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ON PL EMBEDDINGS OF A 2-SPHERE IN THE 4-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. We prove that there is no 2-convex PL embedding of S^2 in E^4 in the form of a polyhedron, each vertex of which is incident to no more than 5 edges.

Keywords: polyhedron, PL embedding, Euclidean space.

1. Introduction

Recall the notion of a k-convex subset of the Euclidean space (see [1]).

Definition 1. *The subset $C \subset E^n$ of the Euclidean space is said to be k-convex if for every point $x \in E^n \setminus C$, there exists a k-dimensional plane that passes through x and does not interject C.*

Note that the classic definition of a convex set corresponds to the case $k = n-1$. The following question was posed by Zelinsky [1]:

Question 1. *Does there exist an embedding of a 2-sphere S^2 in the 4-dimensional Euclidean space E^4 in the form of a 2-convex subset.*

We will call such embeddings 2-convex embeddings.

Remark 1. *A standard Clifford torus is a 2-convex subset of the Euclidean space E^4 (see [1]).*

In our earlier works, it was shown that there is no 2-convex C^2-smooth embedding of S^2 in E^4 [2]. In this paper, we provide a partial generalisation of this result for a class of partially linear (PL) embeddings, i.e. such embeddings, which have a

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polyhedron in E^4 that is homeomorphic to a 2-sphere as their image. The main contribution can be stated as the following theorem:

Theorem 1. *There is no 2-convex PL embedding of S^2 in E^4 in the form of a polyhedron each vertex of which is incident to no more than 5 edges.*

Despite the fact that the general idea and some elements of the proof are similar to those in the "smooth" case considered in [2], we will nonetheless provide the whole proof for the sake of completeness.

2. Preliminaries

Recall that an $n-1$-segmented polygonal chain $\gamma = A_1A_2\dots A_n$ with vertices $A_i,\ i = 1\dots n$, is a connected series of $n-1$ line segments $[A_i, A_{i+1}],\ i = 1,\dots,n$. If $A_1 = A_n$, a polygonal chain is said to be closed. Also, a polygonal chain is said to be *non-degenerate* if no three consecutive vertices of that chain lie on the same line. A natural ordering of vertices of a polygonal chain defines its orientation, i.e. which segment $[A_i, A_{i+1}]$ corresponds to a vector $\overrightarrow{A_i, A_{i+1}}$.

A *k-subchain* of a polygonal chain γ is a polygonal chain that consists of segments of γ. Every k-chain is defined by a sequence of vertices $A_iA_{i+1}\dots A_{i+k}$ or

$$A_iA_{i-1}\dots A_{i-k}.$$

This order may be cyclic if a polygonal chain is closed. In the first case, the orientation of a k-subchain is induced by the orientation of γ and so we say that such a k-subchain is γ-oriented.

If γ is non-degenerate, then every 2-subchain defines a plane. Moreover, a γ-orientation induces a γ-orientation of a plane in the following way: a 2-subchain ABC defines a basis $\{\overrightarrow{AB}, \overrightarrow{BC}\}$ of a plane, that passes through A, B, C.

By L_γ denote a convex hull of γ. We say that a 2-subchain of a non-degenerate polygonal chain γ is a *supporting* subchain, if it belongs to a face of L_γ.

Lemma 1. *Let $\gamma = ABCD$ be a polygonal chain with the convex hull L_γ, which is a tetrahedron. Then γ-oriented 2-subchains that have a common segment set the γ-orientations of the corresponding faces that are induced by diffetent orientations of ∂L_γ [1].*

Proof. The orientations of ∂L_γ are given by the orientation of all of its faces. This means that induced orientations on a common edge of two neighbouring faces must be differenent. It is therefore obvious that γ-orientations of neighbouring faces are not coherent. □

We will call a vertex of a polygonal chain *a boundary vertex* if it is incident to only one segment of that chain (we will also call that segment a boundary segment). The other segments and vertices will be called *inner*.

Let γ be a non-closed non-degenerate 3-segmented plane polygonal chain without self-intersections with a convex hull L_γ. Then the following configurations of γ are possible:

(1) L_γ is a triangle;
  (a) the triangle contains a boundary vertex of γ (fig.1 a));
  (b) the triangle contains an inner vertex of γ (fig.1 b));

[1]Note, that the boundary ∂L_γ is a PL-sphere, and is orientable.

(2) L_γ is a quadrangle;
 (a) all segments of γ belong to ∂L_γ (fig. 1 c));
 (b) one of the diagonals of L_γ is a segment of γ (fig. 1 d)).

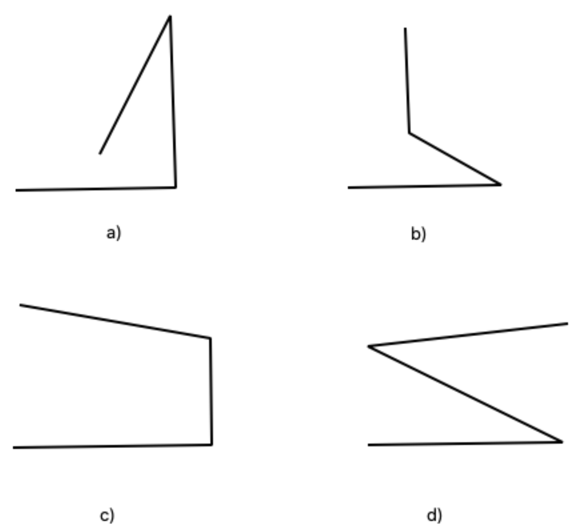

FIG. 1. 3-segmented plane polygonal chain

Noting the fact that the convex hull of 5 points in a 3-dimensional space, whose boundary contains all five points and no four points among them lie in the same plane, is a trigonal bipyramid, we give the following classification of configurations of a non-degenerate embedeed closed polygonal chain γ in a 3-dimensional space that contains no more than five segments.

Classification.

(1) γ is a 3-segmented polygonal chain (in this case it is obvious that it is a plane polygonal chain).
(2) γ is a 4-segmented polygonal chain:
 (a) L_γ is a tetrahedron (a general position case);
 (b) γ is a plane polygonal chain.
(3) γ is a 5-segmented polygonal chain:
 (a) There is no plane that contains 4 vertices of γ (a general position case):
  (i) L_γ is a tetrahedron, with one of the vertices of γ strictly inside of it.
  (ii) L_γ is a trigonal bipyramid. In that case, two possibilites arise:

(A) one of the segments of γ connects two top-vertices of the bipyramidal[2] (fig. 2 b));

(B) all segments of γ are edges of the bipyramid (fig. 2 c)).

(b) γ - is a 5-segmented polygonal chain and at least 4 vertices of γ lie in the same plane (degenerate case):

(i) γ is a plane polygonal chain;

(ii) three segments of γ lie at one plane and define a plane non-close polygonal chain γ'' of one of the configurations described above. (fig. 1), and the only vertex that does not lie at that plane is connected to boundary vertices of γ''.

In the figure 2, one can see closed polygonal chains, which represent various configurations of a 5-segmented space (not plane) polygonal chains.

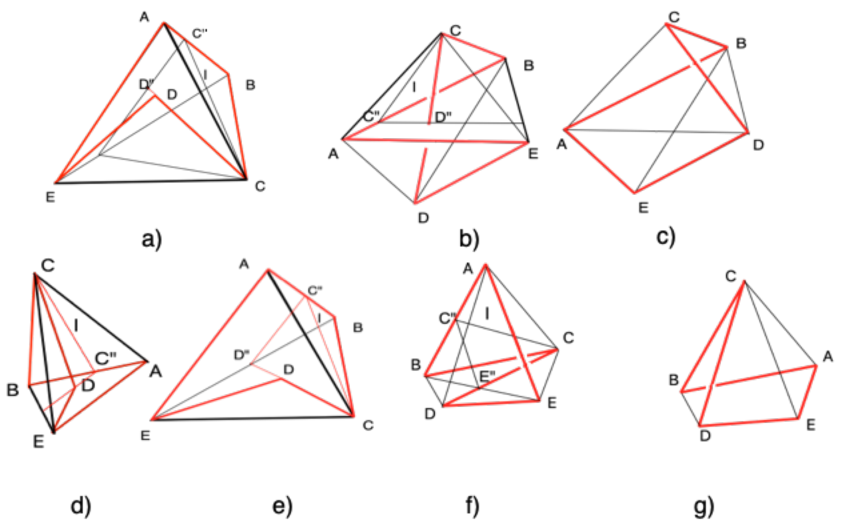

FIG. 2. Configurations of γ

Consider $\gamma = ABCDEA$, a closed 5-segemented space polygonal chain without intersections. By $L_{\gamma_{sym}}$ denote a convex polyhedron with right faces, that is combinatorially equivalent to L_γ. For simplicity we will name its vertices with the same letters that are used for L_γ. For configurations b) and c) (fig. 2) $L_{\gamma_{sym}}$ is a bypiramidal, which is made up of two right tetrahedrons, and for configurations f) and g (fig. 2) it is a half of a right octahedron.

In the case of L_γ being a bypiramidal, combinatorial correspondence of polyhedrons $ABCDEA = L_\gamma \leftrightarrow L_{\gamma_{sym}} = ABCDEA$ defines a polygonal chain $\gamma_{sym} =$

[2]By top-vertices we mean the vertices, that do not lie in the bases of a pyramid and bipyramid. By the base of the bipyramid we mean the common base of two pyramids that make it up. These notions are introduced in order to differentiate the vertices of a pyramid and bipyramid from the vertices of a polyhedron.

$ABCDEA$. If L_γ is a tetrahedon, then one of the vertices γ either lies inside of L_γ or inside of one of the faces of L_γ. In that case, define the corresponding vertex γ_{sym} as a barycenter of $L_{\gamma_{sym}}$ or a barycenter of a respective face of $L_{\gamma_{sym}}$, using the same letter to name it. We will call this vertex a *barycnter vertex*. It is obvious that γ_{sym} and γ have the same configuration.

Consider the following action of a subgroup $G'_\gamma \subset G_\gamma$ of a symmetry group G_γ of a convex hull $L_{\gamma_{sym}}$ on the set of 5-segmented closed space polygonal chains that have the same vertices as γ^3, that leaves the set of vertices γ_{sym} invariant. Then any symmetry $g : L_{\gamma_{sym}} \to L_{\gamma_{sym}},\ g \in G'_\gamma$ induces a permutation of vertices of $L_{\gamma_{sym}}$. If there is a vertex of γ_{sym} that is a barycenter vertex, it is not "moved" by g. Therefore we have a permutation of the set of vertices of a polygonal chain $\gamma_{sym} = ABCDEA$

$$\sigma_g : \{A, B, C, D, E\} \to \{A, B, C, D, E\}.$$

A new polygonal chain

$$g\gamma = \sigma_g(A)\sigma_g(B)\sigma_g(C)\sigma_g(D)\sigma_g(E)\sigma_g(A)$$

is an image of γ under the action of an element g of a symmetry group G_γ.

3. Preliminary results

Proposition 1. *Let γ be a 5-segmented closed non-degenerate space polygonal chain. Then*

(1) *$g\gamma$ has the same configuration as γ and all possible 5-segmented polygonal chains of this configuration which have the same set of vertices can be presented as $g\gamma,\ g \in G'_\gamma$.*
(2) *For $g \in G'_\gamma$, images of supporting 2-subchains of γ are supporting 2-subchains of $g\gamma$.*
(3) *If two segments of γ lie on skew lines then their images at $g\gamma,\ g \in G'_\gamma$ also lie on skew lines.*

Proof. The second statement follows directly from the first one, since the boundary $\partial L_{\gamma_{sym}}$ is invariant under the action G'_γ.

We now consider the configurations of a polygonal chain γ that are presented in figure 2.

In the case a), the first statement follows from the fact that G'_γ contains symmetries that induce a permutation of two vertices of $L_{\gamma_{sym}}$. If we consider any two non-neigbouring vertices of a polygonal chain γ of configuration a), we can say that they lie on skew lines since otherwise we would have that four vertices of γ lie in one plane, which is impossible for this configuration. The third statement follows from the first one and the obvious fact that any symmetry $g \in G'_\gamma$, being a bijection, maps non-neighbouring vertices to non-neighbouring vertices.

In cases b) and c), the group G'_γ fixes or permutes top-vertices of a bypiramid $L_{\gamma_{sym}}$, while leaving its base invariant. From this we can see that the polygonal chain $g\gamma,\ g \in G'_\gamma$, has the same configuration as γ. Moreover, since G'_γ contains the symmetries that permute two given vertices of the base of $L_{\gamma_{sym}}$, while leaving its top-vertices fixed, we can obtain all possible configurations γ for a given case with the same set of vertices, thus proving the first statement. Note that for b) and

[3]Here by a symmetry group of $L_{\gamma_{sym}}$ we understand a subgroup of the isometry group $\mathbb{R}^3$ which leaves the convex hull $L_{\gamma_{sym}}$ invariant.

c) a set of vertices of γ is equal to the set of vertices of a bypiramid L_γ. Using the same reasoning as earlier, we can show that the third statement follows from the first one.

We now consider the cases d) and e). Then the convex hull L_γ is a tetrahedron that can be presented as a pyramid with the base that has 4 vertices of γ. Since the vertex γ_{sym} that lies inside the base of a pyramid should be invariant under the action of G'_γ, we can say that the top-vertex of a pyramid $L_{\gamma_{sym}}$ and its base are also invariant under this action. In that case G'_γ is a symmetry group of a right triangle, which is a base of a pyramid $L_{\gamma_{sym}}$. Since the boundary vertices of γ'' are mapped to boundary vertices of its image $g\gamma''(\ g \in G'_\gamma)$ and the barycenter of a triangle is invariant under the action of G'_γ, we have that $g\gamma''$ has the same configuration as γ'' and the polygonal chain $g\gamma$ has the same configuration as γ. Note that G'_γ contains symmetries, that permute two given vertices of the base of $L_{\gamma_{sym}}$, while other vertices of γ_{sym} remain fixed. This allows us to obtain all possible configurations of γ of this type with the same set of vertices, thus proving the first statement. Now assume that two segments a and b of γ lie on skew lines. This means that one of these segments contains a top vertex of a pyramid L_γ and the other one lies in the base of that pyramid. But since both the vertex and the base of $L_{\gamma_{sym}}$ are invariant under the action of G'_γ, we have that the properties of a and b are also preserved, i.e. one of the images of these segments contains a top-vertex of a pyramid L_γ, while the other one belongs to its base, therefore ga and gb lie on skew lines, hence the third statement holds.

Finally, in cases f) and g) a convex hull L_γ is a pyramid with the quadrangle as its base. The top-vertex of a pyramid $L_{\gamma_{sym}}$ and its base are again invariant under the action of a group G'_γ, since the quadrangle face is unique (other faces are triangles) and therefore should stay invariant under any symmetry $g \in G'_\gamma$. Note that $L_{\gamma_{sym}}$ is a half of a right octahedron. In that case G'_γ is a group of symmetries of a square. Since these symmetries map diagonals to diagonals, we have that the polygonal chain $g\gamma, \ g \in G'_\gamma$, has the same configuration as γ. Since G'_γ contains symmetries that map a given side of a square to any other side of a square and contains a symmetry that permutes the diagonals of a said square while leaving its sides invariant. These symmetries allow us to obtain all possible configurations of γ for the considered cases and with the same set of vertices. Therefore, the first statement holds. The third statement holds by the reasoning, similar to the one used when we considered two previous cases.

□

We will now prove the following proposition

Proposition 2. *Let γ be an embedeed closed non-degenerate space polygonal chain with no more than five segments and lying in a three-dimensional Euclidean space Π. By L_γ denote its convex hull. Then*

(1) *There exists a plane $\tau \subset \Pi$ such that a projection $pr_{\overrightarrow{m}} : \Pi \to \tau$ along a direction $\overrightarrow{m}$ bijectively maps γ onto its image γ' that can be one of the following flat curves:*
   a) *a convex polygonal chain;*
   b) *a 5-segmented non-convex polygonal chain (see fig. 4 d)), which has two parallel segments ($A'E'$ u $C'D'$). In that case one of these segments*

$(C'D')$ forms a unique concave 2-subchain $(B'C'D')$, which forms an angle, that is convex outside of the domain, which is bounded by γ'.

(2) *The polygonal chain γ contains a pair of supporting 2-subchains, which define the γ-orientations, that are induced by different orientations of ∂L_γ, of corresponding faces. The images of these subchains under $pr_{\overrightarrow{m}}$ are non-degenerate 2-subchains which are, in the case that γ' is not convex (see fig. 4 d)) correspond to a subchain that connects the parallel segments $(A'B'C')$ and one of the 2-subchains $(E'A'B'$ or $D'E'A')$ of a 3-subchain that is complement to a convex 2-subchain $(D'E'A'B')$.*

Remark 2. *The subchains in the brackets are corresponding to the figure 4 d). If instead γ has the same configuration as shown in the figure 5 d′), we should pick the subchains accordingly.*

Proof.

Case 1. $\gamma = ABCDA$ *corresponds to the configuration type* $2(a)$ *of our classification.* Consider an affine projection of a tetrahedron L_γ on a plane that is parallel to edges AC and BD that do not belong to γ (see fig. 3). We choose the direction $\overrightarrow{m}$ of a projection as parallel to the segment that connects the middle points of the aforementioned edges.

Under this projection γ will be mapped onto a quadrangle with inresecting diagonals, meaning that the image of γ is a convex polygonal chain. From Lemma 1, it follows that 2-subchains ABC and BCD are the required supporting 2-subchains.

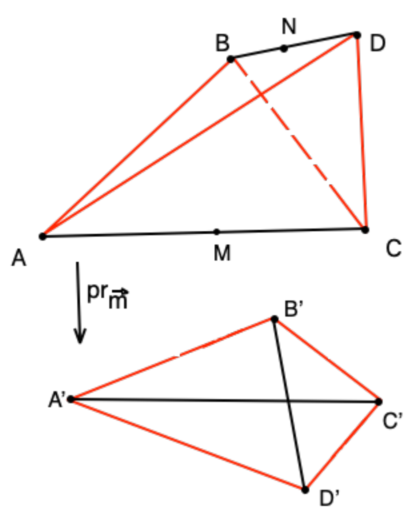

FIG. 3. Projection of a tetrahedron

Case 2. γ *corresponds to the configuration type* $3(a)i$

We will consider the case, presented in figure 2.

Since the vertex D that is located inside of L_γ is connected to two vertices γ, it is clear that the other part of γ is a 3-subchain $(EABC)$ that consists of edges of a tetrahedron L_γ. By Lemma 1, we can find a pair of γ-oriented supporting 2-chains (EAB and ABC) that set the orientations of the corresponding faces L_γ, that are induced by different orientations of ∂L_γ.

Without the loss of generality, consider the subchain ABC. It is connected with segments EA and CD, which lie on skew lines. Continue the segment CD until it intersects the triangle ΔEAB. Denote the intersection point by D''. Now draw a line, passing through D'' that is parallel to EA and by C'" denote its intersection point with AB. From this construction we obtain that the plane that contains the triangle $\Delta CD''C''$ is parallel to the edges EA and CD. Denote by $\overrightarrow{n}$ a normal vector to that plane. Let l be a line, passing through C and C''. Consider an orthogonal projection $pr_{\overrightarrow{v}} : \Pi \to \pi^\perp$ onto a plane $\pi^\perp$ that is orthogonal to l with the direction set by the vector $\overrightarrow{v}$. The image of γ in that case is presented in figure 4 a). From our construction we have that $\overrightarrow{n}$ is parallel to the plane $\pi^\perp$ and is not orthogonal to the plane that contains the triangle ΔABC. For a sufficiently small turn $\phi \in SO(3)$ around the line with the direction set by vector $\overrightarrow{n}$, an orthogonal projection $pr_{\overrightarrow{m}} : \Pi \to \tau$ onto a plane $\tau := \phi(\pi^\perp)$ with the normal vector $\overrightarrow{m}$ defines a bijection of γ to a plane polygonal chain $\gamma' = A'B'C'D'E'A'$ (see fig. 4 d)). The choice of ϕ being a clockwise or a counter-clockwise turn is defined by the necessary condition $\angle E'A'C' < \angle E'A'B'$. In that case we have that $A'B' \cap C'D' = \emptyset$ (see.fig. 4 d)). The absolute value of $|\phi|$ can always be picked to be small enough so that the continuation of a segment $B'C'$ intersected $A'E'$ (see fig. 6). This proves the first statement of our proposition. Additionally, subchains EAB and ABC satisfy the second statement.

Case 3. *γ corresponds to the configuration type* $3(a)iiA$

In this case, L_γ is a bipyramid such that its top-vertices are connected by the segment of γ. We consider the case depicted in the figure 2 b).

For finding the projection $pr_{\overrightarrow{m}} : \Pi \to \tau$ we first act similarly to the previous case: construct a line l with the directional vector $\overrightarrow{v}$ that is orthogonal to the plane $\pi^\perp$. For this, find an intersection point of a segment CD that connects the top-vertices of a bipyramid with its base EAB and denote it by D''. Then draw a line through D'' that is parallel to AE and find an intersection point of this line with AB (dente this point by C''). The segment CC'' defines l. After that, the reasoning used in case 2 can be applied. By using a small turn around the line with the directional vector v_n that is orhogonal to the plane containing $\Delta CC''D''$, we get the projection $pr_{\overrightarrow{m}} : \Pi \to \tau$ that maps γ to the polygonal chain γ' that has a configuration presented in figure 4 d). Note that $\overrightarrow{n}$ is parallel to $\pi^\perp$ and is not orthogonal to the plane, containing ΔABC. Thus, the first statement of the proposition is proved.

By applying Lemma 1, first to the polygonal chain $EABC$ and then to $DEAB$, we obtain that the supporting 2-subchains ABC and DEA are inducing γ-orientations of corresponding faces induced by different orientations of ∂L_γ. The subchains ABC and DEA satisfy the second statement of our proposition.

Case 4. *γ corresponds to the configuration type* $3(b)ii$ *and* γ'' *corresponds to configurations a), b), d) (see figure 1).*

These are the limiting cases of those already considered and the constructions of $pr_{\overrightarrow{m}} : \Pi \to \tau$ are completely similar. As in previous cases, $\gamma' = pr_{\overrightarrow{m}}(\gamma)$ has a configuration, corresponding to the figure 4 d). This proves the first statement of our proposition.

By applying the Lemma 1 to a 3-subchain $EABC$ for configurations d) and e) (fig 2), we get that the EAB and ABC are the supporting chains mentioned in the second statement.

In the case of configuration f) (fig. 2) apply Lemma 1 to tetrahedrons $ABCD$ and $CDEA$. Note that 2-subchains BCD and CDE induce different orientations of the face $BCED$, hence the orientations of faces, induced by 2-subchains DEA and ABC are induced by different orientations of ∂L_γ. This proves the second statement of our proposition in the case f) (fig. 2).

Case 5. *γ corresponds to the configuration type* $3(b)ii$ *γ'' corresponds to configuration c) (fig. 1).*

We will consider the case pictured in the figure 2 g). Note that segments CD and EA lie on skew lanes, since otherwise γ would be a plane polygonal chain. Hence the intersection of a plane, that passes through CD parallel to EA (and has a normal vector $\overrightarrow{n}$) and a plane that contains ΔABC defines a line l with the directional vector v (this is similar th case 2). In that case, there are three possible types of the projection $pr_{\overrightarrow{v}} : \Pi \to \pi^\perp$. If, as before, it has type a) (fig. 4), then, by applying Lemma 1 to the subchain $EABC$ we can show that 2-subchains EAB and ABC are the required supporting chains. Note that in that case all 2-subchains of γ are supporting subchains. If the images of points B and C under $pr_{\overrightarrow{v}}(\gamma)$ are merged (fig. 4 c)), we can, by applying a small rotation around the axis with the directional vector $\overrightarrow{n}$ (similarly to case 2), make it so that the image $\gamma' = pr_{\overrightarrow{m}}(\gamma)$ had a configuration corresponding to the figure 4 f). Note that the images of A and B under $pr_{\overrightarrow{v}}(\gamma)$ cannot be merged (see 5 c')), as in that case the plane containing $ABDE$ must be parallel to CD, but this is impossoble (see fig. 2 g)). Moreover, it can be seen that the case depicted in the figure 5 a') is also impossible.

In the other case, the projection $pr_{\overrightarrow{v}} : \Pi \to \pi^\perp$ can map γ bijectively onto its image, which, in that case, will be a convex quadrangle, presented in figure 4 b). Similarly to the previous cases, by applying a small rotation around the axis with the directional vector $\overrightarrow{n}$ we can make the image $\gamma' = pr_{\overrightarrow{m}}(\gamma)$ have a configuration, corresponding to the one shown in figure 4 e). As in the previous case, we can pick EAB and ABC as our supporting 2-subchains.

Case 6. *γ corresponds to the configuration type* $3(a)iiB$

We will consider the case, shown in figure 2 c). In that case, all 2-subchains but one (CDE) are supporting subhcains. Since we assume that the bypiramid $ABCDE$ is not a pyramid, we have that any two segments that are not adjacent (in particular, EA and CD) lie on skew curves. Hence, the intersection of the plane that passes through CD parallel to EA (and has a normal vector $\overrightarrow{n}$) and the plane containing ΔABC defines the line l with the directional vector $\overrightarrow{v}$ (this is similar to case 2). Here, as in the previous case, three configurations of γ' are possible and all of them can be considered in the similar way as before. Apriori, in this case $pr_{\overrightarrow{v}}(\gamma)$ can merge points A and B (fig. 5 c')). By applying a small rotation around the axis with the directional vector $\overrightarrow{n}$ we can make the image $\gamma' = pr_{\overrightarrow{m}}(\gamma)$ have a configuration that corresponds the figure 5 f'). In that case, the 2-subchains ABC and BCD will be the supporting subchains. Moreover, in that case a situation symmetric to the case 4 d), that is shown in the figure 5 d') is also apriori possible. In this scenario, ABC and BCD will also be the supporting subchains. In other cases

EAB and ABC will be the supporting subchains instead. Hence the proposition is proved for this case.

By Proposition 1, the argument, presented above for the polygonal chains of types, that are depicted in the figure 2 can also be applied if we consider polygonal chains $g'\gamma,\ g' \in G'_\gamma$ without any loss of generality.

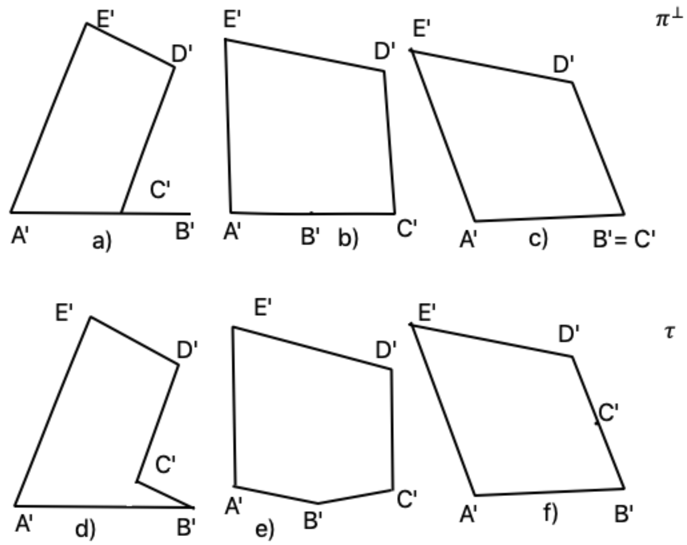

FIG. 4. Projections of γ

□

4. Proof ot Theorem 1

Consider an embedding of a sphere S^2 in the 4-dimensional Euclidean space E^4 in the form of a 2-dimensonal polyhedron $P^2 \subset E^4$. Since the sphere is compact, there exists a ball B^4 in E^4 that contains P^2. We will decrease the radius of the said ball, until its boundary $S^3 := \partial B^4$ will not touch the polyhedron P^2. The inner points of an edge or a face of P^2 cannot touch S^3 as in that case P^2 could not be contained inside B^4, hence only one or a few vertices of P^2 can touch S^3. Let p be a vertex of P^2, such that S^3 and P^2 both contain it. Consider a 3-dimensional plane in E^4, that is tangent to S^3 and another plane Π that is parallel and sufficiently close to the first one, such that Π separates p from other vertices of P^2. This can be done by, for instance, choosing Π to be so close to the tangent plane, that it cuts off a convex body B_p from B^4, with this body having a diameter, that is guaranteed to be smaller than the minimal distance between vertices of P^2. In that case, the intersection $\Pi \cap P^2$ is connected and is a closed polygonal chain $\gamma \subset P^2$, the segments of which are generated by intersections of Π with faces, that are incident to p.

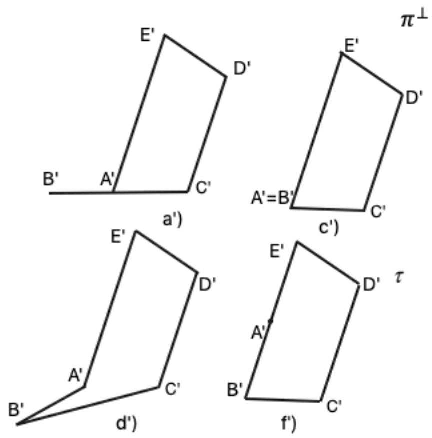

FIG. 5. Projections of γ

Our proof is by contradiction. Assume that for every point $x \in E^3 \setminus P^2$, there exists a two-dimensional plane π_x that passes through x, such that

$$\pi_x \cap P^2 = \emptyset. \qquad (*)$$

Case 1. *γ is a plane polygonal chain that lies in some two-dimensional plane $\alpha \subset \Pi$.*

In that case, by Jordan's theorem, γ splits the plane α into two connected components, one of which is gomeomorphic to a 2-dimensional disk D^2. For every point $x \in \operatorname{int} D^2$, an intersection $\pi_x \cap \Pi$ is a line l_x that intersects D^2 in one point x, where π_x satisfies $(*)$. Otherwise, $\pi_x \cap \Pi$ intersects γ and this contaridcts $(*)$. This means that γ is a non-trivial element of a fundamental group $\pi_1(E^4 \setminus \pi_x)$, since the skew projection $p : E^4 \to \alpha$, parallel to π_x does not move points of α and defines a deformation retraction $E^4 \setminus \pi_x \to \alpha \setminus x$ and, therefore, induces an isomorphism of fundamental groups $\pi_1(E^4 \setminus \pi_x) \cong \pi_1(\alpha \setminus x) = \mathbb{Z}$. It is clear that γ represents a generator of $\pi_1(\alpha \setminus x)$. On the other hand, $\gamma \subset P^2 \subset E^4 \setminus \pi_x$ and γ is contracted over P^2 into a point, since $\pi_1(P^2) = 0$. Thus, we obtain a contradiction.

Case 2. *γ is a non-plane polygonal chain.*

Consider a projection $pr_{\overrightarrow{m}} : \Pi \to \tau$ from Proposition 2. By G' denote an domain in the plane τ that is bounded by the polygonal chain γ' and by $S' \subset G'$ denote a subset of points of G' such that every line, passing through $y \subset S'$, intersects γ' in two points. It is clear that if G' is a convex set, then $S' = \operatorname{int} G'$, and if G' corresponds to the figure 4 d), then S' is an inside of a convex quadrangle $A'F'C'L'$, where $F' = l_{B'C'} \cap l_{A'E'}$ and $L' = l_{D'C'} \cap l_{A'B'}$[4]. Said quadragnle is presented in

[4]By l_{XY} we denote a line that passes through points X and Y.

figure 6. The case of γ' corresponding to the figure 5 d') can be dealt with in a similar way.

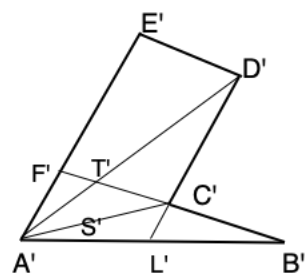

FIG. 6. Domain G'

Since γ is connected, we have that any 2-dimensional plane in Π, that passes through $x \in L_\gamma \setminus \gamma$ must intersect γ. This contradicts our assumption. Hence, if π_x satisfies $(*)$, then $\pi_x \cap \Pi,\ x \in L_\gamma \setminus \gamma$ is a line, which we will denote l_x. Therefore, if $x \in pr^{-1}_{\overrightarrow{m}}(S') \cap L_\gamma$, then the image $l'_{x'} := pr_{\overrightarrow{m}}(l_x)$ is a line or a point, that contains the point $x' = pr_{\overrightarrow{m}}(x) \in S'$. Then $l'_{x'} \cap \gamma'$ either consists of two points, which we will denote x'_1 and x'_2, or is empty. Note that the domain $S := pr^{-1}_{\overrightarrow{m}}(S') \cap L_\gamma$ does not contain γ, since the image of this domain under the map $pr_{\overrightarrow{m}}$ does not contain γ' and, since it is an intersection of two convex sets $pr^{-1}_{\overrightarrow{m}}(S')$ and L_γ, is also a convex set

We will divide all lines $\{l_x,\ x \in S\}$ into 4 types, depending on whether l_x passes "over" or "under"γ in the sence of ordering $(<)$ defined by the directional vector $\overrightarrow{m}$ on lines $pr^{-1}_{\overrightarrow{m}}(x'_i),\ i = 1, 2$. Denote by $\{x_1, x_2\} \in \gamma$ and $\{y_1, y_2\} \in l_x$ preimages of points $\{x'_1, x'_2\}$ under the projection $pr_{\overrightarrow{m}}$, provided that these preimages are not empty sets.

(1) Type $\emptyset : pr_{\overrightarrow{m}}(l_x) = x' \in S'$ (the line l_x is projected onto a point);
(2) Type $+-$: $x_1 < y_1;\ x_2 > y_2$ or $x_1 > y_1;\ x_2 < y_2$;
(3) Type $--$: $x_1 > y_1;\ x_2 > y_2$;
(4) Type $++$: $x_1 < y_1;\ x_2 < y_2$.

If l_x belongs to the type $\emptyset$, then γ can be deformed along the generatrices of a cylinder $pr^{-1}_{\overrightarrow{m}}(\gamma')$, which are, in the case considered, parallel to l_x to turn into γ', that lies in the plane τ. We can choose τ in such a way that $\tau \cap \gamma \neq \emptyset$ and take any of the intersection points as a marked point in the space $E^4 \setminus \pi_x$ и $\tau \setminus x'$. This means that

$[\gamma] = [\gamma'] \in \pi_1(E^4 \setminus \pi_x)$. The polygonal chain γ' represents a generator of the group $\pi_1(\tau \setminus x') = \mathbb{Z}$ and therefore also represents a generator of the group $\pi_1(E^4 \setminus \pi_x)$, since $\tau \setminus x'$ is a deformation retract of $E^4 \setminus \pi_x$ with respect to the projection $E^4 \to \tau$ that is parallel to π_x. On the other hand, $\gamma \subset P^2 \subset E^4 \setminus \pi_x$ and γ can be contracted over P^2 into a point, since $\pi_1(P^2) = 0$. And since $[\gamma] = [\gamma'] \in \pi_1(E^4 \setminus \pi_x)$, we obtain a contradiction.

Now assume that l_x belongs to the type $+-$. By τ_1 denote a plane, passing through the point $\{x_1, x_2\} \in \gamma$ and consider the case when restriction of a projection $pr_{\overrightarrow{m}}|_{\tau_1} : \tau_1 \to \tau$ is a bijective map. Then there exists a deformation $\gamma_t,\ t \in [0,1]$ of a polygonal chain $\gamma =: \gamma_0$ along the generatrices of a cylinder $pr_{\overrightarrow{m}}^{-1}(\gamma')$ onto a polygonal chain $\gamma_1 \subset \tau_1$, such that $\gamma_t \cap l_x = \emptyset\ \forall t \in [0,1]$. Note that the points x_1, x_2, y_1, y_2 lie in the plane $pr_{\overrightarrow{m}}^{-1}(l_{x_1' x_2'})$. Since l_x belongs to the type $+-$, we have that the segments $[y_1, y_2]$ and $[x_1, x_2]$ intersect in some point $z_0 \in (x_1, x_2)$. The reverse function $(pr_{\overrightarrow{m}}|_{\tau_1})^{-1} : \tau \to \tau_1$ maps G' into G_1, which is bounded by the polygonal chain γ_1 and maps the segment $[x_1', x_2'] \subset G'$ into the segment $[x_1, x_2] \subset G_1$. Therefore $l_x \cap \tau_1 = z_0 \in \operatorname{int} G_1$ and $[\gamma_1]$ is a generator of a group $\pi_1(\Pi \setminus l_x) \cong \pi_1(\tau_1 \setminus z_0) \cong \mathbb{Z}$ since $\tau_1 \setminus z_0$ is a deformation rectract of $\Pi \setminus l_x$ under the deformation rectraction $\Pi \to \tau_1$ along the parallel lines ith the directional vector $\overrightarrow{m}$. And since $[\gamma] = [\gamma_1] \in \pi_1(\Pi \setminus l_x)$, we have that γ also represents a generator of a group $\pi_1(\Pi \setminus l_x) \cong \mathbb{Z}$. Now note that the projection $E^4 \setminus \pi_x \to \Pi \setminus l_x$ along any direction, that is transversal to Π and parallel to π_x is a deformation retraction and induces an isomorphism of fundamental groups. Therefore $[\gamma]$ is a generator of a group $\pi_1(E^4 \setminus \pi_x)$. But on the other hand, since $\pi_1(P^2) = 0$ and $\gamma \subset P^2 \subset E^4 \setminus \pi_x$, we have that γ can be contracted over P^2 into a point. The resulting contradiction finishes the proof in the case that the set of lines l_x of type $+-$ is not empty.

Now consider the case, when the sets of lines $l_x,\ x \in S$, corresponding to types $\emptyset$ and $+-$ are empty. Recall that $l_x = \pi_x \cap \Pi$, where π_x is a plane that contains x and satisfies $(*)$, which, by our assumption, exists $\forall x \in E^4 \setminus \gamma$. We will show that in this case the set of $\{l_x,\ x \in S\}$ must contain lines, corresponding to both remaining types $++$ and $--$.

Without the loss of generality we can assume that the configuration corresponds to one of those presented in figure 2. Consider the supporting subchain ABC and let the point $x \in \operatorname{int} \Delta ABC$ be chosen in such a way, that $x' = pr_{\overrightarrow{m}}(x) \in S'$. Such a point exists because $\operatorname{int} \Delta A'B'C' \cap S' \neq \emptyset$. Since the line l_x cannot lie in the plane, containing the triangle ΔABC (as in that case it would intersect $\gamma \subset P^2$), we have that $l_x \cap \Delta ABC = x$. Note that at least one of the points $\{x_1', x_2'\} = l_{x'}' \cap \gamma'$ belongs to the subchain $A'B'C'$ of a polygonal chain γ, since $l_{x'}' \cap \operatorname{int} \Delta A'B'C' \neq \emptyset$. If both points $\{x_1', x_2'\}$ belonged to the said subchain we would get that the type of l_x is $+-$, since l_x would tranversely intersect $\operatorname{int} \Delta ABC$ and points y_1, y_2 would lie on the different sides of a plane, containing ABC thus vectors $\overrightarrow{x_1 y_1}$ и $\overrightarrow{x_2 y_2}$ would be facing in different directions. Therefore only one of the points $\{x_1', x_2'\}$ belongs to $A'B'C'$. For the sake of clarity we will assume that it is x_1'. Since ABC is a supporting subchain, we have that one of the semi-intervals $(x, y_2] \in l_x$ or $[y_1, x) \in l_x$ lies outside of the convex hull L_γ and the other one intersects it. Assume that $(x, y_2]$ lies outside of L_γ. Since the plane containing ΔABC separates the points y_1 and y_2 and $x_1 \in \Delta ABC$, we have that vectors $\overrightarrow{x_2 y_2}$ and $\overrightarrow{x_1 y_1}$ are facing in different directions. But this means that l_x is of type $+-$, which is impossible by our assumption. From this we derive that $\overrightarrow{xy_2}$ is directed inside the convex hull L_γ and y_1 and L_γ lie on the different sides of the plane containing ΔABC.

Now apply similar reasoning to the second supporting subchain. By Proposition 2, it is either the subchain EAB, DEA (then the projection of γ corresponds to the one presented in figure 4), or one of the subchains BCD and CDE (in this case the projection of γ corresponds to the one presented in figure 5). Since similar reasoning can be applied for the cases presented in figure 5, we will assume that one of those shown in figure 4 takes place. Note that $\Delta\ E'A'B' \cap S' \neq \emptyset$ and $\Delta\ D'E'A' \cap S' \neq \emptyset$ (see fig. 6).

Without the loss of generality consider the subchain EAB. From our previous results, it follows that there exists a point $z \in \text{int}\ \Delta\ EAB$, such that $z' = pr_{\overrightarrow{m}}(z) \in S'$.

By $\{z_1, z_2\} \in \gamma$ and $\{w_1, w_2\} \in l_z$ denote the preimages of $\{z_1', z_2'\} = l'_{z'} \cap \gamma'$ under the projection $pr_{\overrightarrow{m}}$, where $l'_{z'} := pr_{\overrightarrow{m}}(l_z)$. Without the loss of generality we can assume that z_1' belongs to the subchain $E'A'B'$, while z_1 belongs to the subchain EAB. By reasoning similar to the one used for y_1, we can say that w_1 and L_γ lie on different sides of the plane containing the triangle ΔEAB. Considering the orientation of supporting subchains (see Proposition 2), we obtain that $\{\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{y_1x_1}\}$ and $\{\overrightarrow{EA}, \overrightarrow{AB}, \overrightarrow{w_1z_1}\}$ are forming bases, that present different orientations of the space Π. Therefore the bases $\{\overrightarrow{A'B'}, \overrightarrow{B'C'}, \overrightarrow{y_1x_1}\}$ and

$$\{\overrightarrow{E'A'}, \overrightarrow{A'B'}, \overrightarrow{w_1z_1}\},$$

that are equivalent to them also represent different orientations of Π. At the same time $\{\overrightarrow{A'B'}, \overrightarrow{B'C'}\}$ and $\{\overrightarrow{E'A'}, \overrightarrow{A'B'}\}$ are forming equivalent bases of the plane τ. From this we conclude that $\overrightarrow{y_1x_1}$ and $\overrightarrow{w_1z_1}$ are facing in different directions and, if the line l_x is of the type $--$ then l_z will be of the type $++$ and conversely if l_x is of the type $++$ then l_z is of the type $--$.

Define the sets $C_+ \subset S$ and $C_- \subset S$ as follows: $x \in C_+(C_-)$ if there exists a line l_x that belongs to the type $++$ $(--)$. We showed that in the case that the subsets of types $\emptyset$ and $+-$ of the set of lines $\{l_x,\ x \in S\}$ are empty, both sets C_+ and C_- are not empty and $S = C_+ \cup C_-$. Since planes $\pi_y,\ y \in S$, that are close and parallel to π_x will still satisfy $(*)$ and the lines $l_y := \pi_y \cap \Pi$ will have the same type as l_x we conclude that C_+ and C_- are open in S. Recall that the set S is convex and therefore it is connected. From this we get that $C_+ \cap C_- \neq \emptyset$ and there exists a point $s \in S$, such that there exist lines $l^+ := \pi^+ \cap \Pi$ and $l^- := \pi^- \cap \Pi$ of types $++$ and $--$ respectively that pass through it. Here $\pi^+ \subset E^4$ и $\pi^- \subset E^4$ are two-dimensional planes that satisfy $(*)$. Moreover, by slightly moving one of the planes, for instance π^+, in such a way that leaves point s fixed, we can always make it so that π^+ and π^- are in the similar position and still satisfy $(*)$. By η denote the plane, that is generated by lines l^+ and l^-. Assume that the restriction of the projection $pr_{\overrightarrow{m}}$ onto η degenerates into a line, that passes through the points x_1', x_2' of a polygonal chain γ'. Let $\{y_1^+, y_2^+\} \in l^+$, $\{y_1^-, y_2^-\} \in l^-$, $\{x_1, x_2\} \in \gamma$ genote the preimages of points $\{x_1', x_2'\}$ under the projection $pr_{\overrightarrow{m}}$. Then, on the other hand, we must have that $y_1^- < x_1 < y_1^+$, while $y_2^- < x_2 < y_2^+$, which is impossible in our case, since $[y_1^- y_2^-] \cap [y_1^+ y_2^+] = s$. Then either $y_1^+ < y_1^-$ or $y_2^+ < y_2^-$, which is impossible (see fig. 7). Hence we conclude that $pr_{\overrightarrow{m}}$ does not degenerate on η.

Consider a variation of parallel planes π_t^- along the vector m:

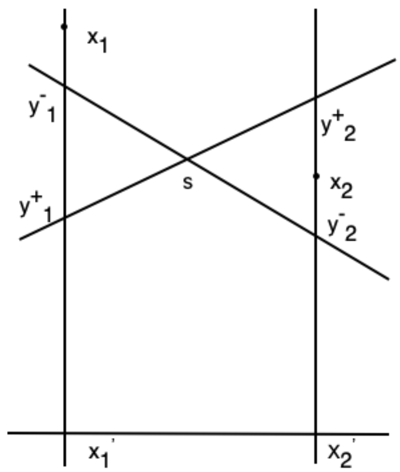

FIG. 7. Plane η

$$\overrightarrow{r}' = \overrightarrow{r} + t\overrightarrow{m},$$

where $\overrightarrow{r}$ is a radius vector of a plane π^-.

If $t = 0$ we have the original plane π^-, while for small $t > 0$ we get that π_t^- and π^+ still satisfy $(*)$ while being in a similar position. In that case, the lines $l_t^- := \pi_t^- \cap \Pi$ and l^+ will not intersect and will still belong to the types $--$ and $++$ respectively. Let $l'^-_t := pr_{\overrightarrow{m}}(l_t^-)$ and $l'^+ := pr_{\overrightarrow{m}}(l^+)$. A polygonal chain γ has a non-trivial link with l_t^- and l^+, i.e. we cannot have an isotropy of γ over the set $\Pi \setminus l_t^- \cup l^+$ into a curve, that completely lies inside the ball $B \subset \Pi \setminus l_t^- \cup l^+$. The diagram of this link is defined by a map $pr_{\overrightarrow{m}} : \Pi \to \tau$ (see fig. 8).

We will show that

$$\pi_1(\Pi \setminus (l_t^- \cup l^+)) = \mathbb{Z} * \mathbb{Z}.$$

Indeed, note that $\Pi \setminus (l_t^- \cup l^+)$ is homotopically equivalent to the Euclidean plane without two points. To construct the corresponding homotopy, we first need to homeomorphically map Π onto itself in such a way, so the lines become parallel and then deform the image $\Pi \setminus (l_t^- \cup l^+)$ onto a plane without to points that is orthogonal to said lines. It is known that the Euclidean plane without two points is homotopically equivalent to the bouqet of 2 circles (also known as a rose with 2 petals) $S^1 \vee S^1$ and $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.

We will now show that γ represents a commutant $\alpha\beta\alpha^{-1}\beta^{-1}$ of generators α, β of a fundamental group $\pi_1(\Pi \setminus (l_t^- \cup l^+))$.

Let $\overrightarrow{s_t} := \overrightarrow{s} + t\overrightarrow{m}$ where $\overrightarrow{s}$ и $\overrightarrow{s_t}$ are radius vectors of points s and s_t respectively. It is clear that $s_t \in l_t^-$. Let $\{x_1^+, x_2^+\} \in \gamma$ and $\{x_1^-, x_2^-\} \in \gamma$ denote the preimages

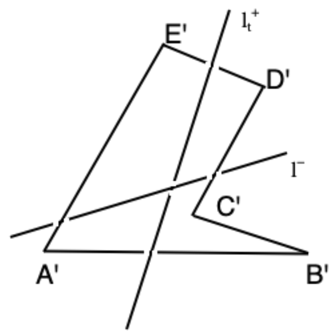

FIG. 8. Link diagram

of the points

$$\{x'^+_1, x'^+_2\} = pr_{\overrightarrow{m}}(l^+) \cap \gamma'$$

and

$$\{x'^-_1, x'^-_2\} = pr_{\overrightarrow{m}}(l^-_t) \cap \gamma' = pr_{\overrightarrow{m}}(l^-) \cap \gamma'$$

under the projection $pr_{\overrightarrow{m}}|_\gamma : \gamma \to \gamma'$.

By $\{y^-_{1t}, y^-_{2t}\} \in l^-_t$ denote the preimages of points $\{x'^-_1, x'^-_2\}$, while $\{y^+_1, y^+_2\} \in l^+$ denote the preimages of – $\{x'^+_1, x'^+_2\}$ under the projection $pr_{\overrightarrow{m}}$. Since $y^-_{1t} < x^-_1$, $y^-_{2t} < x^-_2$, $y^+_1 > x^+_1$, $y^+_2 > x^+_2$, we have that $l^-_t \cap [x^-_1, x^-_2] = \emptyset$ and $l^+ \cap [x^+_1 x^+_2] = \emptyset$. Otherwise, by the above reasoning we obtain the contradiction (see figure 7). Let $s' = pr_{\overrightarrow{m}}(s)$. Define the points $s^+ \in [x^+_1 x^+_2]$, $s^- \in [x^-_1 x^-_2]$ in the following way: $pr_{\overrightarrow{m}}(s^+) = pr_{\overrightarrow{m}}(s^-) = pr_{\overrightarrow{m}}(s) = s'$. From our construction we have that $s_t > s$ and thus we have the following ordering of points on the line $pr^{-1}_{\overrightarrow{m}}(s')$: $s^+ < s < s_t < s^-$. This means that the segments $[x^-_1, x^-_2]$, $[x^+_1, x^+_2]$ do not intersect each other and the lines l^+_t and l^-.

Since $[x'^-_1, x'^-_2] \cap [x'^+_1, x'^+_2] = s' \neq \emptyset$ we conclude that the ordering of points on γ' is cyclic: $x'^-_1 x'^+_1 x'^-_2 x'^+_2$. This implies that the ordering of the points on γ is also cyclic: $x^-_1 x^+_1 x^-_2 x^+_2$. Consider the following paths $a := \gamma_{x^-_1 x^+_1}$, $b = \gamma_{x^+_1 x^-_2}$, $c = \gamma_{x^-_2 x^+_2}$, $d = \gamma_{x^+_2 x^-_1}$, $e = [x^-_2, x^-_1]$, $f = [x^+_1, x^+_2]$. It can be easily seen that closed paths $\alpha = abe$ and $\beta = d^{-1} f^{-1} a^{-1}$ can be chosen as representatives of a fundamental group $\pi_1(\Pi \setminus (l^-_t \cup l^+))$. Here degree -1 means the reverse path. It is clear that the path $\bar{\alpha} = e^{-1} b^{-1} a^{-1}$ is reverse to alpha, while the path $\bar{\beta} = afd$ is

[4] γ_{xy} denote the segment of a path on γ that starts at the point x and ends at the point y.

reverse to β. It is also easy to see that the path $\beta' = e^{-1}cf^{-1}be$ is homotopic to β and that the following homotopy takes place

$$\alpha\beta'\bar{\alpha}\bar{\beta} = abee^{-1}cf^{-1}bee^{-1}b^{-1}a^{-1}afd \sim abcd$$

This means that $[\gamma] = [\alpha][\beta][\alpha]^{-1}[\beta]^{-1}$ and is precisely what was stated.

Let E^4_+ и E^4_- be the half-spaces of the Euclidean space E^4, that are separated by the three-dimensional plane $\Pi \subset E^4$. One of the disks, on which γ splits a polyhedron P^2 lies in E^4_- and the other one lies in E^4_+. Recall that the planes π_t^- and π^+ are in the similar position and are intersecting only by one point, which we denote by o. Withouth the loss of generality we can assume that $o \in E^4_-$. Since the retraction

$$r : E^4_+ \setminus (\pi_t^- \cup \pi^+) \to \Pi \setminus (l_t^- \cup l^+)$$

that maps $e \in E^4_+$ to an intersection point $l_{eo} \cap \Pi$ of a line l_{eo} that passes through points e and o with the plane Π is a deformation retraction, we have that γ is a non-trivial element of a group

$$\pi_1(E^4_+ \setminus (\pi_t^- \cup \pi^+)) \cong \pi_1(\Pi \setminus (l_t^- \cup l^+)) = \mathbb{Z} * \mathbb{Z},$$

more specifically, a commutant of its generators. But, as was stated before, one of the two-dimensional disks, on which γ splits P^2 lies in E^4_+. And since $P^2 \cap (\pi_t^- \cup \pi^+) = \emptyset$ (by our construction), we have that γ bounds the disk in $E^4_+ \setminus (\pi_t^- \cup \pi^+)$, which means that $[\gamma] = 0$ in $\pi_1(E^4_+ \setminus (\pi_t^- \cup \pi^+))$. Thus, we obtain a contradiction. Therefore the assumption that for every point $x \in E^4$ outside of P^2 there exists a plane π_x that passes through x such that $\pi_x \cap P^2 = \emptyset$ is incorrect. The theorem is proved.

5. Conclusion

In this paper we showed that the property "being 2-convex" does not hold for any PL-embeddings of a 2-sphere in the 4-dimensional Euclidean space, if valence of the vertices does not exceed 5. While attempting to apply similar ideas for cases of higher valence, one may encounter additional difficulties: firstly, the polygonal chain γ may be knotted and secondly we lose the precense of two supporting 2-chains at the boundary of the convex hull L_γ, which is a key element in our proof. Therefore, in our opinion, it is interesting to find another proof that can be applied for these cases aswell.

We also note another problem that is still open even in the case of smoothness (at least, the author of this paper is not aware of it being solved).

Does there exist a 2-convex embedding of a surface M^2_g g, $g \geq 2$ in the Euclidean space E^4?

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