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HKSS-COMPLETENESS OF MODAL ALGEBRAS

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Abstract. The paper studies computability-theoretic properties of countable modal algebras. We prove that the class of modal algebras is complete in the sense of the work of Hirschfeldt, Khoussainov, Shore, and Slinko. This answers an open question of Bazhenov [Stud. Log., 104 (2016), 1083–1097]. The result implies that every degree spectrum and every categoricity spectrum can be realized by a suitable modal algebra.

Keywords: modal algebra, computable structure, Boolean algebra with operators, degree spectrum, categoricity spectrum, computable dimension, first-order definability.

1. Introduction

The paper studies algorithmic properties of computable modal algebras. Recall that a countable structure $\mathcal{S}$ is *computable* if the domain of $\mathcal{S}$ is a computable subset of ω, and all signature predicates and functions of $\mathcal{S}$ are uniformly computable. Equivalently, $\mathcal{S}$ is computable if its atomic diagram $D(\mathcal{S})$ is a computable set.

In the paper, we work within the framework developed by Hirschfeldt, Khoussainov, Shore, and Slinko [1]. They introduced the notion of a class of structures which is complete with respect to degree spectra of nontrivial structures, effective dimensions, expansion by constants, and degree spectra of relations. For the sake of brevity, here we follow [2] and call such classes *HKSS-complete*.

Informally speaking, the idea behind HKSS-completeness is as follows. If a countable structure has a particular computability-theoretic property, then for any HKSS-complete class K, there is a structure $\mathcal{S} \in K$ possessing the same property. The formal definition of HKSS-completeness is provided in Section 2.1. We

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note that HKSS-completeness is closely related to the recently developed theory of computable functors [3] (these functors act between classes of countable structures).

The paper [1] established that the following classes are HKSS-complete: undirected graphs, partial orders, (non-modular) lattices, rings, integral domains (of arbitrary characteristic), commutative semigroups, and 2-step nilpotent groups. Further examples of HKSS-complete classes include: fields of arbitrary characteristic [4], projective planes [5], structures with two equivalence relations [6, 7], and contact algebras [8].

It is well-known that the class of Boolean algebras is not HKSS-complete. For example, this follows from the result of Goncharov and Dzgoev [9]: they proved that for a computable Boolean algebra $\mathcal{B}$, the computable dimension of $\mathcal{B}$ is either 1 or ω.

Informally, one can say that the class of Boolean algebras is *not universal* in the computability-theoretic sense. Hence, there is a natural question: how does expanding the language of Boolean algebras affect computability-theoretic properties of the class?

Following this line of research, Khoussainov and Kowalski [10] initiated systematic investigations of computable Boolean algebras with operators.

Definition 1 (see Definition 3.1 in [10])**.** *Suppose that* $L = \{f_k^{n_k} : k \in I\}$ *is a functional language, and* $\mathcal{B}$ *is a Boolean algebra. A structure* $\mathcal{B}^L = (\mathcal{B}, f_k)_{k\in I}$ *is a* Boolean algebra with operators (BAO) *if for any* $k \in I$, $i \le n_k$, *and* $a_j \in \mathcal{B}$, *the following hold:*

(i) $f_k(a_1, \ldots, a_{i-1}, 0^{\mathcal{B}}, a_{i+1}, \ldots, a_{n_k}) = 0^{\mathcal{B}}$, *and*

(ii) $f_k(a_1, \ldots, a_{i-1}, b \vee c, a_{i+1}, \ldots, a_{n_k}) = f_k(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_{n_k}) \vee f_k(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_{n_k})$, *for any* $b, c \in \mathcal{B}$.

Operations that satisfy (i) and (ii) are called operators*. Unary operators are called* modalities. Modal algebra *is a BAO for the language* $\{f^1\}$, *and* polymodal algebra *is a BAO for the language* $\{f_k^1 : k \in I\}$, *where the cardinality of* I *is at least two.*

Khoussainov and Kowalski [10, Theorem 7.3] proved that the class of BAOs for the language $\{f^2, g^1\}$ is HKSS-complete. Theorem 1 of [11] proved that the class of polymodal algebras for the language $\{f_1, f_2, f_3, f_4\}$ is HKSS-complete. Note that in [11], HKSS-complete classes are called DS-complete.

The goal of this paper is to show that the class of modal algebras is rich from the computability-theoretic point of view. We prove that the class of modal algebras is HKSS-complete (Theorem 1). This result gives the positive answer to Question 1 from [11]. We note that our result implies that every degree spectrum of a countable structure can be realized by a modal algebra (to be explained in Section 2.1). Similarly, computable modal algebras realize all possible categoricity spectra [12].

The paper is arranged as follows. Section 2 provides the necessary preliminaries. In Section 3, we prove Theorem 1. The last section contains further discussion.

2. Preliminaries

We consider only finite languages, and structures with domains contained in ω. We identify first-order formulas with their Gödel numbers. For a structure $\mathcal{S}$, $\mathrm{dom}(\mathcal{S})$ denotes the domain of $\mathcal{S}$, and $D(\mathcal{S})$ is the atomic diagram of $\mathcal{S}$. Let

$\deg(\mathcal{S})$ denote the Turing degree of the set $D(\mathcal{S})$. The reader is referred to [13, 14] for the background on computable structures.

Let L_{BA} be the language $\{\vee, \wedge, \mathrm{C}; 0, 1\}$. We treat Boolean algebras as L_{BA}-structures. For a non-zero $n \in \omega$, consider the language

$$L_n = L_{BA} \cup \{f_1, f_2, \dots, f_n\},$$

where f_i are unary functions.

For $n \geq 2$, let PM_n be the class of polymodal algebras in the language L_n. By PM_1 we denote the class of modal algebras (in the language L_1). For a polymodal algebra $\mathcal{S}$, $BA(\mathcal{S})$ denotes the Boolean reduct of $\mathcal{S}$.

If $x_0, x_1, \dots, x_n$ are natural numbers, then $\langle x_0, x_1, \dots, x_n \rangle$ is the Gödel number of the tuple $(x_0, x_1, \dots, x_n)$. In the proof of Theorem 1, we identify tuples and their Gödel numbers.

For a function f, $\mathrm{dom}(f)$ denotes the domain of f and $\mathrm{ran}(f)$ denotes the range of f. For a set X, $\deg_T(X)$ is the Turing degree of X. As usual, we assume that $\{\varphi_e^X\}_{e \in \omega}$ is the standard effective enumeration of all unary partial X-computable functions. We will (sometimes) use $\leq_\omega$ to denote the standard ordering of ω.

2.1. **HKSS-Complete classes.** For a structure $\mathcal{S}$, a *copy* of $\mathcal{S}$ is a structure $\mathcal{A}$ such that $\mathcal{A}$ is isomorphic to $\mathcal{S}$, and the domain of $\mathcal{A}$ is a computable subset of ω. The *degree spectrum* of a structure $\mathcal{S}$ is the set

$$\mathrm{DgSp}(\mathcal{S}) = \{\deg(\mathcal{A}) : \mathcal{A} \text{ is a copy of } \mathcal{S}\}.$$

A structure $\mathcal{S}$ is *trivial* if there is a finite set $X \subseteq \mathrm{dom}(\mathcal{S})$ with the following property: any permutation f of $\mathrm{dom}(\mathcal{S})$, which keeps all elements of X fixed, is an automorphism of the structure $\mathcal{S}$. Knight [15] proved that the degree spectrum of a nontrivial structure is closed upwards.

Let $\mathbf{d}$ be a Turing degree. For a computable structure $\mathcal{S}$, the $\mathbf{d}$-*computable dimension* of $\mathcal{S}$, denoted by $\dim_{\mathbf{d}}(\mathcal{S})$, is the number of computable copies of $\mathcal{S}$ up to $\mathbf{d}$-computable isomorphisms.

Suppose that $\mathcal{S}$ is a computable structure, and R is a relation on the domain of $\mathcal{S}$. The *degree spectrum* of R on $\mathcal{S}$ is the set

$$\mathrm{DgSp}_{\mathcal{S}}(R) = \{\deg_T(f(R)) : \mathcal{A} \text{ is a computable copy of } \mathcal{S}, \text{ and } f \text{ is an isomorphism from } \mathcal{S} \text{ onto } \mathcal{A}\}.$$

A relation R is *invariant* if $f(R) = R$ for any automorphism f of $\mathcal{S}$.

Let $\mathfrak{C} \in \{\Delta_1^0, \Sigma_1^0\}$. A relation R is *intrinsically* $\mathfrak{C}$ on $\mathcal{S}$ if for any computable copy $\mathcal{A}$ of $\mathcal{S}$ and any isomorphism $f \colon \mathcal{S} \cong \mathcal{A}$, the relation $f(R)$ belongs to the class $\mathfrak{C}$. The relation R is called *relatively intrinsically* $\mathfrak{C}$ on $\mathcal{S}$ if for any copy $\mathcal{A}$ of $\mathcal{S}$ and any $f \colon \mathcal{S} \cong \mathcal{A}$, the relation $f(R)$ is in $\mathfrak{C}(D(\mathcal{A}))$.

Definition 2 ([1, Definition 1.21], see also [2, Definition 4.6])**.** *A class of structures K is* HKSS-complete *if for every nontrivial countable graph G, there is a nontrivial structure $\mathcal{A}_G \in K$ with the following properties.*

(1) $\mathrm{DgSp}(\mathcal{A}_G) = \mathrm{DgSp}(G)$.
(2) *If G has a computable copy, then the following hold:*
 (a) *for any Turing degree $\mathbf{d}$, $\dim_{\mathbf{d}}(\mathcal{A}_G) = \dim_{\mathbf{d}}(G)$;*

(b) *for any element* $c \in \mathrm{dom}(G)$, *there is an element* $a \in \mathrm{dom}(\mathcal{A}_G)$ *with*

$$\dim_{\mathbf{0}}(\mathcal{A}_G, a) = \dim_{\mathbf{0}}(G, c);$$

(c) *if* $R \subseteq \mathrm{dom}(G)$, *then there exists a relation* $Q \subseteq \mathrm{dom}(\mathcal{A}_G)$ *such that*

$$\mathrm{DgSp}_{\mathcal{A}_G}(Q) = \mathrm{DgSp}_G(R);$$

in addition, if R *is intrinsically* Σ^0_1, *then so is* Q.

3. The Main Result

Theorem 1. *The class of modal algebras is HKSS-complete.*

Recall that Theorem 1 of [11] proved that the class PM_4 is HKSS-complete. We also note that a Boolean algebra is nontrivial if and only if it is infinite. Therefore, it is sufficient to establish the following:

Proposition 1. *Suppose that* n *is a non-zero natural number and*

$$\mathcal{S} = (\mathrm{dom}(\mathcal{S}), \vee, \wedge, \mathrm{C}; 0, 1; f_1, f_2, \dots, f_n, f_{n+1})$$

is a countably infinite structure from the class PM_{n+1}. *There exists a countably infinite structure* $\mathcal{A}(\mathcal{S}) \in PM_n$ *such that* $\mathrm{DgSp}(\mathcal{A}(\mathcal{S})) = \mathrm{DgSp}(\mathcal{S})$. *In addition, if the structure* $\mathcal{S}$ *is computable, then* $\mathcal{A}(\mathcal{S})$ *satisfies the following:*

(1) *For any degree* $\mathbf{d}$, *we have* $\dim_{\mathbf{d}}(\mathcal{A}(\mathcal{S})) = \dim_{\mathbf{d}}(\mathcal{S})$.

(2) *For any* $c \in \mathrm{dom}(\mathcal{S})$, *there is an element* $a \in \mathrm{dom}(\mathcal{A}(\mathcal{S}))$ *with*

$$\dim_{\mathbf{0}}(\mathcal{A}(\mathcal{S}), a) = \dim_{\mathbf{0}}(\mathcal{S}, c);$$

(3) *For any* $R \subseteq \mathrm{dom}(\mathcal{S})$, *there is a relation* $Q \subseteq \mathrm{dom}(\mathcal{A}(\mathcal{S}))$ *such that*

$$\mathrm{DgSp}_{\mathcal{A}(\mathcal{S})}(Q) = \mathrm{DgSp}_{\mathcal{S}}(R).$$

Moreover, if R *is intrinsically* Σ^0_1, *then so is* Q.

Proof of Proposition 1. Note that it is sufficient to consider the typical case $n = 2$. In other words, here we show how to transform an algebra with three modalities into an algebra with two modalities. The rest of the proof (i.e. for $n \neq 2$) is a straightforward modification of the described construction.

The proof is based on the idea from [10, p. 495]: essentially, we check that the simulation technique of Kracht and Wolter [16] preserves all desired computability-theoretic properties. Our verification follows the outline of the proof from Proposition 2.14 of [1].

Suppose that $\mathbb{B}_1$ is the two-element Boolean algebra. For an algebra $\mathcal{S} \in PM_3$, the corresponding structure $\mathcal{A}(\mathcal{S}) = (\mathrm{dom}(\mathcal{A}(\mathcal{S})), \vee, \wedge, \mathrm{C}; \mathtt{0}, \mathtt{1}; g_1, g_2)$ is defined as follows:

$$BA(\mathcal{A}(\mathcal{S})) = BA(\mathcal{S}) \times BA(\mathcal{S}) \times \mathbb{B}_1;$$

$$g_1(\langle a, b, c\rangle) = \begin{cases} \langle b \vee f_1(a), a \vee f_2(b), 0\rangle, & \text{if } c = 0, \\ \langle 1, a \vee f_2(b), 0\rangle, & \text{if } c = 1; \end{cases}$$

$$g_2(\langle a, b, c\rangle) = \langle f_3(a), f_3(b), 0\rangle;$$

where $a, b \in \mathcal{S}$ and $c \in \mathbb{B}_1$. It is easy to show that $\mathcal{A}(\mathcal{S})$ is a $\deg(\mathcal{S})$-computable polymodal algebra.

The definition of $\mathcal{A}(\mathcal{S})$ is a modification of the definition of the algebra $\mathfrak{A}^{\mathrm{sim}}$ from [16, p. 116]. Informally speaking, the key difference in two definitions is the following: while the modality from [16] is the necessity operator $\Box$, our modality is the possibility operator $\Diamond = \neg\Box\neg$.

First, we list some simple properties of $\mathcal{A}(\mathcal{S})$ that will be used in the subsequent lemmas:

(A) $g_1(\langle 1,1,1\rangle) = \langle 1,1,0\rangle$ and $g_1(\langle 0,0,1\rangle) = \langle 1,0,0\rangle$;
(B) $g_1(\langle a,0,0\rangle) = \langle f_1(a),a,0\rangle$ for any $a \in \mathcal{S}$;
(C) $g_1(\langle 0,b,0\rangle) = \langle b,f_2(b),0\rangle$ for any $b \in \mathcal{S}$;
(D) $g_1(\langle 0,b,1\rangle) = \langle 1,f_2(b),0\rangle$ for any $b \in \mathcal{S}$.

Lemma 1. *The elements $c_1 = \langle 1,0,0\rangle$, $c_2 = \langle 0,1,0\rangle$, and $c_3 = \langle 0,0,1\rangle$ are definable by quantifier-free L_1-formulas in $\mathcal{A}(\mathcal{S})$.*

Proof. We only need to show that c_1 and c_3 are definable. By (A), we have $c_3 = \mathrm{C}(g_1(\mathtt{1}))$ and $c_1 = g_1(c_3)$. $\Box$

We define the following auxiliary $(L_2 \cup \{c_1,c_2,c_3\})$-formulas:

$$\begin{aligned}
Dom(x) &= (x \le c_1),\\
Mod_1(x,y) &= Dom(x)\,\&\,Dom(y)\,\&\,(g_1(x)\wedge c_1 = y),\\
\Phi(x,y) &= (y \le c_2)\,\&\,(g_1(y)\wedge c_1 = x),\\
Mod_2(x,y) &= Dom(x)\,\&\,Dom(y)\,\&\,\exists z\left(\Phi(x,z)\&\Phi(y,g_1(z\vee c_3)\wedge \mathrm{C}(c_1))\right),\\
Mod_3(x,y) &= Dom(x)\,\&\,Dom(y)\,\&\,(g_2(x) = y).
\end{aligned}$$

By Lemma 1, we may assume that Dom, Mod_1, Φ, and Mod_3 are quantifier-free L_2-formulas, and Mod_2 is an existential L_2-formula. The next lemma is a simple consequence of the properties (A)–(D) above.

Lemma 2. (1) *$Dom[\mathcal{A}(\mathcal{S})] = \{\langle a,0,0\rangle : a \in \mathcal{S}\}$.*
(2) *$\Phi[\mathcal{A}(\mathcal{S})] = \{(\langle a,0,0\rangle,\langle 0,a,0\rangle) : a \in \mathcal{S}\}$.*
(3) *For every $i \in \{1,2,3\}$, $Mod_i[\mathcal{A}(\mathcal{S})] = \{(\langle a,0,0\rangle,\langle f_i(a),0,0\rangle) : a \in \mathcal{S}\}$.*
(4) *The relation Mod_2 is definable by the universal formula:*

$$Mod_{2,\forall}(x,y) = Dom(x)\,\&\,Dom(y)\,\&\,\forall z\left(\Phi(x,z)\to\Phi(y,g_1(z\vee c_3)\wedge \mathrm{C}(c_1))\right).$$

Since Mod_2 is definable by both $\exists$- and $\forall$-formulas, we obtain the following:

Corollary 1. *The relations Dom and Mod_i, $i \in \{1,2,3\}$, are relatively intrinsically computable, invariant relations on $\mathcal{A}(\mathcal{S})$.*

For $x \in \omega$, let $F(x) = \langle x,0,0\rangle$. For a structure $\mathcal{B}$ with $\mathrm{dom}(\mathcal{B}) \subseteq \omega$, define the mapping

$$F_{\mathcal{B}} = F \restriction \mathrm{dom}(\mathcal{B}).$$

Lemma 3. *Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are computable copies of $\mathcal{S}$, and h is an isomorphism from $\mathcal{S}_1$ onto $\mathcal{S}_2$. Then there is a unique isomorphism $\tilde{h}$ from $\mathcal{A}(\mathcal{S}_1)$ onto $\mathcal{A}(\mathcal{S}_2)$ such that $\tilde{h} \restriction Dom[\mathcal{A}(\mathcal{S}_1)] = F_{\mathcal{S}_2}\circ h\circ F^{-1}_{\mathcal{S}_1}$. Moreover, the map $\tilde{h}$ is $\deg_T(h)$-computable.*

Proof. For $a,b \in \mathcal{S}_1$ and $c \in \mathbb{B}_1$, set $\tilde{h}(\langle a,b,c\rangle) = \langle h(a),h(b),c\rangle$. It is easy to show that $\tilde{h}$ is a $\deg_T(h)$-computable isomorphism from $\mathcal{A}(\mathcal{S}_1)$ onto $\mathcal{A}(\mathcal{S}_2)$. In addition, we have $\tilde{h}(\langle a,0,0\rangle) = F_{\mathcal{S}_2}\circ h\circ F^{-1}_{\mathcal{S}_1}(\langle a,0,0\rangle)$ for all $a \in \mathcal{S}_1$.

Now assume that $\tilde{h}'$ is an isomorphism satisfying the conditions of the lemma, and $u = \langle a,b,c\rangle$ is an element from $\mathcal{A}(\mathcal{S}_1)$. It is evident that $\tilde{h}'(\langle a,0,0\rangle) =$

$\langle h(a), 0, 0\rangle$. Lemma 1 implies that $\tilde{h}'(\langle 0, 0, c\rangle) = \langle 0, 0, c\rangle$. Let $y = \langle 0, b, 0\rangle$. By Lemma 2, $x = \langle b, 0, 0\rangle$ is the unique element such that $\mathcal{A}(\mathcal{S}_1) \models \Phi(x, y)$. In addition, $\tilde{h}'(x) = \langle h(b), 0, 0\rangle$; thus, $y' = \langle 0, h(b), 0\rangle$ is the unique element with $\mathcal{A}(S_2) \models \Phi(\tilde{h}'(x), y')$. Hence, $\tilde{h}'(\langle 0, b, 0\rangle) = \langle 0, h(b), 0\rangle$ and $\tilde{h}'(u) = \tilde{h}(u)$. □

Let $\mathcal{B}$ be a copy of $\mathcal{A}(\mathcal{S})$. The L_3-structure $\tilde{S}(\mathcal{B})$ is defined as follows. The domain of $\tilde{S}(\mathcal{B})$ is equal to $Dom[\mathcal{B}]$, the Boolean operations of $\tilde{S}(\mathcal{B})$ are induced by the operations of $BA(\mathcal{B})$, and for $i \in \{1, 2, 3\}$, the graph of the function $f_i^{\tilde{S}(\mathcal{B})}$ is defined by $Mod_i[\mathcal{B}]$.

Lemma 4. (1) *The structure $\tilde{S}(\mathcal{B})$ is a $\deg(\mathcal{B})$-computable copy of $\mathcal{S}$.*

(2) *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are computable copies of $\mathcal{A}(\mathcal{S})$ and G is an isomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$, then $G \upharpoonright Dom[\mathcal{A}_1]$ is a $\deg_T(G)$-computable isomorphism from $\tilde{S}(\mathcal{A}_1)$ onto $\tilde{S}(\mathcal{A}_2)$.*

(3) *For a computable copy $\mathcal{S}_1$ of $\mathcal{S}$, $F_{\mathcal{S}_1}$ is a computable isomorphism from $\mathcal{S}_1$ onto $\tilde{S}(\mathcal{A}(\mathcal{S}_1))$.*

(4) *Suppose that $\mathcal{A}_1$ is a computable copy of $\mathcal{A}(\mathcal{S})$. Then there is a computable isomorphism H from $\mathcal{A}_1$ onto $\mathcal{A}(\tilde{S}(\mathcal{A}_1))$ such that $H \upharpoonright Dom[\mathcal{A}_1] = F_{\tilde{S}(\mathcal{A}_1)}$.*

Proof. The first three claims easily follow from Lemma 2, Corollary 1, and the first-order definitions of the relations Dom and Mod_i, $i = 1, 2, 3$.

We now define the isomorphism H from the fourth claim. For an element u from $\mathcal{A}_1$, we find the element v such that $\mathcal{A}_1 \models \Phi(v, u \wedge c_2^{\mathcal{A}_1})$ and we set

$$H(u) = \langle u \wedge c_1^{\mathcal{A}_1}, v, u \wedge c_3^{\mathcal{A}_1}\rangle.$$

It is not difficult to verify that the map H is a desired computable isomorphism. □

Now we are ready to prove that the transformation $\mathcal{S} \mapsto \mathcal{A}(\mathcal{S})$ preserves our computability-theoretic properties. The omitted technical details can be easily recovered from Propositions 2.10–2.13 in [1].

Lemma 5. $\mathrm{DgSp}(\mathcal{A}(\mathcal{S})) = \mathrm{DgSp}(\mathcal{S})$.

Proof. Recall that every infinite polymodal algebra is a nontrivial structure. Hence, the degree spectra of $\mathcal{S}$ and $\mathcal{A}(\mathcal{S})$ are both closed upwards. The rest follows from the $\deg(\mathcal{S})$-computability of $\mathcal{A}(\mathcal{S})$ and Lemma 4.1. □

Lemma 6. *For any Turing degree $\mathbf{d}$, we have $\dim_{\mathbf{d}}(\mathcal{A}(\mathcal{S})) = \dim_{\mathbf{d}}(\mathcal{S})$.*

Proof. Claims 2 and 3 from Lemma 4 imply the following: if $\mathcal{S}_1$ and $\mathcal{S}_2$ are computable copies of $\mathcal{S}$ and there is no $\mathbf{d}$-computable isomorphism from $\mathcal{S}_1$ onto $\mathcal{S}_2$, then $\mathcal{A}(\mathcal{S}_1)$ and $\mathcal{A}(\mathcal{S}_2)$ are not $\mathbf{d}$-computably isomorphic. Thus, $\dim_{\mathbf{d}}(\mathcal{A}(\mathcal{S})) \geq \dim_{\mathbf{d}}(\mathcal{S})$.

Lemma 3 and Lemma 4.4 guarantee the following property: if $\mathcal{A}_1 \cong \mathcal{A}_2 \cong \mathcal{A}(\mathcal{S})$ and $\mathcal{A}_1$ and $\mathcal{A}_2$ are not $\mathbf{d}$-computably isomorphic, then $\tilde{S}(\mathcal{A}_1) \ncong_{\mathbf{d}} \tilde{S}(\mathcal{A}_2)$. Hence, $\dim_{\mathbf{d}}(\mathcal{A}(\mathcal{S})) \leq \dim_{\mathbf{d}}(\mathcal{S})$. □

Lemma 7. *For any $x \in \mathcal{S}$, there is an element $u \in \mathcal{A}(\mathcal{S})$ with $\dim_{\mathbf{0}}(\mathcal{A}(\mathcal{S}), u) = \dim_{\mathbf{0}}(\mathcal{S}, x)$.*

Proof. For an element $x \in \mathcal{S}$, choose $u = \langle x, 0, 0\rangle$. The rest of the proof is similar to Lemma 6. □

Lemma 8. *Suppose that $R \subseteq \mathrm{dom}(\mathcal{S})$. There exists a relation $Q \subseteq \mathrm{dom}(\mathcal{A}(\mathcal{S}))$ such that $\mathrm{DgSp}_{\mathcal{A}(\mathcal{S})}(Q) = \mathrm{DgSp}_{\mathcal{S}}(R)$. Moreover, if R is intrinsically Σ^0_1, then so is Q.*

Proof. For $R \subseteq \mathrm{dom}(\mathcal{S})$, set $Q = \{\langle x, 0, 0\rangle : x \in R\}$. First, suppose that h is an isomorphism from $\mathcal{S}$ onto a computable structure $\mathcal{S}_1$. Define $Q' = \{\langle h(x), 0, 0\rangle : x \in R\}$. Evidently, $(\mathcal{A}(\mathcal{S}_1), Q')$ is isomorphic to $(\mathcal{A}(\mathcal{S}), Q)$. Hence, $\mathrm{DgSp}_{\mathcal{S}}(R) \subseteq \mathrm{DgSp}_{\mathcal{A}(\mathcal{S})}(Q)$.

Assume that h is an isomorphism from $\mathcal{A}(\mathcal{S})$ onto a computable structure $\mathcal{A}_1$. Note that $h(Q) \subseteq Dom[\mathcal{A}_1]$. By Lemma 4.2, the structures $(\tilde{S}(\mathcal{A}(\mathcal{S})), Q)$ and $(\tilde{S}(\mathcal{A}_1), h(Q))$ are isomorphic via $h \restriction Dom[\tilde{S}(\mathcal{A}(\mathcal{S}))]$. In turn, the structures $(\mathcal{S}, R)$ and $(\tilde{S}(\mathcal{A}(\mathcal{S})), Q)$ are isomorphic via $f_{\mathcal{S}}$. This implies that $\deg_T(h(Q))$ lies in the degree spectrum of R. In other words, we have $\mathrm{DgSp}_{\mathcal{S}}(R) \supseteq \mathrm{DgSp}_{\mathcal{A}(\mathcal{S})}(Q)$. □

Lemmas 5–8 together establish Proposition 1. This concludes the proof of Theorem 1. □

4. Further discussion

First, we give an interesting consequence of Theorem 1. The paper [17] proved that the index set of computably categorical structures is Π^1_1-complete — informally speaking, this means that there is no simple syntactic characterization of computably categorical graphs. By employing this result, similarly to Section 4.1 of [11], one can obtain the following:

Corollary 2. *The index set of computably categorical modal algebras is Π^1_1-complete.*

Second, the following question is left largely open:

Problem 1. Study the HKSS-completeness for familiar varieties $\mathcal{V}$ of modal algebras.

For example, a modal algebra $(\mathcal{B}, f)$ is a *closure algebra* if it satisfies the following two additional axioms:

$$a \leq f(a) \text{ and } f(f(a)) \leq f(a).$$

Is the class of closure algebras HKSS-complete?

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