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AN EXAMPLE OF A SIMPLE DOUBLE LIE ALGEBRA

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Abstract. We extend the correspondence between double Lie algebras and skew-symmetric Rota—Baxter operators of weight 0 on the matrix algebra to the infinite-dimensional case. We give the first example of a simple double Lie algebra.

Keywords: double Lie algebra, Rota—Baxter operator.

1. Introduction

In 2008 [1], M. Van den Bergh introduced the notion of a double Poisson algebra developing noncommutative geometry. For this, he followed the Kontsevich—Rosenberg principle saying that a structure on an associative algebra has geometric meaning if it induces standard geometric structures on its representation spaces.

Given a finitely generated associative algebra A and $n \in \mathbb{N}$, consider the representation space $\mathrm{Rep}_n(A) = \mathrm{Hom}(A, M_n(F))$, where F denotes the ground field. To equip A with a structure such that $\mathrm{Rep}_n(A)$ is a Poisson variety for every n, M. Van den Bergh defined a double bracket $\{\!\{\cdot,\cdot\}\!\}\colon A\otimes A \to A\otimes A$ satisfying the analogues of anti-commutativity, Jacobi identity, and Leibniz rule. An associative algebra equipped with such a double bracket is called a double Poisson algebra. One of the crucial examples of such structure is a double Poisson algebra defined on a quiver algebra.

Double Poisson algebras are deeply connected with H_0-Poisson structures [2], pre-Calabi—Yau algebras [3], vertex algebras [4].

The notion of a double Lie algebra naturally appeared from the very definition of double Poisson algebra, it is a vector space endowed with a double bracket satisfying above mentioned anti-commutativity and Jacobi identity. Every double

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Lie algebra structure defined on a vector space V can be uniquely extended to a double Poisson algebra structure on the free associative algebra $\mathrm{As}\langle V\rangle$. Thereby, A. Odesskii, V. Rubtsov, V. Sokolov extended [5] linear and quadratic double Lie algebras defined on an n-dimensional vector space to double Poisson algebras defined on the free n-generated associative algebra.

In [6], M. Goncharov and P. Kolesnikov proved that there are no simple finite-dimensional double Lie algebras. This problem was stated by V. Kac during the conference "Lie and Jordan algebras, their representations and applications" dedicated to Efim Zelmanov's 60th birthday (Bento Gonçalves, Brasil, 2015). After this work the natural question about constructing simple infinite-dimensional double Lie algebras has arisen.

It is known that the structure of a double Lie algebra on a finite-dimensional vector space V is equivalent to a skew-symmetric Rota—Baxter operator of weight 0 on the matrix algebra $M_n(F)$, where $n = \dim(V)$ [5, 6, 7]. Recall that a linear operator R defined on an algebra A is called a Rota—Baxter operator (RB-operator, for short) of weight λ, if

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$

for all $x, y \in A$. This notion for the first time appeared in the article [8] of F. Tricomi in 1951 and further was several times [9, 10] rediscovered, see the monograph [11]. Let us mention the bijection [12, 13, 7] between RB-operators of weight 0 on the matrix algebra $M_n(F)$ and solutions of the associative Yang—Baxter equation (AYBE) on $M_n(F)$ [14, 15, 16].

We generalize this correspondence between double Lie algebras and skew-symmetric Rota—Baxter operators for the infinite-dimensional case. We state such correspondence for a countable-dimensional double Lie algebra V and a Rota—Baxter operator acting from the space of matrices with finite numbers of nonzero elements to $\mathrm{End}(V)$ and satisfying some additional finiteness conditions. This correspondence allows us to construct new double Lie algebras. In particular, we show that the vector space $F[t]$ endowed with the double bracket

$$\{\!\{t^n, t^m\}\!\} = -\frac{(t^n \otimes t^m - t^m \otimes t^n)}{t \otimes 1 - 1 \otimes t}$$

is a simple double Lie algebra. As far as we know it is the first example of a simple double Lie algebra.

In terms of RB-operators we interpret the amazing double Lie algebra of V. Kac (see [6]) whose definition is very close to the definition of the Yangian $Y(gl_N)$.

2. Preliminaries

2.1. **Rota—Baxter operators.**

Definition 1. *A linear operator R defined on a (not necessary associative) algebra A is called a Rota—Baxter operator (RB-operator, for short) of weight $\lambda \in F$, if*

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy) \tag{1}$$

holds for all $x, y \in A$.

Proposition 1 ([13]). *Let A be an algebra, let R be an RB-operator of weight λ on A, and let ψ be either automorphism or antiautomorphism of A. Then the operator $R^{(\psi)} = \psi^{-1}R\psi$ is an RB-operator of weight λ on A.*

As an application of Proposition 1, we will use the conjugation with transpose of an RB-operator defined on the matrix algebra.

The following definition has appeared by the name of relative Rota—Baxter operator or $\mathcal{O}$-operator [17] or as generalized RB-operator in the case of zero weight [18]. For simplicity, we also call it Rota—Baxter operator.

Definition 2. *Let A be an algebra and I be an ideal of A. A linear operator $R\colon I \to A$ is called a Rota—Baxter operator of weight λ, if*

$$R(i)R(j) = R(R(i)j + iR(j) + \lambda ij)$$

holds for all $i, j \in I$.

When $I = A$, this definition coincides with Definition 1.

The next statement follows immediately.

Proposition 2. *Let A be an algebra and let J be an ideal of A. Given an RB-operator $P\colon J \to A$ of weight λ and an algebra B, the operator $Q = P \otimes id_B$ is again an RB-operator of the same weight λ from $I \otimes B$ to $A \otimes B$.*

Henceforth, we consider only Rota—Baxter operators of weight 0. It is well-known that, given an RB-operator R of weight 0 and $\alpha \in F$, the operator αR is again an RB-operator R of weight 0.

2.2. **Double Lie algebras.** Let V be a linear space. Given $u \in V^{\otimes n}$ and $\sigma \in S_n$, u^σ denotes the permutation of tensor factors. By a double bracket on V we call a linear map from $V \otimes V$ to $V \otimes V$. Given an associative algebra A, we consider the outer bimodule action of A on $A \otimes A$: $b(a \otimes a')c = (ba) \otimes (a'c)$.

Definition 3 ([1])**.** *A double Poisson algebra is an associative algebra A equipped with a double bracket satisfying the following identities for all $a, b, c \in A$*

$$\{\!\{a, b\}\!\} = -\{\!\{b, a\}\!\}^{(12)}, \tag{2}$$

$$\{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L - \{\!\{b, \{\!\{a, c\}\!\}\}\!\}_R^{(12)} = \{\!\{\{\!\{a, b\}\!\}, c\}\!\}_L, \tag{3}$$

$$\{\!\{a, bc\}\!\} = \{\!\{a, b\}\!\}c + b\{\!\{a, c\}\!\}, \tag{4}$$

where $\{\!\{a, b \otimes c\}\!\}_L = \{\!\{a, b\}\!\} \otimes c$, $\{\!\{a, b \otimes c\}\!\}_R = (b \otimes \{\!\{a, c\}\!\})^{(12)}$, and $\{\!\{a \otimes b, c\}\!\}_L = (\{\!\{a, c\}\!\} \otimes b)^{(23)}$.

Definition 4 ([4, 5, 7])**.** *A double Lie algebra is a linear space V equipped with a double bracket satisfying the identities* (2) *and* (3)*.*

Due to [6], an *ideal* of a double Lie algebra V is a subspace $I \subseteq V$ such that

$$\{\!\{V, I\}\!\} + \{\!\{I, V\}\!\} \subseteq I \otimes V + V \otimes I.$$

Given an ideal I of a double Lie algebra V, we have a natural structure of a double Lie algebra on the quotient space V/I, i. e., $\{\!\{x+I, y+I\}\!\} = \{\!\{x, y\}\!\} + I \otimes V + V \otimes I$.

Let us define homomorphisms of double Lie algebras as follows. Let L and L' be double Lie algebras and let $\varphi\colon L \to L'$ be a linear map. Then φ is called a *homomorphism* from L to L' if

$$(\varphi \otimes \varphi)(\{\!\{a, b\}\!\}) = \{\!\{\varphi(a), \varphi(b)\}\!\}$$

holds for all $a, b \in L$. Note that the kernel of any homomorphism from L is an ideal of L.

Definition 5 ([6])**.** *A double Lie algebra V is said to be simple if $\{\!\{V, V\}\!\} \neq (0)$ and there are no nonzero proper ideals in V.*

3. Finite-dimensional double Lie algebras

Suppose that V is a finite-dimensional space. In [6], it was shown that every double Lie algebra structure $\{\{\cdot,\cdot\}\}$ on V is determined by a linear operator $R\colon \mathrm{End}(V) \to \mathrm{End}(V)$, precisely,

$$\{\{a,b\}\} = \sum_{i=1}^{N} e_i(a) \otimes R(e_i^*)(b), \quad a,b \in V, \tag{5}$$

where $e_1,\dots,e_N$ is a linear basis of $\mathrm{End}(V)$, $e_1^*,\dots,e_N^*$ is the corresponding dual basis relative to the trace form.

A linear operator P on $\mathrm{End}(V)$ is called skew-symmetric if $P = -P^*$, where P^* is the conjugate operator on $\mathrm{End}(V)$ relative to the trace form.

Theorem 1 ([6])**.** *Let V be a finite-dimensional vector space with a double bracket $\{\{\cdot,\cdot\}\}$ determined by an operator $R\colon End(V) \to End(V)$ by (5). Then V is a double Lie algebra if and only if R is a skew-symmetric RB-operator of weight 0 on $End(V)$.*

Remark 1. Theorem 1 was stated in [7] in terms of skew-symmetric solutions of the associative Yang—Baxter equation (AYBE). Since there is a one-to-one correspondence between solutions of AYBE and Rota—Baxter operators of weight 0 on the matrix algebra [13], Theorem 1 follows from [7]. Actually, Theorem 1 was also mentioned in [5].

Let us consider several examples of double Lie algebras and corresponding RB-operators. We will use the linear basis e_{ij}, $1 \le i,j \le \dim(V)$, of $\mathrm{End}(V)$. So, we have $e_{ij}^* = e_{ji}$ relative to the trace form.

In the case of a one-dimensional double Lie algebra L, we have by (2) only zero double bracket.

Example 1 [1, 5, 6]. The space $F^2 = Fe_1 \oplus Fe_2$ equipped with a double product $\{\{e_1,e_1\}\} = e_1 \otimes e_2 - e_2 \otimes e_1$ (others are zero) is a double Lie algebra. The corresponding RB-operator on $M_2(F)$ is $R_1(e_{11}) = e_{21}$ and $R_1(e_{12}) = -e_{11}$ (others are zero).

Example 2 [5, 6, 19]. The space F^2 with a double product $\{\{e_1,e_2\}\} = e_1 \otimes e_1 = -\{\{e_2,e_1\}\}$ is again a double Lie algebra. The corresponding RB-operator on $M_2(F)$ is $R_2(e_{11}) = e_{12}$ and $R_2(e_{21}) = -e_{11}$.

The RB-operators R_1 and R_2 are conjugate with the transpose of matrices, i. e., $R_2 = R_1^{(T)}$, where T denotes the transpose. However, the algebraic properties of the double Lie algebras from Examples 1 and 2 are quite different, see [6].

Note that all RB-operators (including skew-symmetric) of weight 0 on $M_2(F)$ were classified by M. Aguiar [20] in 2000 and all skew-symmetric RB-operators of weight 0 on $M_3(\mathbb{C})$ were described by V. V. Sokolov [21] in 2013.

Example 3. Consider the restriction of the double bracket defined in [1, §6.5] on the infinite-dimensional path algebra over a field F arisen from the quiver Q with the vertex set $\{1,2\}$ and the edge set $\{e_1,e_2,a,a^*\}$, where $a = (1,2)$ and $a^* = (2,1)$. We put $L = \mathrm{Span}\{e_1,e_2,a,a^*\}$, and the double bracket on L equals

$$\{\{a,a^*\}\} = e_2 \otimes e_1, \quad \{\{a^*,a\}\} = -e_1 \otimes e_2,$$

all other double brackets are zero. Let us identify $e_3 = a$ and $e_4 = a^*$. By (5) we get the RB-operator R on $M_4(F)$ defined as follows, $R(e_{32}) = e_{14}$, $R(e_{41}) = -e_{23}$.

4. Infinite-dimensional double Lie algebras

Consider a countable-dimensional double Lie algebra $\langle V, \{\!\{\cdot,\cdot\}\!\}\rangle$. We fix a linear basis u_i, $i \in \mathbb{N}$, of V. Define $e_{ij} \in \mathrm{End}(V)$ by the formula $e_{ij}u_k = \delta_{jk}u_i$. Let $\varphi \in \mathrm{End}(V)$, then we may write $\varphi = \sum\limits_{ij} a_{ij}e_{ij}$. We identify φ with an infinite matrix $[\varphi] = (a_{ij})_{i,j\geq 0}$. Since $\varphi \in \mathrm{End}(V)$ is well-defined, there is only a finite number of nonzero elements in every column of the matrix $[\varphi]$, i. e., $a_{ik} = 0$ for almost all i when k is fixed.

Let us define the subalgebra $\mathrm{End}_f(V)$ of $\mathrm{End}(V)$ as follows,

$$\mathrm{End}_f(V) = \{\varphi \in \mathrm{End}(V) \mid \text{ for every } i,\ [\varphi]_{ij} = 0 \text{ for almost all } j\}.$$

Introduce I as an ideal in $\mathrm{End}_f(V)$ linearly spanned by matrix unities e_{ij}.

Let $\varphi \in \mathrm{End}_f(V) = \sum\limits_{i,j} a_{ij}e_{ij}$. We define the symmetric non-degenerate bilinear trace form $\langle\cdot,\cdot\rangle$ on $I \times \mathrm{End}_f(V) \cup \mathrm{End}_f(V) \times I$ as follows,

$$\langle e_{kl}, \varphi\rangle = \langle \varphi, e_{kl}\rangle = \mathrm{tr}(e_{kl}\varphi) = a_{lk}.$$

Moreover, the form is associative, i. e., $\langle a, bc\rangle = \langle ab, c\rangle$, where at least one of a, b, c lies in I and others are from $\mathrm{End}_f(V)$.

Given a double bracket $\{\!\{\cdot,\cdot\}\!\}$ on a space V, we may define a linear operator $R\colon I \to \mathrm{End}(V)$ by the formula

$$\{\!\{a, b\}\!\} = \sum_{i,j\geq 0} e_{ij}(a) \otimes R(e_{ji})(b), \quad a, b \in V. \tag{6}$$

Conversely, given an operator $R\colon I \to \mathrm{End}(V)$, one can define a double bracket on V by (6). Note that the correspondence does not work if $R\colon \mathrm{End}(V) \to \mathrm{End}(V)$.

Moreover, we define a conjugate operator $R^*\colon I \to \mathrm{End}(V)$ as follows,

$$\{\!\{b, a\}\!\}^{(12)} = \sum_{i,j\geq 0} e_{ij}(a) \otimes R^*(e_{ji})(b), \quad a, b \in V. \tag{7}$$

Denote $R(e_{st}) = \sum\limits_{k,l} a^{st}_{kl}e_{kl}$. By (6), a^{st}_{kl} equals the coefficient of $u_t \otimes u_k$ in the double product $\{\!\{u_s, u_l\}\!\}$. Analogously, put $R^*(e_{st}) = \sum\limits_{k,l} b^{st}_{kl}e_{kl}$. Then b^{st}_{kl} equals the coefficient of $u_k \otimes u_t$ in the double product $\{\!\{u_l, u_s\}\!\}$. Hence,

$$\langle R(e_{st}), e_{kl}\rangle = \{\!\{u_s, u_k\}\!\}|_{u_t\otimes u_l} = \langle R^*(e_{kl}), e_{st}\rangle.$$

Generally we have

$$\langle R(x), y\rangle = \langle x, R^*(y)\rangle, \quad x, y \in I. \tag{8}$$

Remark 2. It is not clear how to introduce objects defined above in invariant manner. For example, consider a linear basis u_s, $s \in \mathbb{N}$, of V and a linear map $\psi \in \mathrm{End}_f(V)$ defined as follows, $\psi(u_0) = u_0$ and $\psi(u_s) = 0$, $s > 0$. Let us consider the basis w_s, $s \in \mathbb{N}$, of V, where $w_0 = u_0$ and $w_s = u_0 + u_s$, $s > 0$. Then $\psi(w_s) = w_0$ for all s. Thus, $\psi \notin \mathrm{End}_f(V)$. Hence, the change of the basis does not preserve the condition $R\colon I \to \mathrm{End}_f(V)$.

Theorem 2. *Let V be a countable-dimensional vector space with a fixed linear basis u_i and with a double bracket $\{\!\{\cdot,\cdot\}\!\}$ determined by a linear map $R \in End_f(V)$ by (6). Then V is a double Lie algebra if and only if R is a skew-symmetric RB-operator of weight 0 from I to $End_f(V)$.*

Proof. By (6) and (7), the identity (2) holds if and only if $R = -R^*$.

Define $F_{12} \in \mathrm{End}(V^{\otimes 3})$ by

$$F_{12}(a\otimes b\otimes c) = \{\!\{a, \{\!\{b,c\}\!\}\}\!\}_L = \sum_{i,j} e_j(a)\otimes R(e_j^*)(e_i(b))\otimes R(e_i^*)(c), \quad a,b,c\in V.$$

For $x, y \in I$, we compute applying associativity of the form $\langle \cdot,\cdot\rangle$

$$(9)\quad (\langle x,\cdot\rangle\otimes\langle y,\cdot\rangle\otimes \mathrm{id})F_{12} = \sum_{i,j}\langle x,e_j\rangle\langle y, R(e_j^*)e_i\rangle R(e_i^*)$$
$$= \sum_i \left\langle y, \sum_j \langle x,e_j\rangle R(e_j^*)e_i\right\rangle R(e_i^*) = \sum_i \langle y, R(x)e_i\rangle R(e_i^*)$$
$$= \sum_i \langle yR(x), e_i\rangle R(e_i^*) = R(yR(x)).$$

Analogously, put

$$F_{23}(a\otimes b\otimes c) = \{\!\{b, \{\!\{a,c\}\!\}\}\!\}_R^{(12)} = \sum_{i,j} e_j(a)\otimes e_i(b)\otimes R(e_i^*)(R(e_j^*)(c)),$$
$$G_{12}(a\otimes b\otimes c) = \{\!\{\{\!\{a,b\}\!\},c\}\!\}_L = \sum_{i,j} e_i(e_j(a))\otimes R(e_j^*)(b)\otimes R(e_i^*)(c).$$

Then for $x, y \in I$ we have

$$(\langle x,\cdot\rangle\otimes\langle y,\cdot\rangle\otimes \mathrm{id})F_{23} = R(y)R(x), \quad (\langle x,\cdot\rangle\otimes\langle y,\cdot\rangle\otimes \mathrm{id})G_{12} = R(R^*(y)x).$$

Thus, the identities (2), (3) hold if and only if R is a skew-symmetric RB-operator of weight 0 from I to $\mathrm{End}_f(V)$. □

Remark 3. We restrict R in Theorem 2 as an operator from I to $\mathrm{End}_f(V)$ instead of $\mathrm{End}(V)$, since otherwise the term $R(yR(x))$ in (1) is not well-defined.

Example 4 [6, 1]. The space $V = F[t]$ equipped with

$$\{\!\{t^n, t^m\}\!\} = \frac{(t^n\otimes 1 - 1\otimes t^n)(t^m\otimes 1 - 1\otimes t^m)}{t\otimes 1 - 1\otimes t}$$

is a double Lie algebra L_1.

Compute the operator $R_1\colon I \to \mathrm{End}(V)$ corresponding to the double Lie algebra L_1,

$$(10)\qquad R_1(e_{ij}) = \begin{cases} -(e_{i,j+1} + e_{i+1,j+2} + \ldots), & i > j \\ e_{0,j-i+1} + e_{1,j-i+2} + \ldots + e_{i-1,j}, & i \le j, \end{cases}$$

where the sum is infinite when $i > j$. By the formula and by Theorem 2, R_1 is a skew-symmetric RB-operator from I to $\mathrm{End}_f(V)$.

Let us identify the matrix algebra $M_n(F)$ of order n with $e_{ij} \in I$, $0 \le i, j \le n-1$. Given an operator P from I to $\mathrm{End}_f(V)$, by the projection P_n we mean a linear operator of the space $\mathrm{Span}\{e_{ij} \mid 0 \le i,j \le n-1\}$ acting as follows: $P(e_{ij}) - P_n(e_{ij}) \in \mathrm{Span}\{e_{kl} \mid n \le k \text{ or } n \le l\}$.

One can check that the linear operator $(R_1)_n$ on $M_n(F)$ is an RB-operator of weight 0 on $M_n(F)$ for each n. Moreover, $(((R_1)_n)^{(\psi_n)})^{(T)}$ coincides with the RB-operator from [13, Example 5.15] and it appears in [15, Example 2.3.3] in terms of the solution of associative Yang—Baxter equation. Here ψ_n is the automorphism of $M_n(F)$ defined as follows,

$$\psi(e_{ij}) = e_{n-1-i,n-1-j}.$$

Example 5. Consider $R_2\colon I \to \mathrm{End}_f(V)$ such that

$$R_2(e_{ij}) = \begin{cases} -(e_{i-1,j} + e_{i-2,j-1} + \ldots + e_{i-1-j,0}), & i > j, \\ e_{i,j+1} + e_{i+1,j+2} + \ldots, & i \leq j. \end{cases} \tag{11}$$

We have defined R_2 in such a way that $(R_2)_n = (((R_1)_n)^{(\psi_n)})^{(T)}$. The definition does not guarantee that we necessarily obtain a skew-symmetric RB-operator of weight 0 from I to $\mathrm{End}_f(V)$. Thus, we have to state this property of R_2.

Proposition 3. *The operator R_2 is a skew-symmetric RB-operator of weight 0 from I to $\mathrm{End}_f(V)$.*

Proof. Firstly, we check the identity $R_2(e_{ij})R_2(e_{kl}) = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl}))$ considering different cases of the values of indices.

Case 1: $i > j$, $k > l$. Then

$$\alpha = R_2(e_{ij})R_2(e_{kl}) = (e_{i-1,j} + \ldots + e_{i-1-j,0})(e_{k-1,l} + \ldots + e_{k-1-l,0}),$$

$$\beta = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl})) \\ = -R_2((e_{i-1,j} + \ldots + e_{i-1-j,0})e_{kl} + e_{ij}(e_{k-1,l} + \ldots + e_{k-1-l,0})).$$

Let $k > j$, i.e., $j = k-1-p$ for some $p \geq 0$. Then

$$\beta = -\chi_{j+l+1-k\geq 0}R_2(e_{ij}e_{k-1-p,l-p}) = -\chi_{j+l+1-k\geq 0}R_2(e_{i,j+l+1-k}) \\ = \chi_{j+l+1-k\geq 0}(e_{i-1,j+l+1-k} + \ldots + e_{i-j-l+k-2,0}),$$

since $i > j+l+1-k$. Here $\chi_P = 1$, if P is true, and $\chi_P = 0$, else. Moreover, $\alpha = \chi_{j+l+1-k\geq 0}(e_{i-1,j+l+1-k} + \ldots + e_{i-j-l+k-2,0}) = \beta$.

Let $k \leq j$, then applying the inequality $i > j+l+1-k$, we compute

$$\beta = -R_2(e_{i-1-j+k,k}e_{kl}) = -R_2(e_{i-1-j+k,l}) = e_{i-2-j+k,l} + \ldots + e_{i-2-j+k-l,0}.$$

On the other hand,

$$\alpha = e_{i-2-j+k,l} + e_{i-2-j+k-1,l-1} + \ldots + e_{i-2-j+k-l,0} = \beta.$$

Case 2: $i \leq j$, $k \leq l$. Then

$$\alpha = R_2(e_{ij})R_2(e_{kl}) = (e_{i,j+1} + \ldots)(e_{k,l+1} + \ldots),$$
$$\beta = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl})) = R_2((e_{i,j+1} + \ldots)e_{kl} + e_{ij}(e_{k,l+1} + \ldots)).$$

Let $k > j$, i.e., $j = k-p$ for some $p > 0$. Then $\alpha = e_{i+p-1,l+1} + \ldots = e_{i+k-j-1,l+1} + \ldots$. Also,

$$\beta = R_2(e_{i+p-1,k}e_{kl}) = R_2(e_{i+k-j-1,l}) = e_{i+k-j-1,l+1} + \ldots = \alpha,$$

since $i+k \leq j+l+1$.

Let $j \geq k$, i.e., $j = p+k$ for some $p \geq 0$. Then $\alpha = e_{i,l+p+2} + \ldots = e_{i,j+l+2-k} + \ldots$. Further,

$$\beta = R_2(e_{ij}e_{k+p,l+p+1}) = R_2(e_{i,l+p+1}) = R_2(e_{i,l+j-k+1}) = e_{i,l+j-k+2} + \ldots = \alpha,$$

since $i+k < l+j+2$.

CASE 3: $i > j$, $k \le l$. Then

$$\alpha = R_2(e_{ij})R_2(e_{kl}) = -(e_{i-1,j} + \ldots + e_{i-1-j,0})(e_{k,l+1} + \ldots),$$

$$\beta = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl})) \\ = R_2(-(e_{i-1,j} + \ldots + e_{i-1-j,0})e_{kl} + e_{ij}(e_{k,l+1} + \ldots)).$$

When $k > j$, we get $\alpha = \beta = 0$. Let $j = k + p$ for some $p \ge 0$. We compute $\alpha = -(e_{i-1,l+j-k+1} + \ldots + e_{i-1-j+k,l+1})$. On the other hand, $\beta = R_2(-e_{i-1-j+k,l} + e_{i,l+j-k+1})$. If $i + k \le j + l + 1$, then

$$\beta = -(e_{i-1-j+k,l+1} + \ldots) + (e_{i,l+j-k+2} + \ldots) \\ = -(e_{i-1,l+j-k+1} + \ldots + e_{i-1-j+k,l+1}) = \alpha.$$

Else,

$$\beta = (e_{i-2-j+k,l} + \ldots + e_{i-2-j+k-l,0}) - (e_{i-1,l+j-k+1} + \ldots + e_{i-2-j+k-l,0}) = \alpha.$$

CASE 4: $i \le j$, $k > l$. Then

$$\alpha = R_2(e_{ij})R_2(e_{kl}) = -(e_{i,j+1} + \ldots)(e_{k-1,l} + \ldots + e_{k-1-l,0}),$$

$$\beta = R_2(R_2(e_{ij})e_{kl} + e_{ij}R_2(e_{kl})) = R_2((e_{i,j+1} + \ldots)e_{kl} - e_{ij}(e_{k-1,l} + \ldots + e_{k-1-l,0})).$$

When $j + 1 \ge k$, we have $\alpha = 0 = \beta$.

Let $j + 2 \le k$, i. e., $k = j + 2 + p$ for some $p \ge 0$. Thus, $\alpha = -(e_{i,l-k+j+2} + \ldots + e_{i+k-j-2,l})$. Also, $\beta = R_2(e_{i+k-j-1,l}) - R_2(e_{i,l-k+j+1})$. If $i + k > j + l + 1$, we have

$$\beta = -(e_{i+k-j-2,l} + \ldots + e_{i+k-j-l-2,0}) + (e_{i-1,l-k+j+1} + \ldots + e_{i+k-j-l-2,0}) \\ = -(e_{i+k-j-2,l} + \ldots + e_{i,l-k+j+2}) = \alpha.$$

If $i + k \le j + l + 1$, we have

$$\beta = (e_{i+k-j-1,l+1} + \ldots) - (e_{i,l-k+j+2} + \ldots) = -(e_{i+k-j-2,l} + \ldots + e_{i,l-k+j+2}) = \alpha.$$

Now, we check that R_2 is also a skew-symmetric operator from I to $\mathrm{End}_f(V)$. Therefore, we have to show that $R_2(e_{ij}) + R_2^*(e_{ij}) = 0$ for all $i, j \ge 0$. By the definition,

$$R_2^*(e_{ij}) = \sum_{k,l\ge 0} R(e_{lk})|_{e_{ji}} e_{kl},$$

where $R(e_{lk})|_{e_{ji}} = (R(e_{lk}), e_{ij})$ denotes the e_{ji}-coordinate of $R(e_{lk})$.

CASE 1: $i \le j$. Then $R(e_{lk})|_{e_{ji}}$ is nonzero only when $(l, k) \in \{(j + 1 + p, i + p) \mid p \ge 0\}$. Thus, $R_2^*(e_{ij}) = -(e_{i,j+1} + \ldots) = -R_2(e_{ij})$, as required.

CASE 2: $i > j$. Then $R(e_{lk})|_{e_{ji}}$ is nonzero only for $(l, k) \in \{(j - p, i - 1 - p) \mid p = 0, \ldots, j\}$. Hence, $R_2^*(e_{ij}) = e_{i-1,j} + \ldots + e_{i-1-j,0} = -R_2(e_{ij})$. □

Corollary 1. *We have a double Lie algebra structure L_2 on V defined due to* (6) *by R_2,*

$$\{\{t^n, t^m\}\} = -\frac{(t^n \otimes t^m - t^m \otimes t^n)}{t \otimes 1 - 1 \otimes t}.$$

Now, we prove that the obtained double Lie algebra L_2 is simple. It is the first example of a simple double Lie algebra.

Theorem 3. *The double Lie algebra L_2 is simple.*

Proof. Suppose that J is a nonzero proper ideal in L_2. Define n as the minimal degree in t of elements from J. Let us show that $n=0$. If $n>0$, then consider $f = t^n + \sum\limits_{j=0}^{n-1}\alpha_j t^j \in J$. We have that the product

$$\{\!\{1,f\}\!\} = t^{n-1}\otimes 1 + t^{n-2}\otimes t + \ldots + 1\otimes t^{n-1} + \sum_{j=1}^{n-1}\alpha_j(t^{j-1}\otimes 1 + \ldots + 1\otimes t^{j-1})$$

lies in $V\otimes J + J\otimes V$.

Consider the map $\psi\colon V\otimes V \to V/J\otimes V/J$ acting as follows: $\psi(v\otimes w)=(v+J)\otimes(w+J)$. On the one hand, $1+J, t+J, \ldots, t^{n-1}+J$ are linearly independent elements of V/J. On the other hand, $J\otimes V + V\otimes J = \ker(\psi)$. Thus, $\{\!\{1,f\}\!\}$ is at the same time zero and nonzero element of $V/J\otimes V/J$. We obtain a contradiction. So, $n=0$ and $1\in J$.

Let us prove by induction on $s\geq 0$ that $t^s\in J$. For $s=0$, it is true. Suppose that $s>0$ and we have proved that $t^j\in J$ for all $q<s$. Since $1\in J$, we have

$$\{\!\{1,t^{2s+1}\}\!\} = t^{2s}\otimes 1 + t^{2s-1}\otimes t + \ldots + t^{s+1}\otimes t^{s-1} + t^s\otimes t^s + \ldots + 1\otimes t^{2s} \in V\otimes J + J\otimes V.$$

So, $t^s\otimes t^s \in V\otimes J + J\otimes V$. Hence, $\psi(t^s\otimes t^s)=0$, it means that $t^s\in J$. □

In the next two examples we consider conjugation of R_1 and R_2 with transpose and corresponding double Lie algebras.

Example 6. For $R_3 = R_1^{(T)}$, we get a double Lie algebra L_3 with the double bracket

$$\{\!\{t^n,t^m\}\!\} = \frac{(t^{n+1}\otimes t^{m+1} - t^{m+1}\otimes t^{n+1})}{t\otimes 1 - 1\otimes t}.$$

Example 7 [1]. For $R_4 = R_2^{(T)}$, we get a double Lie algebra L_4 with the double bracket

$$\{\!\{t^n,t^m\}\!\} = -\frac{(t^{n+1}\otimes 1 - 1\otimes t^{n+1})(t^{m+1}\otimes 1 - 1\otimes t^{m+1})}{t\otimes 1 - 1\otimes t}.$$

In [1], it was stated that each homogeneous double Poisson algebra on $F[t]$ up to an equivalence is either L_1 or L_4. It is easy to show that the double Lie algebras L_2 and L_3 do not satisfy (4), for example, since $\{\!\{t,1\}\!\}\neq 0$, and therefore do not define the structure of a double Poisson algebra on $F[t]$. Note the following connections between double brackets in L_1, L_2, L_3, L_4:

$$\{\!\{t^n,t^m\}\!\}_{L_3} = -(t\otimes t)\{\!\{t^n,t^m\}\!\}_{L_2}, \quad \{\!\{t^n,t^m\}\!\}_{L_4} = -\{\!\{t^{n+1},t^{m+1}\}\!\}_{L_1}.$$

Example 8 (V. Kac, see [6]). Consider the double Poisson algebra $dY(N) = F[t]\otimes M_N(F)$. Its double bracket relative to the basis $T_n^{ij} = t^n\otimes e_{ij}$, $n\geq 0$, $i,j=1,\ldots,N$, has the following form:

$$\{\!\{T_m^{ij},T_n^{kl}\}\!\} = \sum_{r=0}^{\min\{m,n\}-1}\big(T_r^{kj}\otimes T_{m+n-r-1}^{il} - T_{m+n-r-1}^{kj}\otimes T_r^{il}\big),$$

the inner bimodule $dY(N)$-action is the associative product.

It is worth mentioning that these relations are similar to the defining relations of the Yangian $Y(gl_N)$:

$$[T_m^{ij}, T_n^{kl}] = \sum_{r=0}^{\min\{m,n\}-1} \left(T_r^{kj} T_{m+n-r-1}^{il} - T_{m+n-r-1}^{kj} T_r^{il}\right).$$

We get an RB-operator $R\colon I \otimes M_N(F) \to \mathrm{End}_f(V) \otimes M_N(F)$ such that the double bracket on $dY(N)$ is defined by (6) with the help of R. We have

$$R(e_{ij} \otimes e_{st}) = \begin{cases} (e_{i,j+1} + e_{i+1,j+2} + \ldots) \otimes e_{st}, & i > j \\ -(e_{0,j-i+1} + e_{1,j-i} + \ldots + e_{i-1,j}) \otimes e_{st}, & i \leq j, \end{cases}$$

where $e_{ij} \in I$, $e_{st} \in M_N(F)$. Actually, $R = (-R_1) \otimes \mathrm{id}$.

Remark 4. We may extend the double Lie algebra structures L_1, L_2, L_3, L_4 on $F[t, t^{-1}]$ and $dY(N)$ on $F[t, t^{-1}] \otimes M_N(F)$ respectively. It is enough to let both sums in the definition of the corresponding RB-operator R_1, R_2, R_3, R_4, and R be infinite. Introduce $e_{ij} \in \mathrm{End}(V)$, where $V = F[t, t^{-1}]$, $i, j \in \mathbb{Z}$, in such a way that $e_{ij} t^k = \delta_{jk} t^i$. For example, let us extend R_2. We define

$$\widetilde{R}_2(e_{ij}) = \begin{cases} -\sum\limits_{p=0}^{\infty} e_{i-1-p,j-p}, & i > j, \\ \sum\limits_{p=0}^{\infty} e_{i+p,j+1+p}, & i \leq j. \end{cases}$$

Analogously to the proof of Proposition 3, one can check that $\widetilde{R}_2$ is a skew-symmetric RB-operator from $\mathrm{Span}\{e_{ij} \mid i, j \in \mathbb{Z}\}$ to $\mathrm{End}_f(V)$. By Theorem 2, we get a double Lie algebra structure $\widetilde{L}_2$ on $F[t, t^{-1}]$. Analogously we get double Lie algebras $\widetilde{L}_1$, $\widetilde{L}_3$, $\widetilde{L}_4$, and $\widetilde{dY}(N)$.

Remark 5. The operator R_2 is injective. Moreover, $I \subset \mathrm{Im}(R_2)$. So, we may define the inverse map $d = R_2^{-1}$ from I to I. Then $d(e_{ij}) = e_{i,j-1} - e_{i+1,j}$ is a derivation of I. By [22], every such derivation is an inner derivation, i. e., of the form $x \to xa - ax$, where $a \in \mathrm{End}(V)$ such that $a^{-1}[U]$ is finite-dimensional for each finite-dimensional subspace $U \subset V$.

It is easy to show that $d(x) = Ax - xA$ for $A = e_{10} + e_{21} + \ldots$. Given a positive integer k, we may define $d_k\colon I \to I$ such that $d_k(x) = A^k x - x A^k$. By d_k, we may define $P_k \in \mathrm{End}'(V)$ as an analogue of the operator R_2. And all of P_k provide by (6) some double Lie algebra structures on $F[t]$.

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