} \mathbb{E}X_i^2 = x^2\sigma^2$.

Remark 1. *It can be seen that the result in Theorem 1 extends and complements the MDP in Theorem 3.1 and Corollary 3.2 of Gao and Xu* [8] *with $d = 1$ and $Y_i = 0$.*

Consider an example. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbf{P}_{i,\theta_i^2})$, $i \in \mathbb{N}$, $\theta_i^2 \in (0, 1 + 1/i]$ be a family of probability spaces. Suppose that $X_i \sim N(0, \theta_i^2)$ on the probability space

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbf{P}_{i,\theta_i^2})$$

for any fixed $\theta_i^2 \in (0, 1 + 1/i]$. Then we have for any fixed

$$\theta_1^2 \in (0, 2], \ldots, \theta_n^2 \in (0, 1 + 1/n]$$

that $X_1, X_2, \ldots$ is a sequence of independent random variables with normal distribution on the probability space

$$(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mathbf{P}_{1,\theta_1^2} \times \cdots \times \mathbf{P}_{n,\theta_n^2}) := (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mathbf{P}_{n,\boldsymbol{\theta}_n^2}),\ \boldsymbol{\theta}_n^2 := (\theta_1^2, \ldots, \theta_n^2).$$

Let $\mathcal{H}$ be a linear space of all random variables X such that

$$\sup_{\boldsymbol{\theta}^2} \int_\Omega |X(\omega)| \mathbf{P}_{\boldsymbol{\theta}^2}(d\omega) < \infty,$$

where $\Omega := \mathbb{R}^\infty$, $\boldsymbol{\theta}^2 := (\theta_1^2, \ldots, \theta_n^2, \ldots)$,

$$\mathbf{P}_{\boldsymbol{\theta}^2} := \prod_{i=1}^{\infty} \mathbf{P}_{i,\theta_i^2},\ \theta_i^2 \in (0, 1 + 1/i].$$

Let's define

$$\mathbb{E}X := \sup_{\boldsymbol{\theta}^2} \int_\Omega |X(\omega)| \mathbf{P}_{\boldsymbol{\theta}^2}(d\omega),$$

for $X \in \mathcal{H}$. It's obvious that $(\Omega, \mathcal{H}, \mathbb{E})$ is a sublinear expectation space and $X_1, X_2, \ldots$ is a sequence of independent random variables such that

$$\mathbb{E}X_i = \mathcal{E}X_i = 0,$$

and

$$\sigma_i^2 = \mathbb{E}X_i^2 = \sup_{\theta_i^2 \in (0, 1+1/i]} \theta_i^2 = 1 + 1/i.$$

It is easy to see that

$$\ln n < \sum_{i=1}^{n} \frac{1}{i} \le 1 + \ln n,$$

therefore

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2=\lim_{n\to\infty}\frac{1}{n}\left(n+\sum_{i=1}^{n}\frac{1}{i}\right)=1.$$

For any $q>0$

$$\mathbb{E}e^{q|X_i|}=\sup_{0<|\theta_i|\le\sqrt{1+\frac{1}{i}}}\frac{1}{\sqrt{2\pi}\theta_i}\int_{-\infty}^{+\infty}e^{q|x|}e^{-\frac{x^2}{2\theta_i^2}}\,dx$$

$$=\sup_{0<|\theta_i|\le\sqrt{1+\frac{1}{i}}}\sqrt{\frac{2}{\pi}}\frac{1}{\theta_i}e^{\frac{q^2\theta_i^2}{2}}\int_0^{+\infty}e^{-\frac{(x-q\theta_i^2)^2}{2\theta_i^2}}\,dx$$

$$\le\sup_{0<|\theta_i|\le\sqrt{1+\frac{1}{i}}}e^{\frac{q^2\theta_i^2}{2}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}}\,dx=2\sup_{0<|\theta_i|\le\sqrt{1+\frac{1}{i}}}e^{\frac{q^2\theta_i^2}{2}}<2e^{\frac{q^2(1+1/i)}{2}}\le 2e^{q^2}.$$

Therefore, the hypotheses (2) and (3) of our result are satisfied. Thus the MDP of Theorem 1 is applicable with the rate function

$$I(y)=\frac{y^2}{2}.$$

3. The proof of Theorem 1

In this section we prove the Theorem 1.

1) Let F be any given closed set. If $F=\varnothing$ then the result is obvious. Thus we assume here that $F\neq\varnothing$. Denote

$$y_-:=\sup\{y\in F:y<0\}\le 0,\ \ y_+:=\inf\{y\in F:y\ge 0\}\ge 0,$$

then $F\subseteq(-\infty,y_-]\cup[y_+,+\infty)$. Here we assume that $y_-=-\infty$, if $F\cap(-\infty,0]=\varnothing$ and $y_+=+\infty$, if $F\cap[0,+\infty)=\varnothing$. Therefore, it is easy to see that if $F\neq\varnothing$, then at least one of the values y_- or y_+ is finite. Then, from the above statements we have that

$$\begin{aligned}\ln\mathbb{V}(s_n\in F)&=\ln\mathbb{E}\mathbf{I}_{\{s_n\in F\}}\\&\le\ln\left(\mathbb{E}\mathbf{I}_{\{s_n\in(-\infty,y_-]\}}+\mathbb{E}\mathbf{I}_{\{s_n\in[y_+,+\infty)\}}\right)\\&\le\ln\left(2\max\left(\mathbb{E}\mathbf{I}_{\{s_n\in(-\infty,y_-]\}},\ \mathbb{E}\mathbf{I}_{\{s_n\in[y_+,+\infty)\}}\right)\right).\end{aligned}\tag{9}$$

We first estimate $\mathbb{E}\mathbf{I}_{\{s_n\in(-\infty,y_-]\}}$. The following Chebyshev inequality follows from [3, Proposition 2.1]: let $f(x)$ be a nondecreasing positive function on $\mathbb{R}$, then for any $x\in\mathbb{R}$

$$\mathbb{V}(X\ge x)\le\frac{\mathbb{E}f(X)}{f(x)}.\tag{10}$$

Therefore, for all $\lambda>0$, applying inequality (10) with $f(x)=e^x$ we get

$$\mathbb{E}\mathbf{I}_{\{s_n\in(-\infty,y_-]\}}=\mathbb{E}\mathbf{I}_{\left\{-\lambda\frac{x(n)}{n}S_n\ge-\lambda\frac{x^2(n)}{n}y_-\right\}}\le\frac{\mathbb{E}e^{-\lambda\frac{x(n)}{n}S_n}}{e^{-\lambda\frac{x^2(n)}{n}y_-}}.$$

It follows from Lemma 2 (see section 4) that

$$\mathbb{E}e^{-\lambda\frac{x(n)}{n}S_n}\le e^{\frac{x^2(n)}{n}(\frac{\lambda^2}{2}\sigma^2+o(1))}.$$

Then

$$\mathbb{E}\mathbf{I}_{\{s_n\in(-\infty,y_-]\}}\le e^{\frac{x^2(n)}{n}(\frac{\lambda^2}{2}\sigma^2+\lambda y_-+o(1))}.\tag{11}$$

Choose $\lambda = -\frac{y_-}{\sigma^2}$ in (11) then

$$\mathbb{E}\mathbf{I}_{\{s_n \in (-\infty, y_-]\}} \le e^{-\frac{x^2(n)}{n}(\frac{y_-^2}{2\sigma^2}+o(1))}. \tag{12}$$

By using the similar method we can also prove that

$$\mathbb{E}\mathbf{I}_{\{s_n \in [y_+, +\infty)\}} \le e^{-\frac{x^2(n)}{n}(\frac{y_+^2}{2\sigma^2}+o(1))}. \tag{13}$$

Therefore, it follows from (4), (9), (12) and (13) that

$$\limsup_{n\to+\infty} \frac{n}{x^2(n)} \ln\big(\mathbb{V}(s_n \in F)\big) \le -\min\left(\frac{y_-^2}{2\sigma^2}, \frac{y_+^2}{2\sigma^2}\right).$$

It remains to prove that $\inf\limits_{y\in F} I(y) = \min(\frac{y_-^2}{2\sigma^2}, \frac{y_+^2}{2\sigma^2})$. In fact, it follows from the form of $I(y)$ in (8) such that

$$\inf_{y\in F} I(y) = \min(I(y_-), I(y_+)) = \min\left(\frac{y_-^2}{2\sigma^2}, \frac{y_+^2}{2\sigma^2}\right).$$

Thus, the first inequality (6) in definition 4 is obtained.

2) Now it is left to prove that for any open set G inequality (7) holds with $I(y)$ defined in (8). Let G be any open set. If $G = \varnothing$, the result is obvious. Therefore, we will assume that $G \neq \varnothing$. It is easy to see that for any $l \ge 0$ the set

$$K_l := \{y : I(y) \le l\}$$

is compact. Since $G \neq \varnothing$, there exists $l_G > 0$ such that $G \cap K_{l_G} \neq \varnothing$.

Since G is an open set, for any $\varepsilon > 0$ there exists $y \in G \cap K_{l_G}$ such that

$$\inf_{x\in G} I(x) \ge I(y) - \varepsilon. \tag{14}$$

For any $\delta > 0$ denote

$$y^{(\delta)} := (y - \delta, y + \delta).$$

For any $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$, define

$$A(\lambda, n) := \ln \mathbb{E} e^{\lambda \frac{x(n)}{n} S_n}.$$

It is easy to see that

$$\mathbf{I}_{\{s_n \in y^{(\delta)}\}} e^{\lambda \frac{x(n)}{n} S_n} = \mathbf{I}_{\{s_n \in y^{(\delta)}\}} e^{\lambda \frac{x^2(n)}{n} s_n} \ge \mathbf{I}_{\{s_n \in y^{(\delta)}\}} e^{\frac{x^2(n)}{n}(\lambda y - |\lambda|\delta)}. \tag{15}$$

Using the inequalities (15) and $\mathbb{E}[X-Y] \ge \mathbb{E}X - \mathbb{E}Y$, for any $\lambda \in \mathbb{R}$ and sufficiently small $\delta > 0$ we have that

$$\begin{aligned}
\ln \mathbb{V}(s_n \in G) &\ge \ln \mathbb{V}(s_n \in y^{(\delta)}) = \ln \mathbb{E}\mathbf{I}_{\{s_n \in y^{(\delta)}\}} \\
&\ge \ln \mathbb{E}\left(\mathbf{I}_{\{s_n \in y^{(\delta)}\}} e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)} e^{-\lambda \frac{x^2(n)}{n} y + A(\lambda,n)} e^{-|\lambda| \frac{x^2(n)}{n}\delta}\right) \\
&= -\lambda \frac{x^2(n)}{n} y + A(\lambda, n) - |\lambda| \frac{x^2(n)}{n}\delta + \ln \mathbb{E}\left((1 - \mathbf{I}_{\{s_n \notin y^{(\delta)}\}}) e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)}\right)
\end{aligned}$$

$$\ge -\lambda \frac{x^2(n)}{n} y + A(\lambda, n) - |\lambda| \frac{x^2(n)}{n}\delta + \ln\left(1 - \mathbb{E}\mathbf{I}_{\{s_n \notin y^{(\delta)}\}} e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)}\right). \tag{16}$$

For all $r > 0$ it can be seen that

$$\begin{aligned}
\mathbb{E}\mathbf{I}_{\{s_n \notin y^{(\delta)}\}} e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)} &\leq \mathbb{E}\mathbf{I}_{\{s_n \geq y+\delta\}} e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)} + \mathbb{E}\mathbf{I}_{\{s_n \leq y-\delta\}} e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)} \\
&\leq \frac{\mathbb{E} e^{\frac{rx(n)}{n\sigma^2} S_n} e^{\lambda \frac{x(n)}{n} S_n}}{e^{\frac{rx^2(n)(y+\delta)}{\sigma^2 n}} e^{A(\lambda,n)}} + \frac{\mathbb{E} e^{-\frac{rx(n)}{n\sigma^2} S_n} e^{\lambda \frac{x(n)}{n} S_n}}{e^{\frac{rx^2(n)(\delta-y)}{\sigma^2 n}} e^{A(\lambda,n)}} \\
&= \frac{\mathbb{E} e^{(\frac{r}{\sigma^2}+\lambda) \frac{x(n)}{n} S_n}}{e^{\frac{rx^2(n)(y+\delta)}{\sigma^2 n}} e^{A(\lambda,n)}} + \frac{\mathbb{E} e^{(\lambda-\frac{r}{\sigma^2}) \frac{x(n)}{n} S_n}}{e^{\frac{rx^2(n)(\delta-y)}{\sigma^2 n}} e^{A(\lambda,n)}} \\
&= I_1 + I_2.
\end{aligned}$$

Next, we are going to estimate I_1 and I_2, respectively.

It follows from Lemma 2 (see section 4) that

$$\begin{aligned}
I_1 &\leq \frac{e^{\frac{x^2(n)}{n}\left(\frac{(r/\sigma^2+\lambda)^2}{2}\sigma^2 + o(1)\right)}}{e^{\frac{x^2(n)}{n}\left(\frac{ry+r\delta}{\sigma^2} + \frac{\lambda^2}{2}\sigma^2\right)}} \\
&= \exp\left\{\frac{x^2(n)}{n}\left(\frac{r^2}{2\sigma^2} - \frac{r\delta}{\sigma^2} + r\left(\lambda - \frac{y}{\sigma^2}\right) + o(1)\right)\right\}.
\end{aligned}$$

Choose $r = \delta$ then

$$I_1 \leq \exp\left\{\frac{x^2(n)}{n}\left(-\frac{\delta^2}{2\sigma^2} + \delta\left(\lambda - \frac{y}{\sigma^2}\right) + o(1)\right)\right\}.$$

By employing the same method, we can also prove that

$$I_2 \leq \exp\left\{\frac{x^2(n)}{n}\left(-\frac{\delta^2}{2\sigma^2} + \delta\left(\frac{y}{\sigma^2} - \lambda\right) + o(1)\right)\right\}.$$

Thus,

$$\mathbb{E}\mathbf{I}_{\{s_n \notin y^{(\delta)}\}} e^{\lambda \frac{x(n)}{n} S_n - A(\lambda,n)} \leq 2\exp\left\{\frac{x^2(n)}{n}\left(-\frac{\delta^2}{2\sigma^2} + \delta\left|\frac{y}{\sigma^2} - \lambda\right| + o(1)\right)\right\}. \tag{17}$$

Therefore, for $\lambda = \frac{y}{\sigma^2}$ from (16), (17) and Lemma 2 (see section 4) we have that

$$\liminf_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{V}(s_n \in G) \geq -\frac{y^2}{2\sigma^2} - \frac{|y|}{\sigma^2}\delta. \tag{18}$$

Since $y \in K_{l_G}$ and the set K_{l_G} does not depend on δ, let $\delta \to 0$ on both sides of (18), we obtain that

$$\liminf_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{V}(s_n \in G) \geq -\frac{y^2}{2\sigma^2}. \tag{19}$$

Using (14) we have for any $\varepsilon > 0$

$$\liminf_{n\to\infty} \frac{n}{x^2(n)} \ln \mathbb{V}(s_n \in G) \geq -\frac{y^2}{2\sigma^2} = -I(y) \geq -\inf_{x\in G} I(x) - \varepsilon. \tag{20}$$

Therefore, the proof is derived from (20) and $\varepsilon \to 0$. □

4. Auxiliary results

Let's prove several auxiliary results. For any $\lambda \in \mathbb{R}$ and all $i \in \mathbb{N}$, denote

$$\varphi_{i,n}(\lambda) := \mathbb{E}e^{\lambda \frac{x(n)}{n} X_i}.$$

It follows from conditions (4) and (3) that $\varphi_{i,n}(\lambda) < \infty$ exists with sufficiently large n.

Lemma 1. *For each $\varphi_{i,n}(\lambda)$ the following inequalities*

$$\varphi_{i,n}(\lambda) \le 1 + \frac{\lambda^2 x^2(n)}{2n^2}\sigma_i^2 + f(n,\lambda), \tag{21}$$

$$\varphi_{i,n}(\lambda) \ge 1 + \frac{\lambda^2 x^2(n)}{2n^2}\sigma_i^2 - f(n,\lambda), \tag{22}$$

hold, where $f(n,\lambda) = o\big(\frac{x^2(n)}{n^2}\big)$ when $n \to \infty$.

Proof. We first prove inequality (21). It is easy to see that for any $u \in \mathbb{R}$

$$\frac{|u|^3}{3!} \le e^{|u|}. \tag{23}$$

Taking Taylor expansion of e^x at $x := \lambda \frac{x(n)}{n} X_i = 0$ and using the inequality (23), for any $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}
\varphi_{i,n}(\lambda) &\le \mathbb{E}\Big[1 + \frac{\lambda x(n)}{n}X_i + \frac{\lambda^2 x^2(n)}{2n^2}X_i^2 + \sum_{k=3}^{\infty}\frac{|\lambda X_i|^k x^k(n)}{n^k k!}\Big] \\
&\le \mathbb{E}\Big[1 + \frac{\lambda x(n)}{n}X_i + \frac{\lambda^2 x^2(n)}{2n^2}X_i^2 + \frac{|\lambda X_i|^3 x^3(n)}{3! n^3} e^{\frac{x(n)}{n}|\lambda X_i|}\Big] \\
&= \mathbb{E}\Big[1 + \frac{\lambda x(n)}{n}X_i + \frac{\lambda^2 x^2(n)}{2n^2}X_i^2 + \frac{8|\lambda|^3 x^3(n)}{q^3 n^3}\frac{|qX_i/2|^3}{3!} e^{\frac{x(n)}{n}|\lambda X_i|}\Big] \\
&\le \mathbb{E}\Big[1 + \frac{\lambda x(n)}{n}X_i + \frac{\lambda^2 x^2(n)}{2n^2}X_i^2 + \frac{8|\lambda|^3 x^3(n)}{q^3 n^3} e^{\left(|\lambda|\frac{x(n)}{n} + \frac{1}{2}q\right)|X_i|}\Big],
\end{aligned} \tag{24}$$

where q satisfies (3).

From condition (4) for any $\lambda \in \mathbb{R}$ when n is large enough we have $\lambda\frac{x(n)}{n} \le q/2$. Thus, by using the sub-additive property of sublinear expectation, condition (2) and inequality (24) we can obtain that

$$\begin{aligned}
\varphi_{i,n}(\lambda) &\le 1 + \mathbb{E}\frac{\lambda x(n)}{n}X_i + \frac{\lambda^2 x^2(n)}{2n^2}\mathbb{E}X_i^2 + \frac{8|\lambda|^3 x^3(n)}{q^3 n^3}\mathbb{E}e^{q|X_i|} \\
&= 1 + \frac{\lambda^2 x^2(n)}{2n^2}\sigma_i^2 + \frac{8|\lambda|^3 x^3(n)}{q^3 n^3}\mathbb{E}e^{q|X_i|}.
\end{aligned}$$

Furthermore, condition (3) implies that there exists a constant $C > 0$ such that for all $i \in \mathbb{N}$

$$\mathbb{E}e^{q|X_i|} \le C. \tag{25}$$

Thus (21) is proved.

Next, it remains to prove (22). Note that from (*ii*) and (*iii*) (see Definition 1) the following properties are derived

$$\mathbb{E}X - \mathbb{E}Y \le \mathbb{E}[X - Y], \quad \mathbb{E}[X + c] = \mathbb{E}X + c. \tag{26}$$

Therefore from the inequality (23) and property (26), for any $\lambda \in \mathbb{R}$ we get

$$\begin{aligned}
\varphi_{i,n}(\lambda) &\geq \mathbb{E}\Big[1 + \frac{\lambda x(n)}{n} X_i + \frac{\lambda^2 x^2(n)}{2n^2} X_i^2 - \sum_{k=3}^{\infty} \frac{|\lambda X_i|^k x^k(n)}{n^k k!}\Big] \\
&\geq \mathbb{E}\Big[1 + \frac{\lambda^2 x^2(n)}{2n^2} X_i^2 - \Big(\frac{\lambda x(n)}{n}(-X_i) + \frac{|\lambda X_i|^3 x^3(n)}{3! n^3} e^{\frac{x(n)}{n}|\lambda X_i|}\Big)\Big] \\
&\geq 1 + \mathbb{E}\frac{\lambda^2 x^2(n)}{2n^2} X_i^2 - \mathbb{E}\Big[\frac{\lambda x(n)}{n}(-X_i) + \frac{8|\lambda|^3 x^3(n)}{q^3 n^3} e^{\left(|\lambda|\frac{x(n)}{n} + \frac{1}{2}q\right)|X_i|}\Big] \\
&\geq 1 + \frac{\lambda^2 x^2(n)}{2n^2}\sigma_i^2 - \frac{8|\lambda|^3 x^3(n)}{q^3 n^3} \mathbb{E} e^{\left(|\lambda|\frac{x(n)}{n} + \frac{1}{2}q\right)|X_i|}, \qquad (27)
\end{aligned}$$

where q is the constant which satisfies condition (3).

Using (4) and (25) again, for large enough n we have

$$\varphi_{i,n}(\lambda) \geq 1 + \frac{\lambda^2 x^2(n)}{2n^2}\sigma_i^2 - \frac{8|\lambda|^3 x^3(n)}{q^3 n^3} C,$$

where C is a constant. Thus all the inequalities are proved. □

Lemma 2. *For all $\lambda \in \mathbb{R}$ the equality*

$$\mathbb{E} e^{\lambda \frac{x(n)}{n} S_n} = e^{\frac{x^2(n)}{n}\left(\frac{\lambda^2}{2}\sigma^2 + o(1)\right)}$$

is true when $n \to \infty$.

Proof. It is easy to see that for any $u \in \mathbb{R}$

$$\frac{u^2}{2} \leq e^{|u|}.$$

Therefore, for all $i \in \mathbb{N}$, from condition (3) there exist constants q and C such that

$$\sigma_i^2 = \mathbb{E} X_i^2 = \frac{2}{q^2} \mathbb{E} \frac{(q X_i)^2}{2} \leq \frac{2}{q^2} \mathbb{E} e^{q|X_i|} \leq \frac{2}{q^2} C. \qquad (28)$$

It is well known that

$$\ln(1 + u) = u + o(u), \qquad (29)$$

when $u \to 0$. And since for any $n \geq 2$ random variable X_n does not depend on $(X_1, \ldots, X_{n-1})$, then when $n \to +\infty$ from Lemma 1, the definition of independence in (1), condition (2) and the above statements we have

$$\mathbb{E} e^{\lambda \frac{x(n)}{n} S_n} = \prod_{i=1}^{n} \varphi_{i,n}(\lambda) = \prod_{i=1}^{n} \left(1 + \frac{\lambda^2 x^2(n)}{2n^2}\sigma_i^2 + o\left(\frac{x^2(n)}{n^2}\right)\right)$$

$$= \exp\left\{\sum_{i=1}^{n} \ln\left(1 + \frac{\lambda^2 x^2(n)}{2n^2}\sigma_i^2 + o\left(\frac{x^2(n)}{n^2}\right)\right)\right\} = \exp\left\{\sum_{i=1}^{n}\left(\frac{\lambda^2 x^2(n)}{2n^2}\sigma_i^2 + o\left(\frac{x^2(n)}{n^2}\right)\right)\right\}$$

$$= \exp\left\{\frac{x^2(n)}{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\lambda^2 \sigma_i^2}{2} + o(1)\right)\right)\right\} = e^{\frac{x^2(n)}{n}\left(\frac{\lambda^2}{2}\sigma^2 + o(1)\right)}.$$

□

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Qian Qian Zhou
School of Sciences,
Nanjing University of Posts and Telecommunications,
Nanjing, 210023, China
Email address: qianqzhou@yeah.net

Logachov Artem
Lab. of Probability Theory and Math. Statistics,
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Dep. of High Math.,
Siberian State University of Geosystems and Technologies,
10, Plahotnogo str.,
Novosibirsk, 630108, Russia
Dep. of Computer Science in Economics,
Novosibirsk State Technical University
20, pr. K. Marks ave.,
Novosibirsk, 630073, Russia
Email address: omboldovskaya@mail.ru