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INVERSE PROBLEM FOR THE STURM–LIOUVILLE EQUATION WITH PIECEWISE ENTIRE POTENTIAL AND PIECEWISE CONSTANT WEIGHT ON A CURVE

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Abstract. A Sturm–Liouville equation with a piecewise entire potential and a non-zero piecewise constant weight function on a curve of an arbitrary shape lying on the complex plane is considered. For such equation, the inverse spectral problem is posed with respect to the ratio of elements of one column or one row of the transfer matrix along the curve. The uniqueness of the solution to the problem is proved with the help of the method of unit transfer matrix using the study of asymptotic solutions of the Sturm–Liouville equation for large values of the absolute value of the spectral parameter. The obtained results allowed to consider inverse problem for a previously unexplored class of Sturm–Liouville equations with three unknown coefficients on a segment of the real axis.

Keywords: inverse spectral problem on a curve, the method of unit transfer matrix, asymptotics of solutions.

1. Introduction. Problem statement and main results

Inverse spectral problems for the Sturm–Liouville equation of the standard form

$$u''(z) + (Q(z) - \lambda^2)u(z) = 0 \tag{1}$$

on a segment of the real axis are well-studied in different statements [1, 2, 3]. But until recently, for curves on a complex plane only the problem on Sturm–Liouville monodromy-free equations (1) with a potential summable on a piecewise closed curve that constitutes a boundary of some bounded convex domain has been studied [4]. The solution of that problem considerably supplemented the results, obtained

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earlier, on potential classes having a power-low growth or disappearing at infinity, with a trivial monodromy on the entire plane [5, 6, 7]. However, the requirement on convexity of a closed curve significantly limits the area of possible application of the results of work [4], which makes their usage impossible, in particular, for solving the inverse problem for the Sturm–Liouville equation on a non-closed curve with given starting and ending points.

The first step in direction of removing the restrictions on the form of the curve (first of all, in the sense that it is a priori unknown) when considering Sturm–Liouville operators of the standard form was performed in work [8] by narrowing the class of the considered potentials to piecewise entire functions, that is, functions that coincide on different regions of the curve with different entire functions of the complex variable z. That narrowing allowed to state the inverse problem for Sturm–Liouville equation of the standard form by column or row of a transfer matrix along non-given continuous rectifiable curve of an arbitrary form (including self-intersecting one) and to formulate the conditions on uniqueness of its solution. Moreover, apart from traditional study of the asymptotics of solutions to the Sturm–Liouville equation, for large values of the absolute value of the spectral parameter, for the first time the method of the unit matrix was used. The method focuses on finding necessary and sufficient conditions for the equation of the class under research on some unknown curve to have a unit transfer matrix for any values of the spectral parameter. When that is done, obtaining the conditions on the uniqueness of solution of the inverse problem by the transfer matrix becomes relatively simple. In works [8, 9], it was additionally proved that the transfer matrix of Sturm–Liouville equation, continuously differentiable along a rectifiable curve with a piecewise entire potential, is identically equal to the unit matrix if and only if at least one of its elements is bounded on the entire complex plane of the spectral parameter. From here, it is easy to obtain that the transfer matrix is uniquely defined by the relation of the elements of its single column or single row, given on the set of values of the spectral parameter that has at least one finite limit point, and therefore, to specify the conditions on uniqueness of solution of the inverse problem by this relation. In work [10], the inverse problem by the whole transfer matrix for the Sturm–Liouville equation (1) with a piecewise entire potential and conditions of solution break, that do not depend on the spectral parameter, on a continuous rectifiable curve γ with a given starting point was stated and solved. Moreover, the curve γ, the potential Q, the position of the break points on the curve and the transfer matrix into them was a priori considered unknown.

In this paper, the results obtained in [8, 9] are generalized to the Sturm–Liouville equations with a piecewise entire potential and nonzero peicewise constant weight, which also allowed the study in Section 2 of the inverse problem on an interval of the real axis for a class of Sturm–Liouville equations with three unknown coefficients that has not been studied before from this point of view. Such inverse problems have a great practical value, since they emerge, for example, in spectroscopy of one-dimentional non-homogenious media [11]. Section 3 of the paper is dedicated to proving a number of auxiliary lemmas, and in Section 4, the main results formulated in Theorems 1 and 2 are proved. Note that the piecewise entirety of the potential Q is used in this paper only to have a possibility to deform the curves without changing the transfer matrices. Hence, all results obtained in this article are also true if each of the functions Q_i $(i = \overline{0,N})$ in (3) is bounded on the region γ_i

of the curve γ, that connects characteristic points z_i and z_{i+1}, and constitutes a monodromy-free potential [5, 12] of Sturm–Liouville equation (2) in every finite region of the complex plane. Section 5 contains a reflection on the possibility of weakening of the requirement on piecewise entirety of the potential Q to a condition of its piecewise analyticity, at the same time remaining in the framework of the auxiliary results obtained in Section 3. It turns out that such weakening is possible for polygonal chains possessing an additional a priory information (see definitions 16, 17). However, that does not mean that generalization of Theorem 2 for a priori unknown curves in the case of piecewise analytical functions is completely impossible. It merely means that it requires non-trivial additional research that is out of the scope of our work. In the concluding sixth section, it is proved that the statement of the inverse problem used in this article is mainly equivalent to the classical inverse problem by two spectra, but in the case of a priori non-given curves, it is more convenient and natural.

Definition 1. *We will call the function F piecewise constant (piecewise entire) on the curve $\gamma \subset \mathbf{C}$, defined parametrically by the function $z = V(t), t \in [t_0, t_f]$, if there exists an integer $M \geq 0$ and a set of numbers $\Omega = \{\tau_j\}_0^{M+1} : t_0 = \tau_0 < \tau_1 < \ldots < \tau_{M+1} = t_f$, such that*

$$F(z) = F_i(z), \ \text{if } z = V(t), t \in (\tau_i, \tau_{i+1}), i = \overline{0, M},$$

where all F_i are constant (entire) functions, moreover, if $M \geq 1$, then for every $m \in \{1, \ldots, M\}$ functions F_m and F_{m-1} are distinct.

Definition 2. *Let $\gamma \subset \boldsymbol{C}$ be a continuous rectifiable curve, function R be nonzero and piecewise constant on γ, function Q be piecewise entire on γ. Then we will refer as an equation of class G to the following equation, considered on the curve γ:*

$$u''(z) + (Q(z) - \lambda^2 R(z))u(z) = 0. \tag{2}$$

In equation (2) and further, the prime will denote a derivative along some continuous rectifiable curve, given parametrically by the function $z = V(t)$, that is, it is supposed that $f'(z) = \lim_{\Delta t \to 0}[f(V(t + \Delta t)) - f(V(t))]/[V(t + \Delta t) - V(t)]$. If the function $f(z)$ is analytical in some domain of the complex plane, then in every point of this domain it has derivatives along each of the rectifiable curves passing through that point, which coincide and equal a usual derivative $df(z)/dz$ of the function $f(z)$ at this point.

Note that the results obtained in the article can be easily generalized to the case when the functions R and Q coincide almost everywhere on the curve with piecewise constant and piecewise entire functions, respectively.

The solutions of equation (2), obviously, depend on the parameter λ. However, in our paper, this dependence will be explicitly mentioned only if it is necessary to emphasize the presence of such dependence.

Definition 3. *Suppose that on the continuous rectifiable curve $\gamma \subset \boldsymbol{C}$, determined parametrically by the function $z = V(t)$ $(t \in [t_0, t_f])$, a nonzero piecewise constant function R and piecewise entire function Q are defined, that is, due to definition 1, there exists an integer $N \geq 0$ and a set of numbers $T = \{t_j\}_0^{N+1} : t_0 < t_1 < \ldots < t_{N+1} = t_f$, such that*

$$Q(z) = Q_i(z), \ R(z) = R_i, \ \text{if } z = V(t), \ t \in (t_i, t_{i+1}) \quad (i = \overline{0, N}). \tag{3}$$

In (3)*, all* Q_i *are entire functions and all* R_i *are nonzero complex constants. Moreover, if* $N \geq 1$*, then for every number* $n \in \{1,\dots,N\}$ *ordered pairs* $\{Q_n, R_n\}$ *and* $\{Q_{n-1}, R_{n-1}\}$ *are distinct.*

Then we say that the points $z_i := V(t_i)$ $(i = \overline{0,N})$*,* $z_{N+1} \equiv z_f := V(t_f)$ *are characteristic, and the ordered set* $W := \{N, \{z_j\}_0^{N+1},\ \{Q_i, R_i\}_0^N\}$ *is the set of characteristic data of the curve* γ *and the corresponding equation* (2) *of class* G *on* γ*.*

Proposition 1. *The curve* γ *may have regions* (*points*)*, that are passed through more than one time, that is, they correspond to two or more intervals of values* (*values*) *of the parameter* t*. We will make a distinction between such regions* (*points*) *by the order of priority of the passing through, and we will consider the curves that geometrically coincide, having a different order of passing through the regions, as distinct ones.*

Definition 4. *Let* $u_1(z), u_2(z)$ *be a continuously differentiable solutions of equation* (2) *of class* G *along the rectifiable curve* γ *and*

$$u_1(z_b) = 1,\ u_1'(z_b) = 0,\ u_2(z_b) = 0,\ u_2'(z_b) = 1 \qquad (z_b \in \gamma). \tag{4}$$

We will refer as a transfer and reduced matrices of equation (2) *between the points* z_b *and* z *of the curve* γ *respectively to the matrices*

$$\hat{P}(\gamma, z, z_b) = \begin{pmatrix} u_1(z) & u_2(z) \\ u_1'(z) & u_2'(z) \end{pmatrix},\quad \hat{S}(\gamma, z, z_b) = \begin{pmatrix} u_2(z)/u_1(z) & u_2(z)/u_2'(z) \\ u_1'(z)/u_1(z) & u_1'(z)/u_2'(z) \end{pmatrix}.$$

Lemma 1. *The elements of the transfer matrix* $\hat{P}(\gamma, z_f, z_0)$ *of equation* (2) *of class* G *are entire functions of the parameter* ρ $(\rho := \lambda^2)$*, which are uniquely determined by assignment of the set of characteristic data* W *of the curve* γ*;* $det\hat{P} \equiv 1$ *and for every* $z_c, z \in \gamma$

$$\hat{P}(\gamma, z, z_0) = \hat{P}(\gamma, z, z_c)\hat{P}(\gamma, z_c, z_0), \tag{5}$$

$$\hat{P}(\gamma, z_0, z) = \hat{P}^{-1}(\gamma, z, z_0) = \begin{pmatrix} u_2'(z) & -u_2(z) \\ -u_1'(z) & u_1(z) \end{pmatrix}. \tag{6}$$

Proof. Let $u_\alpha^{(i)}(z)$ $(\alpha \in \{1,2\}, i \in \{0,\dots,N\})$ be the entire solutions of the auxiliary Sturm–Liouville equation

$$\frac{d^2u^{(i)}}{dz^2} + (Q_i - \lambda^2 R_i)u^{(i)} = 0, \qquad z \in \mathbf{C}, \tag{7}$$

with the initial conditions of the form (4) at the point z_i. We will determine the entire functions $\widetilde{u}_\alpha^{(i)}(z)$ by the following recurrent relations

$$\widetilde{u}_\alpha^{(0)}(z) := u_\alpha^{(0)}(z),$$

$$\widetilde{u}_\alpha^{(n)}(z) := \widetilde{u}_\alpha^{(n-1)}(z_n)u_1^{(n)}(z) + \frac{d\widetilde{u}_\alpha^{(n-1)}(z)}{dz}\Big|_{z=z_n} u_2^{(n)}(z),$$

where $n \in \{1,\dots,N\}$ given $N \geq 1$. Then, if γ_i is a region of an arbitrary continuous rectifiable curve γ, connecting the points z_i and z_{i+1}, then due to (3), the functions $u_\alpha(z)$, such that for every $i \in \{0,\dots,N\}$ with $z \in \gamma_i$ the equality $u_\alpha(z) := \widetilde{u}_\alpha^{(i)}(z)$ is true, are the continuously differentiable solutions of equation (2) of class G along γ, satisfying conditions (4) at the point z_0. Hence, Lemma 1 follows from the similar properties of solutions of linear differential equations with holomorphic coefficients (see, for example, §§2, 24 of work [13]). □

Definition 5. *We call a loop of the curve γ with a knot at the point $z^{(d)}$ a region of the curve γ that starts and ends at the point of its self-intersection $z^{(d)}$. We will refer to all the points of the loop apart from its knot as the inner ones.*

Definition 6. *Suppose that on the curve γ, a piecewise constant function R and a piecewise entire function Q are given. We call the loop of the curve γ an „invisible loop", if its knot coincides with two sequential characteristic points of γ.*

Lemma 2. *Adding to or removing from the curve its „invisible loop" does not change the transfer and reduced matrices of the corresponding equation* (2) *of class G along that curve, and also its starting and ending points.*

Proof. The lemma follows from Definitions 4, 6, Formula (5), and the fact that the transfer matrix of equation (2) with entire coefficients along every loop equals the unit matrix [13, 14]. □

Definition 7. *We will call a curve with a piecewise constant and a piecewise entire functions given on it ordinary, if there are no „invisible loops" on the curve.*

Lemma 3. *The curve γ is ordinary if and only if all its characteristic points satisfy the following conditions:*

$$\Delta z_i := z_{i+1} - z_i \neq 0 \quad (i = \overline{0, N}). \tag{8}$$

Proof. The lemma directly follows from Definitions 5 – 7, since due to Definition 6, the fulfillment of all conditions (8) is equivalent to the absence of the „invisible loops". □

Definition 8. *Suppose that after successive removal of all „invisible loops", the curve γ turns into an ordinary curve γ_{min} with a set of characteristic data $W_{min} := \{N, \{z_j\}_0^{N+1}, \{Q_i, R_i\}_0^N\}$. We call W_{min} the set of basic data, and the points $\{z_j\}_0^{N+1}$ basic points of the curve γ and the corresponding equation* (2) *of class G on γ. If after successive removal of all „invisible loops" γ degenerates into a point, then we will assume that the curve γ does not have basic points and its set of basic data is empty.*

After removing all the „invisible loops" from the initial curve γ and renumbering the characteristic points of the new curve, it can show up its own „invisible loops". For example, the curve γ with a set of characteristic data $\{2, \{z_0, z_1,\ z_2 = z_1, z_3 = z_0\}, \{Q_0, R_0; Q_1, R_1; Q_2 = Q_0, R_2 = R_0\}\}$ has one „invisible loop" with a knot at the point z_1. On the curve that results from its removal, the point z_1 will not be a characteristic one, and this curve also will have one „invisible loop" with a knot at the point z_0. Hence, in Definition 8 and further, we mention successive removal of all „invisible loops". Note also that for an ordinary curve, the sets of characteristic and basic data coincide.

After successively removing all the „invisible loops", the original curve either transforms into an ordinary curve or degenerates into a point. Due to Lemma 2, in the latter case we have $\hat{P} \equiv \hat{I}$ and $\hat{S} \equiv \hat{0}$, where $\hat{I}$ and $\hat{0}$ are unit and zero matrices, respectively.

The main results of this article are formulated in two following theorems, proved in Section 4.

Theorem 1. *A transfer matrix $\hat{P}$ of equation* (2) *of class G along some* (*non-given*) *curve γ equals the unit matrix with all $\rho \in \boldsymbol{C}$ if and only if after successive removal of all the „invisible loops" γ degenerates into a point. In all the remaining cases, all elements of $\hat{P}$ are entire functions ρ of order $1/2$ and of normal type.*

Theorem 2. *Suppose that two equations of class G have sets of characteristic data $W^{(1)}$ and $W^{(2)}$ respectively on two ordinary curves $\gamma^{(1)}$ and $\gamma^{(2)}$ with a common starting point, and also reduced matrices $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ along these curves. Then, if there exist numbers $\alpha, \beta \in \{1, 2\}$ such that the functions $S^{(1)}_{\alpha\beta}(\rho)$ and $S^{(2)}_{\alpha\beta}(\rho)$ meromorphic in $\boldsymbol{C}$ coincide, then $W^{(1)} = W^{(2)}$.*

Due to Lemma 1, $det\hat{P} \equiv 1$, and therefore, the elements of the matrix $\hat{P}$, located in the same row or column do not simultaneously turn into zero, that is they do not possess common zeros as function ρ, hence, by Definition 4 and Theorem 1, the following proposition is true.

Proposition 2. *All elements $S_{\alpha\beta}$ $(\alpha, \beta \in \{1, 2\})$ of the reduced matrix of the equation* (2) *of class G are either meromorphic functions ρ or identically equal zero. Therefore, by uniqueness theorem* (*see, for example, part* 20 *in book* [15]), *in Theorem* 2 *and in all following statements it suffices to require that the elements of the reduced matrices on the set of points of a complex ρ-plane that has at least one finite limit point ρ_s are given* (*coincide*). *Moreover, if ρ_s coincides with the pole of order n of the meromorphic function $S_{\alpha\beta}$, then the uniqueness theorem can be applied to the function $(\rho - \rho_s)^n S_{\alpha\beta}$.*

Corollary 1. *If at least one pair of corresponding elements of the reduced matrices of two equations of class G along two curves, for which only one common starting point is given, coincides on the complex ρ–plane, then these curves have similar sets of basic data.*

Proof. If after successive removal of all the „invisible loops" at least one of the curves degenerates into a point, then by Theorem 1 under the conditions of the corollary, the second curve also degenerates into a point, that is, by Definition 8, the sets of basic data of both curves are empty. Suppose that after successive removal of all the „invisible loops" the initial curves transform into ordinary curves $\gamma^{(1)}_{min}$ and $\gamma^{(2)}_{min}$, possessing different sets of characteristic data. Then due to Lemma 2 under the conditions of the corollary, this contradicts Theorem 2. Therefore, the sets of characteristic data of ordinary curves $\gamma^{(1)}_{min}$ and $\gamma^{(2)}_{min}$ coincide, and hence, by Definition 8, the original curves have similar sets of basic data. □

Definition 9. *We will refer to piecewise entire* (*piecewise constant*) *functions on the curve γ as equivalent ones, if they coincide everywhere on γ, apart from, possibly, its characteristic points.*

Corollary 2. *Let $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ be reduced matrices of two equations of class G along **given** curve γ, and there exist numbers $\alpha, \beta \in \{1, 2\}$ such that the functions $S^{(1)}_{\alpha\beta}(\rho)$ and $S^{(2)}_{\alpha\beta}(\rho)$ are known and coincide in $\boldsymbol{C}$. Then the ordered sets of basic points of these equations coincide and are uniquely defined, and their corresponding coefficients are equivalent everywhere on the curve γ, excluding, possibly, its loops that are uniquely defined, among whose inner points there are no basic points of these equations.*

Proof. By Corollary 1, the sets of basic points of the two mentioned equations coincide and are uniquely defined. It is well known that the transfer matrix of equation (2) with entire coefficients along every loop equals the unit matrix [13, 14]. Due to that fact and property (5), substitution of the potential Q and (or) the value of the weight function R on the loop of the curve γ, among whose inner points there are no basic points, respectively by any entire function and (or) number does not change the transfer matrix of the equation of class G along γ. Therefore, on such loops, corresponding coefficients of the two equations mentioned in the corollary can differ (or can coincide). If after removal of all loops, among whose inner points there are no basic points, the curve γ degenerates into a point, then the corollary is proved; if its transforms into some curve γ_{min}, then this curve is an ordinary one, because by construction it does not have any „invisible loops". Non-equivalency of the corresponding coefficients of two equations mentioned in the corollary on the ordinary curve γ_{min} by Definitions 3, 9 would mean the sets of characteristic data of these equations on γ_{min} to be distinct, which is impossible under the conditions of the corollary due to Lemma 2 and Theorem 2. □

Corollary 3. *If at least one pair of corresponding elements of the reduced matrices* $\hat{S}(\gamma, z_f, z_0)$ *and* $\hat{S}^{(1)}(\gamma^{(1)}, z_f^{(1)}, z_0^{(1)})$ *of two equations of class* G *coincide on a complex* ρ*–plane, then the transfer matrices* $\hat{P}(\gamma, z_f, z_0)$ *and* $\hat{P}^{(1)}(\gamma^{(1)}, z_f^{(1)}, z_0^{(1)})$ *of these equations also coincide.*

Proof. Suppose that $\hat{P}$ and $\hat{S}$ are respectively the transform and reduced matrices of equation (2) of class G along the curve γ. Performing in (2) a substitution of the variable $x = z + \tau$ ($\tau := z_0^{(1)} - z_0$), after inverse redesignation of x on z we obtain that $\hat{P}$ and $\hat{S}$ are respectively the transform and reduced matrices of the equation of class G with a potential $Q^{(2)}(z) = Q(z-\tau)$ and weight function $R^{(2)}(z) = R(z-\tau)$ along the curve $\gamma^{(2)}$ starting at the point $z_0^{(1)}$, which is obtained from the curve γ by a parallel transfer. Since the matrices $\hat{S}$ and $\hat{S}^{(1)}$ possess a common element, by Corollary 1, the sets of basic data of the curves $\gamma^{(2)}$ and $\gamma^{(1)}$ coincide, and therefore, $\hat{P} \equiv \hat{P}^{(1)}$ due to Lemmas 1, 2 and Definition 8. □

2. Inverse problem for one class of Sturm–Liouville equations on the interval of the real axis

Theorems 1, 2 and Corollaries 1 – 3 are helpful in studying different kinds of inverse problems for equations or systems of equations, reducible into equation (2) of class G. Consider, for example, the system of two equations of first order on the interval of the real axis:

$$\left\{ \begin{aligned} \frac{dy}{dx} &= g(x)w(x), \\ \frac{dg}{dx} &= -w(x)\left(q(x) - r(x)\lambda^2\right)y(x) \end{aligned} \right. \qquad (x \in [x_0, x_f]) \tag{9}$$

and the Sturm–Liouville equation of a general form equivalent to it given $w(x) \neq 0$:

$$\frac{d}{dx}\left(\frac{1}{w(x)}\frac{dy}{dx}\right) + w(x)\left(q(x) - r(x)\lambda^2\right)y(x) = 0. \tag{10}$$

Note that if $w(x)$ equals zero almost everywhere on the interval $[x_s, x_s + l] \subseteq [x_0, x_f]$, then system (9) has a constant solution on it, and it can be excluded from consideration by simultaneous substitution of the variable x by the variable

$$\widetilde{x} = \begin{cases} x, & \text{if } x \in [x_0, x_s]; \\ x - l, & \text{if } x \in (x_s + l, x_f] \end{cases}$$

and of the coefficients $w(x),\ r(x),\ q(x)$ by the coefficients $\widetilde{w}(\widetilde{x}),\ \widetilde{r}(\widetilde{x}),\ \widetilde{q}(\widetilde{x})$ using the following rule:

$$\widetilde{f}(\widetilde{x}) = \begin{cases} f(\widetilde{x}), & \text{if } \widetilde{x} \in [x_0, x_s]; \\ f(\widetilde{x} + l), & \text{if } \widetilde{x} \in (x_s, x_f - l]. \end{cases}$$

This procedure can be obviously generalized to the case when $w(x)$ equals zero almost everywhere on any number of parts of the interval $[x_0, x_f]$, and we will assume that it has already been done.

Definition 10. *Suppose that the function $w(x)$ is measurable, bounded, and almost everywhere nonzero on the interval $[x_0, x_f]$, and that the functions $r(x)$ and $q(x)$ can be represented in the form*

$$r(x) := R(V(x)),\ q(x) := Q(V(x)) \qquad (x \in [x_0, x_f]). \tag{11}$$

Here, the function R is nonzero and piecewise constant on the curve γ, defined parametrically by the relation

$$z = V(x),\ \text{where}\ V(x) := \int_{x_0}^{x} w(v)dv \qquad (x \in [x_0, x_f]), \tag{12}$$

and the function Q is piecewise entire on γ. Then we will refer to the system of equations (9) *as a system of class A on the interval $[x_0, x_f]$, equivalent to equation* (2) *of class G on the curve γ.*

System of equations (9) is a system of class A on $[x_0, x_f]$, if, for example, on this interval the functions w, r are piecewise constant and nonzero, and the function q is piecewise entire. It is worth noting that every system of class A on some given interval is equivalent to one equation of class G on the uniquely defined curve γ. However, the reverse case in incorrect, since parametric representation of a curve is not unique. That is, when studying one equation of class G on a given curve, we study the family of systems of equations of class A. That means that the described approach is technically inherently more convenient compared to direct investigation of systems of equations of class A on the interval of the real axis.

Definition 11. *Let (y_α, g_α) $(\alpha \in \{1, 2\})$ be the continuous solutions of systems of equations* (9) *of class A on the interval $[x_0, x_f]$, and*

$$y_1(x_0) = 1,\ g_1(x_0) = 0,\ y_2(x_0) = 0,\ g_2(x_0) = 1.$$

We will call the transfer and reduced matrices of this system between the points x_0 and x of the interval $[x_0, x_f]$ respectively the matrices

$$\hat{P}_r(x, x_0) = \begin{pmatrix} y_1(x) & y_2(x) \\ g_1(x) & g_2(x) \end{pmatrix} \quad \text{and} \quad \hat{S}_r(x, x_0) = \begin{pmatrix} y_2(x)/y_1(x) & y_2(x)/g_2(x) \\ g_1(x)/y_1(x) & g_1(x)/g_2(x) \end{pmatrix}.$$

Lemma 4. *Transfer matrices of system of equations* (9) *on the interval $[x_0, x_f]$ and equation* (2) *of class G equivalent to it* (*in the sense of Definition* 10) *coincide for all $\lambda \in \boldsymbol{C}$.*

Proof. Suppose that (y, g) is a continuous solution of system of equations (9) of class A on the interval $[x_0, x_f]$ with the initial conditions (y_0, g_0) at the point x_0, the curve γ is defined by relation (12), and $u(z)$ is a solution of equation (2) of class G, equivalent to the system of equations (9), continuously differentiable along γ, with the initial conditions $u(0) = y_0, u'(0) = g_0$. Since the function w is measurable and bounded, it is summable on the interval $[x_0, x_f]$, and hence, the function $V(x)$ defined in (12) is absolutely continuous, therefore, the curve γ is continuous and rectifiable, moreover, $dV/dx = w(x)$ almost everywhere on $[x_0, x_f]$ [16]. Since $u(z)$ is a solution of equation (2) of class G, continuously differentiable along γ, then the function $v(z) := u'(z)$ is continuous on γ, and therefore, the functions $\widetilde{y}(x) := u(z)$ and $\widetilde{g}(x) := u'(z)$, where $z = V(x)$, are continuous on the interval $[x_0, x_f]$, and almost everywhere on this interval we have $d\widetilde{y}/dx = u'(z)w(x) = w(x)\widetilde{g}(x)$, $d\widetilde{g}/dx = u''(z)w(x) = -w(x)\left(Q(V(x)) - R(V(x))\lambda^2\right)\widetilde{y}(x)$. The latter fact due to (11) means that $(\widetilde{y}, \widetilde{g})$ is a continuous solution of system of equations (9) of class A with the initial conditions (y_0, g_0) at the point x_0. But it is well known [14] that this system, under the given initial conditions, has a unique continuous solution, and hence, $\widetilde{y} \equiv y$ and $\widetilde{g} \equiv g$ on the interval $[x_0, x_f]$, that is, $u(z) = y(x), u'(z) = g(x)$ for every $z = V(x)$ $(x \in [x_0, x_f])$, which, taking into account Definitions 4 and 11, proves the lemma. □

Definition 12. *Let W be the set of basic data of equation* (2) *of class G, equivalent* (*in the sense of Definition* 10) *to system of equations* (9) *of class A. Then we will refer to W as the set of basic data of this system of equations* (9) *on the interval* $[x_0, x_f]$, *and to the basic points γ as the basic points of system* (9), *and to the points of the interval* $[x_0, x_f]$, *mapped* (12) *into basic points, as the preimages of the basic points.*

Definition 13. *We will call a system of equations of class A on the interval* $[0, 1]$ *a system of class B in two following cases:* 1) *its set of basic data is not empty, the function w is piecewise constant with a constant absolute value, and the polygonal chain* (*possibly degenerated into an interval*) *defined by* (12) *coincides with a polygonal chain that results from successively connecting of the basic points of the system of equations;* 2) $w(x) = 1$ *given* $x \in [0, 1/2]$, $w(x) = -1$ *given* $x \in (1/2, 1]$, $q(x) \equiv 0$ *and* $r(x) \equiv 1$. *In the latter case, we will refer to such system as a unit system of class B.*

Lemma 5. *A system of equations of class B has an empty set of basic data if and only if it is a unit one.*

Proof. By Definition 13, a set of basic data of any system of equations of class B, possibly excluding the unit one, is not empty. On the other hand, a unit system of equations of class B is equivalent (in the sense of Definition 10) to equation (2) of class G with $Q \equiv 0$, $R \equiv 1$ on the curve γ, generated by successive passing of the intervals $[0, 1/2]$ and $[1/2, 0]$. Obviously, γ consists of one „invisible loop”, which proves the lemma. □

Definition 14. *Two systems of equations of class B are called coinciding on an interval, if their corresponding coefficients coincide everywhere on this interval, possibly apart from the preimages of the basic points.*

Lemma 6. *For every ordered set of data of the form* $W := \{N, \{z_j\}_0^{N+1}, \{Q_i, R_i\}_0^N\}$, *where $N \geq 0$ is an integer,* $z_0 = 0$, *the points* $\{z_j\}_0^{N+1}$ *satisfy conditions* (8), *all*

Q_i are entire functions, all $R_i \neq 0$ are complex constants, and, if $N \geq 1$, then for every $n \in \{1, \dots, N\}$ the ordered pairs $\{Q_n, R_n\}$ and $\{Q_{n-1}, R_{n-1}\}$ are distinct, there exists exactly one system of equations of class B with such set of basic data.

Proof. By Definition 13, for every system of equations of class B, apart from unit one, the function w is piecewise constant on the interval $[0, 1]$, has a constant absolute value, and the polygonal chain defined by (12) coincides with the polygonal chain L resulting from successive connecting of the basic points of the system of equations. From the latter condition, the fact that $|w|$ is constant and (12), it follows that $|w| = p_0$, where $p_0 := \sum_{i=0}^{N} |z_{i+1} - z_i|$ is the length of the polygonal chain L. We put

$$x_0 = 0,\ w_i := p_0 \frac{z_{i+1} - z_i}{|z_{i+1} - z_i|},\ x_{i+1} := x_i + \frac{|z_{i+1} - z_i|}{p_0} \quad (i = \overline{0, N}),$$

$$\begin{gathered} w(x) = w_i,\ q(x) = Q_i(V(x)),\ r(x) = R_i,\ \text{if } x \in [x_i, x_{i+1})\ \ (i = \overline{0, N}), \\ w(1) = w_N,\ q(1) = Q_N(z_{N+1}),\ r(1) = R_N, \end{gathered} \tag{13}$$

where the function $V(x)$ is defined in (12). It is easy to make sure that the set of basic data of system of equations (9) with the coefficients determined by relation (13) coincides with the given in the statement of the lemma, the function $w(x)$ is piecewise constant, its absolute value is constant and equals p_0, $x_{N+1} = 1$, and the polygonal chain defined by (12) coincides with the polygonal chain L resulting from successive connecting of the basic points $\{z_j\}_0^{N+1}$. The existence is proved. Assume that there exists a second system of equations of class B satisfying the conditions of the lemma with the coefficients $\widetilde{q}(x)$, $\widetilde{r}(x)$ and piecewise constant function $\widetilde{w}(x)$, such that $|\widetilde{w}(x)| = p_0$. Considering successively all links of the polygonal chain, successively connecting the basic points, we obtain that $\widetilde{w}(x) = w(x)$ on the whole interval $[0, 1]$, possibly apart from the points $\{x_i\}_0^{N+1}$, which are the preimages of the basic points. Therefore, due to relations (3), (11), (12) the function $\widetilde{V}(x) \equiv V(x)$, and the functions $\widetilde{q}(x), \widetilde{r}(x)$ coincide, respectively, with the functions $q(x), r(x)$ up to the values at the points $\{x_i\}_0^{N+1}$, and hence, by Definition 14 the lemma is proved. □

Lemma 7. *Defining on the set of points of a complex ρ–plane, that possesses at least one finite limit point, of any element of the reduced matrix $\hat{S}_r$ of system of equations (9) of class A on some interval uniquely determines the transfer matrix $\hat{P}_r$ of that system on that interval.*

Proof. Assume that there exist two systems of equations (9) of class A on some two intervals, such that they have coinciding corresponding elements of reduced matrices, but distinct transfer matrices. Then due to Lemma 4, equations (2) of class G, equivalent to them (in the sense of Definition 10) also have coinciding corresponding elements of the reduced matrices, but distinct transfer matrices. But this contradicts Corollary 3, which, taking into account Proposition 2, proves the lemma. □

Proposition 3. *Using Lemma 7, further for brevity we will be using the term defining* (*coinciding*) *for the transfer matrices of system of equations (9), meaning defining of one of the elements* (*coinciding of the corresponding elements*) *of the reduced matrices on the set of points of the complex ρ–plane having at least one finite limit point.*

Theorem 3. *Two systems of equations (9) of class A on some two intervals have similar* (*in the sense of Proposition* 3) *transfer matrices if and only if their sets of basic data coincide.*

Proof. Let the sets of basic data of the two systems of equations of class A coincide. Then due to Lemmas 1, 2, 4 and Definitions 8, 12, their transfer matrices coincide. The inverse is true due to Lemma 4, Corollary 1, and Definition 12. □

Theorem 4. *Let a system of equations of class A have on some interval a transfer matrix $\hat{P}_r$. Then there exists a unique system of equations of class B with a similar* (*in the sense of Proposition 3*) *transfer matrix.*

Proof. Since every system of equations of class B is also a system of class A, then by Theorem 3 the sets of basic data of all the systems of equations of class B with a transfer matrix $\hat{P}_r$ coincide. If the set of basic data is empty, that is, $\hat{P}_r \equiv \hat{I}$, then existence and uniqueness of the system of equations of class B with a similar transfer matrix follows from Lemma 5, or from Lemma 6 for the opposite case. □

Consider two systems of equations of class A. We will mark all the values belonging to the first and the second systems by upper index of one or two, respectively. For definiteness, suppose that $x_f^{(2)} \geq x_f^{(1)}$.

Theorem 5. *Suppose that the transfer matrices of the two systems of equations of class A on the intervals with a common starting point coincide and are known* (*in the sense of Proposition* 3)*; the function $w^{(1)}$ is defined on the interval $[x_0, x_f^{(1)}]$, and $w^{(2)} = w^{(1)}$ almost everywhere on $[x_0, x_f^{(1)}]$ $(x_f^{(2)} \geq x_f^{(1)})$. Then $\hat{P}_r^{(2)}(x_f^{(2)}, x_f^{(1)}) \equiv \hat{I}$, and $r^{(1)} = r^{(2)}$, $q^{(1)} = q^{(2)}$ on the interval $[x_0, x_f^{(1)}]$, possibly apart from the preimages of the characteristic points and the regions of this interval that are uniquely defined, each of which transfers while mapping* (*12*) *with $w := w^{(1)}$ into a loop of the curve $\gamma^{(1)}$ which does not have any basic points among its inner points.*

Proof. Consider two equations of class G, equivalent (in the sense of Definition 10) to the two systems of equations of class A described in the theorem. By Lemma 4 and the condition of the theorem, they have similar transfer matrices on the curves $\gamma^{(1)}$ and $\gamma^{(2)}$, defined by relations (12) with the functions $w^{(1)}$ and $w^{(2)}$, respectively. By Corollary 1, the sets of basic data of these curves coincide, and since $w^{(2)} = w^{(1)}$ almost everywhere on $[x_0, x_f^{(1)}]$, then either $\gamma^{(2)} = \gamma^{(1)}$ or the curve $\gamma^{(2)}$ results from the curve $\gamma^{(1)}$ by adding to their common finite point a region, degenerating into a point if we successively remove the „invisible" loops. In any of these cases, $\hat{P}_r^{(2)}(x_f^{(2)}, x_f^{(1)}) \equiv \hat{I}$ by Lemma 4, and therefore, the transfer matrices of the two considered equations of class G on the curve $\gamma^{(1)}$ are equal. Hence, by Corollary 2, the functions $R^{(2)}$, $Q^{(2)}$ are equivalent, respectively, to the functions $R^{(1)}$, $Q^{(1)}$ on the given curve $\gamma^{(1)}$, possibly except from its uniquely defined loops, among whose inner points there are no basic points. Since the function $w^{(1)}$ is defined on the interval $[x_0, x_f^{(1)}]$, then from (12) the preimages of these loops on $[x_0, x_f^{(1)}]$ can also be uniquely defined (taking into account Proposition 1). On the other hand, due to relations (11) and Definition 9, equivalence of the functions $R^{(2)}$, $Q^{(2)}$ respectively to the functions $R^{(1)}$, $Q^{(1)}$ on some region of the curve $\gamma^{(1)}$ yields that $r^{(1)} = r^{(2)}$, $q^{(1)} = q^{(2)}$ on the preimage of this region, possibly except from the preimages of the characteristic points. Moreover, since $w^{(1)}$ is almost everywhere nonzero on the interval $[x_0, x_f^{(1)}]$, then the preimage of every point of the curve $\gamma^{(1)}$ is a point on the interval $[x_0, x_f^{(1)}]$. □

Note that the class of equations (10), which is equivalent for $w(x) \neq 0$ to system of equations (9) of class A, considerably differs from the class of Sturm–Liouville equations of the general form with piecewise analytical coefficients, considered in paper [17]. In particular, in this work, bounded and measurable functions w are considered, but at the same time the weight function and the function w are interrelated strongly enough, and in paper [17], the case when they are piecewise analytical, but meanwhile completely independent, is studied. However, if we consider such systems of equations of class A for which both Theorem 5 and the results of work [17] are applicable, then the conditions following from Theorem 5, that are sufficient for providing uniqueness of solution of the inverse problem by a known (in the sense of Proposition 3) transfer matrix and a known function $w(x)$ will be noticeably weaker. In accordance with the results of work [17], for that, it suffices that all the values of the functions $w(x)$ and $w(x)r(x)$ belong to partially open halves of the complex plane with boundaries passing through zero (see condition A from work [17] and propositions to it), and additionally to $w(x)$ one of the functions $r(x)$ or $q(x)$ is defined. By Theorem 5, for that, it suffices that the curve defined by (12) does not have self-intersections. Another important advantage of Theorem 5 is that it is applicable also to the case when the conditions of uniqueness of solution of the inverse problem are not fulfilled.

3. Asymptotics of the transfer matrix of the Sturm–Liouville equation with a piecewise entire potential and nonzero piecewise constant weight on the curve

Lemma 8. *For every $i \in \{0, \dots, N\}$, $K \in \boldsymbol{N}$ and every set of numbers $\{\mu_{i,k}\}_1^K$, there exist numbers $\lambda_{i,K} > 0, C_{i,K}^{(0)} > 0$ and two continuously differentiable solutions $F_{\pm K}^{(i)}(z, \lambda)$ of corresponding equation (7), which for all $\lambda \neq 0$ and $z \in \boldsymbol{C}$ can be represented in the form:*

$$F_{\pm K}^{(i)} = C_{\pm K}^{(i)}(z, \lambda) \exp\{\pm\lambda\theta_i(z - z_i)\},$$
$$\frac{dF_{\pm K}^{(i)}}{dz} = \pm\lambda\theta_i E_{\pm K}^{(i)}(z, \lambda) \exp\{\pm\lambda\theta_i(z - z_i)\}, \tag{14}$$

where

$$\theta_i = (R_i)^{1/2} \neq 0,\ arg(\theta_i) \in (-\frac{\pi}{2}; \frac{\pi}{2}], \tag{15}$$

$$C_{\pm K}^{(i)}(z, \lambda) := 1 + \sum_{k=1}^{K} (\pm 1)^k \frac{C_{i,k}(z)}{(\lambda\theta_i)^k} + (\pm 1)^{K+1} \frac{B_{\pm K}^{(i)}(z, \lambda)}{(\lambda\theta_i)^{K+1}}, \tag{16}$$

$$E_{\pm K}^{(i)}(z, \lambda) := 1 + \sum_{k=1}^{K} \left(\pm\frac{1}{\lambda\theta_i}\right)^k \left(C_{i,k}(z) + \frac{dC_{i,k-1}(z)}{dz}\right) + +(\pm 1)^{K+1} \frac{H_{\pm K}^{(i)}(z, \lambda)}{(\lambda\theta_i)^{K+1}}. \tag{17}$$

Here all $C_{i,k}(z)$ *are entire functions* z, $C_{i,0}(z) := 1$, $C_{i,j}(z_i) = \mu_{i,j}$ $(j = \overline{1, K})$,

$$\frac{dC_{i,k}}{dz} := -\frac{1}{2}\left(\frac{d^2C_{i,k-1}}{dz^2} + Q_i(z)C_{i,k-1}(z)\right) \qquad (k = \overline{1, K+1}), \tag{18}$$

$$H^{(i)}_{\pm K}(z,\lambda) := \frac{dC_{i,K}(z)}{dz} + B^{(i)}_{\pm K}(z,\lambda) \pm \frac{1}{\lambda\theta_i}\frac{dB^{(i)}_{\pm K}(z,\lambda)}{dz}, \tag{19}$$

and $B^{(i)}_{\pm K}(z,\lambda)$ *for every fixed* λ *are analytical by* z *solutions of the following Cauchy problems:*

$$\begin{gathered} \frac{d^2B^{(i)}_{\pm K}}{dz^2} \pm 2\lambda\theta_i\frac{dB^{(i)}_{\pm K}}{dz} + Q_iB^{(i)}_{\pm K} = \pm 2\lambda\theta_i\frac{dC_{i,K+1}(z)}{dz}, \qquad z \in \boldsymbol{C}, \\ B^{(i)}_{\pm K}(a_{\pm i},\lambda) = \left.\frac{dB^{(i)}_{\pm K}(z,\lambda)}{dz}\right|_{z=a_{\pm i}} = 0, \end{gathered} \tag{20}$$

where

$$\begin{aligned} &a_{+i} = z_i, \;\; a_{-i} = z_{i+1}, \quad \text{if } Re\{\lambda\theta_i\,(z_{i+1} - z_i)\} \geq 0; \\ &a_{+i} = z_{i+1},\; a_{-i} = z_i, \quad \text{if } Re\{\lambda\theta_i\,(z_{i+1} - z_i)\} < 0. \end{aligned} \tag{21}$$

Moreover, if $|\lambda| \geq \lambda_{i,K}$*, then first, on the segment* L_i*, connecting the points* z_i *and* z_{i+1}*, the following inequalities hold:*

$$|B^{(i)}_{\pm K}(z,\lambda)| \leq C^{(0)}_{i,K}, \quad |H^{(i)}_{\pm K}(z,\lambda)| \leq C^{(0)}_{i,K} \quad (z \in L_i), \tag{22}$$

and second, $F^{(i)}_{\pm K}(z,\lambda)$ *are the linearly independent solutions of equation* (7) *for all* $z \in \boldsymbol{C}$*.*

Proof. Formulae (14), (16) – (20) can be checked by substitution into (7). Since the last statement of the lemma follows from the fact that Wronskian is constant for every two solutions of equation (7) and its estimate for the solutions of (14) for a given large $|\lambda|$ at the point $z = z_i$ with the help of formulae (16), (17), (22), then the estimates of (22) are key in this lemma. To prove them, we will use the fact that the solutions of Cauchy problems (20) on the segment L_i are at the same time the solutions of the following integral equations on this segment ($z \in L_i,\ t \in L_i$):

$$B^{(i)}_{\pm K}(z,\lambda) = \int\limits_{a_{\pm i}}^{z} \left(\exp\{\mp 2\lambda\theta_i(z-t)\} - 1\right)\left(\pm\frac{Q_i(t)}{2\lambda\theta_i}B^{(i)}_{\pm K}(t,\lambda) - \frac{dC_{i,K+1}(t)}{dt}\right)dt, \tag{23}$$

where the values of the parameter $a_{\pm i}$ are defined in (21). It is well known that the solution of each of equations (23) exists and is unique given any value of λ and can be found by the method of successive approximations. Taking into account (21), it follows from (23) that given any λ the following inequalities hold:

$$B^{max}_{\pm i,K}(\lambda) \leq \frac{Q_{i0}}{|\lambda\theta_i|}B^{max}_{\pm i,K}(\lambda) + 2C^{int}_{i,K+1}, \tag{24}$$

where

$$B^{max}_{\pm i,K}(\lambda) := \max\{|B^{(i)}_{\pm K}(z,\lambda)|, z \in L_i\},$$
$$Q_{i0} := \int\limits_{L_i} |Q_i(t)||dt|, \quad C^{int}_{i,K+1} := \int\limits_{L_i}\left|\frac{dC_{i,K+1}(t)}{dt}\right||dt|.$$

We put $\lambda_{i,K} := 2Q_{i0}/|\theta_i|$. Then due to (24) given $|\lambda| \geq \lambda_{i,K}$, we have:

$$B^{max}_{\pm i,K}(\lambda) \leq 4C^{int}_{i,K+1}. \tag{25}$$

Further, from (23) we obtain that $(z \in L_i,\ t \in L_i)$

$$\frac{dB^{(i)}_{\pm K}(z,\lambda)}{dz} = \mp 2\lambda\theta_i \int\limits_{a_{\pm i}}^{z} \exp\{\mp 2\lambda\theta_i(z-t)\} \left(\pm \frac{Q_i(t)}{2\lambda\theta_i} B^{(i)}_{\pm K}(t,\lambda) - \frac{dC_{i,K+1}(t)}{dt}\right) dt. \tag{26}$$

From relations (21), (25) and (26) for $|\lambda| \geq \lambda_{i,K}$, the following estimates follow:

$$\left|\frac{dB^{(i)}_{\pm K}(z,\lambda)}{dz}\right| \leq 4C^{int}_{i,K+1}Q_{i0} + 2|\lambda\theta_i|C^{int}_{i,K+1} \leq 4|\lambda\theta_i|C^{int}_{i,K+1}. \tag{27}$$

Putting $C^{(0)}_{i,K} := \max\left\{\left|\frac{dC_{i,K}(z)}{dz}\right|, z \in L_i\right\} + 8C^{int}_{i,K+1}$, we obtain that given $|\lambda| \geq \lambda_{i,K}$, inequalities (22) follow from relations (19), (25), and (27). □

We will denote by the symbols $O(1)$ and $\hat{O}(1)$ the functions and matrices of the parameter λ, respectively, whose particular form is not important for us, bounded for $|\lambda| > \lambda_{cr}$, where λ_{cr} is the finite value (different for distinct functions and matrices).

Lemma 9. *There exists an integer $K_0 \geq 2$, such that for any integer $K \geq K_0$ there exists such finite $\lambda_K > 0$, that given $|\lambda| \geq \lambda_K$, the transfer matrix $\hat{P}(\gamma, z_f, z_0)$ of equation (2) of class G with a set of characteristic data $W = \{N, \{z_j\}_0^{N+1}, \{Q_i, R_i\}_0^N\}$ can be written in the form*

$$\hat{P}(\gamma, z_f, z_0) = \hat{C}^{(f)}\hat{T}^{(N)}\hat{T}^{(N-1)}\dots\hat{T}^{(2)}\hat{T}^{(1)}\hat{T}^{(0)}\hat{A}^{(0)}, \tag{28}$$

where $\hat{T}^{(0)} = \hat{I}$ (unit matrix),

$$\hat{A}^{(0)} = -\frac{1}{D^{(0)}_K}\begin{pmatrix} \lambda\theta_0 E^{(0)}_{-K}(z_0) & C^{(0)}_{-K}(z_0) \\ \lambda\theta_0 E^{(0)}_{+K}(z_0) & -C^{(0)}_{+K}(z_0) \end{pmatrix}, \tag{29}$$

$$\hat{C}^{(f)} = \begin{pmatrix} C^{(N)}_{+K}(z_{N+1})\exp(\lambda\varepsilon_N) & C^{(N)}_{-K}(z_{N+1})\exp(-\lambda\varepsilon_N) \\ \lambda\theta_N E^{(N)}_{+K}(z_{N+1})\exp(\lambda\varepsilon_N) & -\lambda\theta_N E^{(N)}_{-K}(z_{N+1})\exp(-\lambda\varepsilon_N) \end{pmatrix}, \tag{30}$$

$$\hat{T}^{(n)} = \begin{pmatrix} t^{(n)}_1 \exp(\lambda\varepsilon_{n-1}) & t^{(n)}_- \exp(-\lambda\varepsilon_{n-1}) \\ t^{(n)}_+ \exp(\lambda\varepsilon_{n-1}) & t^{(n)}_2 \exp(-\lambda\varepsilon_{n-1}) \end{pmatrix} \ (n = \overline{1,N} \text{ given } N \geq 1). \tag{31}$$

Here $D^{(0)}_K = -\lambda\theta_0\left(C^{(0)}_{+K}(z_0)E^{(0)}_{-K}(z_0) + C^{(0)}_{-K}(z_0)E^{(0)}_{+K}(z_0)\right) = -2\lambda\theta_0(1 + O(1)/\lambda)$,

$$\varepsilon_i = \theta_i(z_{i+1} - z_i) \ \ (i = \overline{0,N}), \tag{32}$$

and with $N \geq 1$ for $t^{(n)}_1, t^{(n)}_2, t^{(n)}_\pm$ $(n = \overline{1,N})$, the following relations are true:

$$t^{(n)}_1 = \frac{\theta_{n-1} + \theta_n}{2\theta_n} + \frac{O(1)}{\lambda}, \ \ t^{(n)}_2 = \frac{\theta_{n-1} + \theta_n}{2\theta_n} + \frac{O(1)}{\lambda}, \tag{33}$$

(34) $$t_{\pm}^{(n)}=\begin{cases}\dfrac{\theta_n-\theta_{n-1}}{2\theta_n}+\dfrac{O(1)}{\lambda}, & \theta_n\neq\theta_{n-1};\\ -\delta_n\left(\mp\dfrac{1}{2\lambda\theta_n}\right)^{m_n+2}\left(1+\dfrac{O(1)}{\lambda}\right), & \theta_n=\theta_{n-1},\end{cases}$$

where the integers m_n ($m_n\in[0,K_0-2]$) and the complex numbers δ_n ($\delta_n\neq 0$) do not depend on λ.

Proof. Due to Lemma 8 (see also the proof of Lemma 1), for every $K\in\mathbf{N}$ there exist numbers $\lambda_{i,K}>0$ ($i=\overline{0,N}$), such that given $|\lambda|\geq\lambda_{i,K}$, the continuously differentiable solutions $u_1(z),u_2(z)$ of equation (2) of class G on the region γ_i of the curve γ, connecting the points z_i and z_{i+1}, can be represented as a linear combination of solutions $F_{\pm K}^{(i)}(z,\lambda)$ with

(35) $$\begin{gathered}\{C_{0,k}(z_0)\}_1^K=0 \text{ (when } i=0),\\ C_{i,k}(z_i)=C_{i-1,k}(z_i) \text{ (when } N\geq 1, i=\overline{1,N}, k=\overline{1,K}).\end{gathered}$$

Hence, when $|\lambda|\geq\lambda_K:=\max\{\lambda_{i,K},i=\overline{0,N}\}$, taking into account the initial conditions of the form (4) at the point z_0 and the fact that the solutions $u_1(z)$, $u_2(z)$ are continuously differentiable at the characteristic points of the curve γ, we obtain that for $\hat{P}(\gamma,z_f,z_0)$ the representation (28) holds, in which the matrices $\hat{A}^{(0)},\hat{C}^{(f)}$ and $\hat{T}^{(n)}$($n=\overline{1,N}$ given $N\geq 1$) satisfy the relations (29) – (31). Moreover,

(36) $$\begin{pmatrix} t_1^{(n)} & t_-^{(n)}\\ t_+^{(n)} & t_2^{(n)}\end{pmatrix}=\begin{pmatrix}1+C_{-K}^{(n-1)}(z_n)W_{+K}^{(n)} & C_{-K}^{(n-1)}(z_n)W_{-K}^{(n)}\\ -C_{+K}^{(n-1)}(z_n)W_{+K}^{(n)} & 1-C_{+K}^{(n-1)}(z_n)W_{-K}^{(n)}\end{pmatrix}+\frac{\hat{O}(1)}{\lambda^{K+1}},$$

where

(37) $$W_{\pm K}^{(n)}=\pm\frac{\lambda}{D_K^{(n)}}\left(\theta_nE_{\pm K}^{(n)}(z_n)-\theta_{n-1}E_{\pm K}^{(n-1)}(z_n)\right)\ (n=\overline{1,N}\text{ given }N\geq 1).$$

Here $D_K^{(n)}=-\lambda\theta_n(C_{+K}^{(n)}(z_n)E_{-K}^{(n)}(z_n)+C_{-K}^{(n)}(z_n)E_{+K}^{(n)}(z_n))$, moreover, from formulae (16), (17), (22), given $|\lambda|\geq\lambda_K$, we have that:

(38) $$D_K^{(i)}=-2\lambda\theta_i\left(1+\frac{O(1)}{\lambda}\right)\neq 0\ \ (i=\overline{0,N}).$$

We will prove that (33), (34) are true. When $N\geq 1$, for every $n\in\{1,\ldots,N\}$ by Definition 3 either the numbers R_{n-1} and R_n are distinct or the entire functions Q_{n-1} and Q_n are so, or both the numbers and the functions differ. From (17), (37), (38), we have:

(39) $$W_{\pm K}^{(n)}=\pm\frac{\theta_{n-1}-\theta_n}{2\theta_n}+\frac{O(1)}{\lambda}\ \ (n=\overline{1,N}\text{ when }N\geq 1).$$

Substitution of (39) into (36) taking into account (16) completely proves (33), and also (34) given $\theta_n\neq\theta_{n-1}$, that is, due to (15) in the case when $R_n\neq R_{n-1}$.

Now let $R_n=R_{n-1}$, and therefore, due to (15), we have that $\theta_n=\theta_{n-1}$, and the functions Q_n and Q_{n-1} are distinct. Hence, there exist an integer $m_n\geq 0$ and a complex number $\delta_n\neq 0$, such that given $z=z_n$,

$$\frac{d^mQ_n}{dz^m}-\frac{d^mQ_{n-1}}{dz^m}=\begin{cases}0, & \text{if } m_n\geq 1, m=\overline{0,m_n-1};\\ \delta_n, & \text{if } m=m_n.\end{cases}$$

In work [9], it was proved that in this case

$$\frac{d\left(C_{n,k}(z)-C_{n-1,k}(z)\right)}{dz}\Big|_{z=z_n}=\begin{cases}0, & 0\le k\le m_n;\\ \left(-\frac{1}{2}\right)^k\delta_n, & k=m_n+1.\end{cases} \tag{40}$$

Substituting (17) into (37) and using (35) and (40), and also (16), (22), (36), and (38), we obtain that relation (34) also holds when $\theta_n=\theta_{n-1}$. □

We will denote $(N+1)-$dimensional vectors by a letter with an arrow above it, and their dot product by round brackets. For example, $\vec{\varepsilon}:=(\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N)$; $\vec{\alpha}_s:=(\alpha_0^{(s)},\alpha_1^{(s)},\dots,\alpha_N^{(s)})$, where $\alpha_i^{(s)}\in\{\pm1\}$ $(i=\overline{0,N})$, moreover, $s=1+\sum\limits_{i=0}^{N}(1+\alpha_i^{(s)})2^{i-1}$, that is, $s\in\{1,\dots,2^{N+1}\}$ (as in the binary system); $(\vec{\varepsilon},\vec{\alpha}_s)=\sum\limits_{i=0}^{N}\alpha_i^{(s)}\varepsilon_i$.

Since due to (15) $\theta_i+\theta_j\neq0$ for every $i,j\in\{0,\dots,N\}$, then Lemma 9 yields the following corollary.

Corollary 4. *Under the conditions of Lemma 9 given $K\ge K_0$ and $|\lambda|\ge\lambda_K$, the elements of the matrix $\hat{P}(\gamma,z_f,z_0)$ can be written in the form*

$$p_{\alpha\beta}=\sum_{s=1}^{2^{N+1}}d_{\alpha\beta}^{(s)}(\lambda)\exp\{\lambda h_s\}\quad(\alpha,\beta\in\{1,2\}). \tag{41}$$

Here the coefficients $h_s=(\vec{\varepsilon},\vec{\alpha}_s)$ do not depend on λ and all functions $d_{\alpha\beta}^{(s)}(\lambda)$ can be represented in the form

$$d_{\alpha\beta}^{(s)}(\lambda)=\left(\frac{1}{\lambda}\right)^{m_{\alpha\beta}^{(s)}}\delta_{\alpha\beta}^{(s)}\left(1+\frac{O(1)}{\lambda}\right)\neq0\quad(\alpha,\beta\in\{1,2\},s=\overline{1,2^{N+1}}),$$

where the integers $m_{\alpha\beta}^{(s)}$ and the complex numbers $\delta_{\alpha\beta}^{(s)}$ $(\delta_{\alpha\beta}^{(s)}\neq0)$ do not depend on λ.

A part of exponents in the right-hand sides of equalities (41) can coincide, and there emerges a question whether in (41) the senior terms of coefficients are reduced for the exponential functions that grow the fastest with the increase of $|\lambda|$. A negative answer to this question for the elements of the transfer matrix of equation (2) of class G along an ordinary curve is given by Lemma 10, which coincides up to notation with Lemma 4 from work [9].

Lemma 10. *Suppose that $h_{max}:=\max\{|h_s|,s=\overline{1,2^{N+1}}\}$, where $h_s:=(\vec{\varepsilon},\vec{\alpha}_s)$. Then there exist at least two distinct numbers $s_0\in\{1,\dots,2^{N+1}\}$ such that $|h_{s_0}|=h_{max}$, moreover, if $\vec{\varepsilon}\neq\vec{0}$, then $h_{max}>0$. Apart from that, if $\varepsilon_i\neq0$ for all values of $i=\overline{0,N}$, then for every coefficient h_{s_m} such that $|h_{s_m}|=h_{max}$, and for every number $s\in\{1,\dots,2^{N+1}\}\backslash\{s_m\}$, the inequality $h_{s_m}\neq h_s$ holds.*

4. Proof of Theorems 1 and 2

Due to Lemma 1, all elements of the transfer matrix $\hat{P}$ are entire functions of the parameter ρ. If the curve γ after successive removal of all the „invisible loops" degenerates into a point, then by Lemma 2, the transfer matrix $\hat{P}\equiv\hat{I}$.

Suppose that after successive removal of all the „invisible loops" γ turns into an ordinary curve γ_{min} with a set of characteristic data $W := \{N, \{z_j\}_0^{N+1}, \{Q_i, R_i\}_0^N\}$. Then $\hat{P}$ can be calculated along γ_{min}.

In this case, from relations (15), (32) and Lemmas 3, 10, it follows that among the coefficients $h_s := |h_s| \exp\{i\Psi_s\}\,(s = \overline{1, 2^{N+1}})$ (i is an imaginary unit), belonging in the formula (41) to the exponents, there is a coefficient h_{s_0} such that $|h_{s_0}| = h_{max} > 0$ and for every number $s \in \{1, \ldots, 2^{N+1}\} \backslash \{s_0\}$ at least one of the two inequalities hold: $\Psi_{s_0} \neq \Psi_s$, $|h_{s_0}| > |h_s|$. Consider $\lambda = |\lambda| \exp\{-i\Psi_{s_0}\}$, where $|\lambda| > 0$. Then $\lambda h_{s_0} = |\lambda| h_{max} > 0$ and for every number $s \in \{1, \ldots, 2^{N+1}\} \backslash \{s_0\}$ the following inequality holds:

$$\Re\{\lambda h_s\} = |\lambda||h_s| \cos(\Psi_s - \Psi_{s_0}) < |\lambda| h_{max}.$$

Due to the latter inequality and the finiteness of the number N, we obtain that given $|\lambda| \to \infty$ along the considered ray, formula (41) can be represented in the form

$$p_{\alpha\beta} = d_{\alpha\beta}^{(s_0)}(\lambda) \exp\{h_{max}|\lambda|\} \left(1 + \frac{O(1)}{|\lambda|}\right) \quad (\alpha, \beta \in \{1, 2\}).$$

Hence, all elements of the transfer matrix $\hat{P}$ are entire functions ρ of order $1/2$ and of normal type $\sigma = h_{max} > 0$ (see, for example, Section 1 of Chapter I in [18]). Theorem 1 is proved.

Note that formulae (41) up to denotations coincide with formula (46) for a characteristic function of the boundary value problem, studied in detail in work [19]. Hence, using the results obtained in [19], we can find the indicator of elements of the matrix $\hat{P}$ and the angular density of distribution of their zeroes on the complex plane of the parameter λ.

To prove Theorem 2, we will need two simple lemmas.

Lemma 11. *Suppose that in* $\boldsymbol{C}$ *the meromorphic functions* S_{11} *and* $\widetilde{S}_{11}$ *or* S_{22} *and* $\widetilde{S}_{22}$ *coincide, where* $\hat{S}$ *and* $\hat{\widetilde{S}}$ *are the reduced matrices. Then at least one of non-diagonal elements of the matrix* $\hat{P}^{(0)} = \hat{\widetilde{P}}(\hat{P})^{-1}$ *equals zero for all* $\rho \in \boldsymbol{C}$*, where* $\hat{P}$ *and* $\hat{\widetilde{P}}$ *are the corresponding transfer matrices.*

Proof. Multiplying the matrices, we obtain:

$$\hat{P}^{(0)} = \begin{pmatrix} p_{22}\widetilde{p}_{11} - p_{21}\widetilde{p}_{12} & p_{11}\widetilde{p}_{12} - p_{12}\widetilde{p}_{11} \\ p_{22}\widetilde{p}_{21} - p_{21}\widetilde{p}_{22} & p_{11}\widetilde{p}_{22} - p_{12}\widetilde{p}_{21} \end{pmatrix}.$$

From here, taking into account Definition 4, we have that: if $S_{11} \equiv \widetilde{S}_{11}$, then $p_{12}^{(0)} \equiv 0$, and if $S_{22} \equiv \widetilde{S}_{22}$, then $p_{21}^{(0)} \equiv 0$. □

Lemma 12. *Suppose that in* $\boldsymbol{C}$ *the meromorphic functions* S_{12} *and* $\widetilde{S}_{12}$ *or* S_{21} *and* $\widetilde{S}_{21}$ *coincide, where* $\hat{S}$ *and* $\hat{\widetilde{S}}$ *are the reduced matrices. Then at least one of non-diagonal elements of the matrix* $\hat{P}^{(0)} = (\hat{P})^{-1}\hat{\widetilde{P}}$ *equals zero given all* $\rho \in \boldsymbol{C}$*, where* $\hat{P}$ *and* $\hat{\widetilde{P}}$ *are the corresponding transfer matrices.*

Proof. Multiplying the matrices, we obtain:

$$\hat{P}^{(0)} = \begin{pmatrix} p_{22}\widetilde{p}_{11} - p_{12}\widetilde{p}_{21} & p_{22}\widetilde{p}_{12} - p_{12}\widetilde{p}_{22} \\ p_{11}\widetilde{p}_{21} - p_{21}\widetilde{p}_{11} & p_{11}\widetilde{p}_{22} - p_{21}\widetilde{p}_{12} \end{pmatrix}.$$

From here, taking into account Definition 4, we have that if $S_{12} \equiv \widetilde{S}_{12}$, then $p_{12}^{(0)} \equiv 0$, and if $S_{21} \equiv \widetilde{S}_{21}$, then $p_{21}^{(0)} \equiv 0$. □

Suppose that

$$W^{(1)} := \{N^{(1)}, \{z_j^{(1)}\}_0^{N^{(1)}+1}, \{Q_i^{(1)}, R_i^{(1)}\}_0^{N^{(1)}}\},$$
$$W^{(2)} := \{N^{(2)}, \{z_j^{(2)}\}_0^{N^{(2)}+1}, \{Q_i^{(2)}, R_i^{(2)}\}_0^{N^{(2)}}\}$$

are the sets of characteristic data of the two considered equations along the ordinary curves $\gamma^{(1)}$ and $\gamma^{(2)}$, respectively, $\hat{P}^{(1)}$ and $\hat{P}^{(2)}$ are the corresponding transfer matrices, moreover, by the condition of the theorem, $z_0^{(1)} = z_0^{(2)} := z_0$. We will prove Theorem 2 by considering two cases.

Case 1. Suppose that for the elements of the reduced matrices $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ at least one of the two identities $S_{11}^{(1)} \equiv S_{11}^{(2)}$, $S_{22}^{(1)} \equiv S_{22}^{(2)}$ is fulfilled. From (5), (6), it follows that the transfer matrix of equation (2) of class G along the curve γ, resulting after a successive traverse, first, through the curve $\gamma^{(1)}$ from the point $z_{N^{(1)}+1}^{(1)}$ to the point z_0, and then through the curve $\gamma^{(2)}$ from the point z_0 to the point $z_{N^{(2)}+1}^{(2)}$, will be equal $\hat{P} = \hat{P}^{(2)}(\hat{P}^{(1)})^{-1}$. Then due to Lemma 11 and Theorem 1, the curve γ is not ordinary, since one of the elements of the transfer matrix $\hat{P}$ along it equals zero.

By the condition of the theorem, $z_0^{(1)} = z_0^{(2)} := z_0$. Hence, there exists an integer $i_0 \geq 0$ such that $z_i^{(2)} = z_i^{(1)} = z_i$ $(i = \overline{0, i_0})$, $Q_i^{(2)} = Q_i^{(1)}$, $R_i^{(2)} = R_i^{(1)}$ $(i = \overline{0, i_0 - 1}$ given $i_0 \geq 1)$, but at the same time if $i_0 \leq \min\{N^{(1)}, N^{(2)}\}$, then the ordered triplets $\{z_{i_0+1}^{(2)}, Q_{i_0}^{(2)}, R_{i_0}^{(2)}\}$ and $\{z_{i_0+1}^{(1)}, Q_{i_0}^{(1)}, R_{i_0}^{(1)}\}$ are distinct. Further we will assume for definiteness that $N^{(1)} \leq N^{(2)}$.

Assume that $i_0 \leq N^{(1)}$ and the ordered pairs $\{Q_{i_0}^{(2)}, R_{i_0}^{(2)}\}$ and $\{Q_{i_0}^{(1)}, R_{i_0}^{(1)}\}$ are distinct. Then after successive removal from the curve γ the „invisible loops", generated by regions of the curves $\gamma^{(1)}$ and $\gamma^{(2)}$ from the point z_0 to the point z_{i_0}, the remaining curve γ_{min} will first coincide with a region of the ordinary curve $\gamma^{(1)}$ from the point $z_{N^{(1)}+1}^{(1)}$ to the point z_{i_0}, and then with a region of the ordinary curve $\gamma^{(2)}$ from the point z_{i_0} to the point $z_{N^{(2)}+1}^{(2)}$. Moreover, due to Lemma 3 the transfer matrix along the curve γ_{min} still equals $\hat{P}$, and the point z_{i_0} will be the characteristic one for the curve γ_{min}, since by the assumption the ordered pairs $\{Q_{i_0}^{(2)}, R_{i_0}^{(2)}\}$ and $\{Q_{i_0}^{(1)}, R_{i_0}^{(1)}\}$ are distinct. Therefore, the curve γ_{min} is ordinary, which, due to Theorem 1, contradicts the fact that one of the elements of the transfer matrix $\hat{P}$ along it equals zero. Hence,

$$Q_{i_0}^{(1)} = Q_{i_0}^{(2)},\ R_{i_0}^{(1)} = R_{i_0}^{(2)}. \tag{42}$$

Suppose that $i_0 \leq N^{(1)}$ and (42) is fulfilled, but $z_{i_0+1}^{(2)} \neq z_{i_0+1}^{(1)}$. Then due to (42), the point z_{i_0} will not be the characteristic one of the curve γ_{min}, and the distinct points $z_{i_0+1}^{(1)}$, $z_{i_0+1}^{(2)}$ will be the successive characteristic points. Therefore, the curve γ_{min} will again be ordinary, which contradicts the fact that one of the elements of the transfer matrix $\hat{P}$ along it equals zero. Hence,

$$z_{i_0+1}^{(1)} = z_{i_0+1}^{(2)}. \tag{43}$$

Formulae (42), (43) contradict the definition of the number i_0 given $i_0 \leq \min\{N^{(1)}, N^{(2)}\}$. Therefore, $i_0 = N^{(1)} + 1$. If at the same time $N^{(1)} < N^{(2)}$, then after successive removal of all the „invisible loops", the curve γ will only have a region of the ordinary curve $\gamma^{(2)}$, connecting the points $z^{(1)}_{N^{(1)}+1} = z^{(2)}_{N^{(1)}+1}$ and $z^{(2)}_{N^{(2)}+1}$, which contradicts the fact that one of the elements of the transfer matrix $\hat{P}$ along it equals zero. To summarize, $N^{(1)} = N^{(2)}$, $i_0 = N^{(1)} + 1$, that is, the sets of characteristic data $W^{(1)}$ and $W^{(2)}$ coincide, which proves the theorem for the first case.

Case 2. Suppose that at least one of the two identities $S^{(1)}_{12} \equiv S^{(2)}_{12}$, $S^{(1)}_{21} \equiv S^{(2)}_{21}$ is fulfilled. Let $\tau := z^{(1)}_f - z^{(2)}_f$. Performing in equation (2) of class G with a potential $Q^{(2)}(z)$ and a weight function $R^{(2)}(z)$ a substitution of the variable: $x = z + \tau$, after inverse redesignation of x by z, we obtain that the matrix $\hat{S}^{(2)}$ is reduced for equation (2) of class G with a potential $Q^{(2)}(z - \tau)$ and a weight function $R^{(2)}(z - \tau)$ between the points $z^{(2)}_0 + \tau$ and $z^{(2)}_f + \tau = z^{(1)}_f$ of the ordinary curve $\widetilde{\gamma}^{(2)}$, resulting from $\gamma^{(2)}$ by parallel transfer. Consider the curve γ, resulting after successive traverse, first, through the curve $\gamma^{(1)}$ from the point $z^{(1)}_0$ to the point $z^{(1)}_{N^{(1)}+1} \equiv z^{(1)}_f$, and later through the curve $\widetilde{\gamma}^{(2)}$ from the point $z^{(2)}_{N^{(2)}+1} + \tau \equiv z^{(1)}_f$ to the point $z^{(2)}_0 + \tau$. Applying to it the reasonings used for considering Case 1, with a substitution of Lemma 11 by Lemma 12, we obtain that

$$N^{(2)} = N^{(1)} = N,$$

and for all $i = \overline{0, N}$

$$Q^{(2)}_i(z - \tau) = Q^{(1)}_i(z),\ R^{(2)}_i = R^{(1)}_i,\ z^{(2)}_i + \tau = z^{(1)}_i. \tag{44}$$

By the condition of the theorem, $z^{(2)}_0 = z^{(1)}_0$, hence, from (44) given $i = 0$ it follows that $\tau = 0$, and therefore, $W^{(1)} = W^{(2)}$, which proves Theorem 2 for the second case.

5. Inverse problem for a Sturm–Liouville equation with a piecewise analytical potential and piecewise constant weight on a standard polygonal chain

In this section, we consider a case when it is a priori known that the curve γ is a polygonal chain with some a priori known characteristics, and the potential Q in equation (2) is piecewise analytical.

Definition 15. *Suppose that on a continuous rectifiable curve $\gamma \subset \boldsymbol{C}$, given parametrically by the function $z = V(t)$ $(t \in [t_0, t_f])$, a nonzero piecewise constant R and a piecewise analytical function Q are defined, that is, there exists an integer $N \geq 0$ and a set of numbers $T = \{t_j\}_0^{N+1} : t_0 < t_1 < \ldots < t_{N+1} = t_f$ such that*

$$Q(z) = Q_i(z),\ R(z) = R_i,\ \text{if } z = V(t),\ t \in (t_i, t_{i+1}) \quad (i = \overline{0, N}). \tag{45}$$

In (45), all functions Q_i are analytical in some neighbourhood Φ_i of the region $\gamma_i := \{V(t), t \in (\tau_i, \tau_{i+1})\}$ of the curve γ and all R_i are nonzero complex constants. Moreover, if $N \geq 1$, then for every number $n \in \{1, \ldots, N\}$ the ordered pairs $\{Q_n, R_n\}$ and $\{Q_{n-1}, R_{n-1}\}$ are distinct.

Then we will refer to equation (2) considered on the curve γ as an equation of class GA, to the points $z_i := V(t_i)$ $(i = \overline{0, N})$, $z_{N+1} \equiv z_f := V(t_f)$ as the characteristic points, and to the ordered set $W := \{N, \{z_j\}_0^{N+1}, \{Q_i, R_i\}_0^N\}$ as the set of characteristic data of the curve γ and the corresponding equation (2) of class GA on γ.

Definitions 4 as well as Lemmas 1, 11, and 12 are transferred to equations of class GA without any changes.

Definition 16. *We will refer to a polygonal chain with a piecewise analytical and a piecewise constant functions defined on it as standard one, if the ordered set of its vertices coincides with the ordered set of its characteristic points.*

Obviously, for a standard polygonal chain, relations (8) are fulfilled. Moreover, Lemma 8 remains true for the neighbourhood Φ_i of the region $\gamma_i := \{V(t), t \in (\tau_i, \tau_{i+1})\}$ of a standard polygonal chain. Hence, for a standard polygonal chain Lemma 9 and Corollary 4 continue to hold as well. All of that means that the following theorem similar to the statement of the second part of Theorem 1 is true.

Theorem 6. *All elements of the transfer matrix $\hat{P}$ of equation* (2) *of class GA along some* (*unknown*) *standard polygonal chain γ are entire functions ρ of order* 1/2 *and of normal type.*

Unfortunately, Theorem 2 in the framework of the results already obtained in this work does not transfer on two arbitrary standard polygonal chains. But it can be proved for the case with additional a priori information given for the two considered polygonal chains.

Definition 17. *We will call a vertex of a polygonal chain a turning vertex, if it is a common end of two successive links of the polygonal chain that do not lie on one straight line.*

We will refer to two polygonal chains as mainly coinciding if they have common turning vertices, starting and ending points and, moreover, all the turning vertices during the movement from the start of each polygonal chain to its end are passed by in a similar order and the number of characteristic points between every corresponding turning vertices to the two polygonal chains coincide. If the polygonal chain does not have any turning vertices, then we will call them mainly coinciding even if only their starting points coincide and both of these degenerate polygonal chains belong to one straight line.

Note that it is possible that a part of the links of one of the two standard mainly coinciding polygonal chains does not overlap the links of the second polygonal chain, and in the absence of turning vertices, such degenerate polygonal chains can have a different number of vertices and even a single common point (the starting one).

Theorem 7. *Suppose that two equations of class GA have sets of characteristic data $W^{(1)}$ and $W^{(2)}$ respectively on two standard mainly coinciding polygonal chains $\gamma^{(1)}$ and $\gamma^{(2)}$, and also the reduced matrices $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ along these polygonal chains. Then, if there exist numbers $\alpha, \beta \in \{1,2\}$ such that meromorphic in $\boldsymbol{C}$ functions $S^{(1)}_{\alpha\beta}(\rho)$ and $S^{(2)}_{\alpha\beta}(\rho)$ coincide, we have that $W^{(1)} = W^{(2)}$.*

Proof. If the considered polygonal chains do not have any turning vertices, then the proof of Theorem 7 completely repeats the proof of Theorem 2 provided above, since all the emerging „invisible loops" of the curve (now the polygonal chain) γ described in that proof inherently belong to one straight line in the areas where the coefficients of equation (2) are analytical, and therefore, their removal does not change the transfer matrix (this circumstance makes the reference to Lemma 3 still relevant). One thing to note is that in the text of the proof of Theorem 2, we should substitute the references to Theorem 1 by the references to Theorem 6, the word „curve” by the word „polygonal chain”, and the notion of an „ordinary curve” by the notion of a „standard polygonal chain”. If the considered polygonal chains have turning vertices, then all the „invisible loops” emerging in this case belong to the same straight line in the areas where the coefficients of equation (2) are analytical due to the condition on coinciding of the starting and ending points of these polygonal chains, all their turning vertices and similarity of the number of characteristic points of each of the polygonal chains between the corresponding turning vertices of these polygonal chains. Hence, in this case too, the proof of Theorem 7 is obtained from the proof of Theorem 2 by the method described above. □

6. Inverse problem for equations of class G by two spectra

Along with equation (2) of class G on the curve γ, defined parametrically by the function $z = V(t), t \in [t_0, t_f]$, consider the equation

$$u''(z) + (Q^{(1)}(z) - \lambda^2 R^{(1)}(z))u(z) = 0 \tag{46}$$

on the curve $\gamma^{(1)}$, defined parametrically by the function $z = z_0 + \eta(V(t) - z_0), t \in [t_0, t_f]$, where η is a nonzero complex number. In (46),

$$Q^{(1)}(z) = \frac{1}{\eta^2} Q\left(z_0 + \frac{z - z_0}{\eta}\right), \quad R^{(1)}(z) = \frac{1}{\eta^2} R\left(z_0 + \frac{z - z_0}{\eta}\right).$$

Obviously, equation (46) is an equation of class G on the curve $\gamma^{(1)}$, moreover, the starting points of the curves γ and $\gamma^{(1)}$ coincide.

Lemma 13. *Let $\hat{P}$ and $\hat{P^{(1)}}$ be the transfer matrices of equations (2) and (46) on the curves γ and $\gamma^{(1)}$ respectively. Then*

$$P_{11}^{(1)} = P_{11}, \quad P_{21}^{(1)} = \frac{1}{\eta} P_{21}, \quad P_{12}^{(1)} = \eta P_{12}, \quad P_{22}^{(1)} = P_{22}. \tag{47}$$

Proof. Let $u_1(z), u_2(z)$ be the continuously differentiable solutions of equation (2) of class G along the rectifiable curve γ, satisfying the boundary conditions (4) given $z_b := z_0$. Then using direct substitution it is easy to see that the functions

$$u_1^{(1)}(z) = u_1\left(z_0 + \frac{z - z_0}{\eta}\right), \quad u_2^{(1)}(z) = \eta u_2\left(z_0 + \frac{z - z_0}{\eta}\right) \tag{48}$$

are the continuously differentiable solutions of equation (46) of class G along the rectifiable curve $\gamma^{(1)}$, satisfying the boundary conditions (4) given $z_b := z_0$. The statement of the lemma directly follows from Definition 4 and relations (48). □

Definition 18. *We will refer as boundary value problems L and L_1 to equation* (2) *of class G along the rectifiable curve γ, supplemented respectively by the boundary conditions*

$$u(z_0) = 0, \quad u(z_f) = 0. \tag{49}$$

and

$$u(z_0) = 0, \quad u'(z_f) = 0. \tag{50}$$

Suppose that $\{\lambda_n\}_{n \geq 0}$ and $\{\mu_n\}_{n \geq 0}$ are the sets of all eigenvalues (taking into account their multiplicity) of the boundary value problems L and L_1 respectively. Along with the boundary value problems L and L_1, we will consider the boundary value problems $\widetilde{L}$ and $\widetilde{L}_1$ of the same form, but with different coefficients in equation (2), and, possibly, on different curves. Moreover, let us agree that if some symbol σ denotes an object relevant to the problem L (L_1), then the symbol $\widetilde{\sigma}$ will denote a similar object relevant to the problem $\widetilde{L}$ ($\widetilde{L}_1$).

Theorem 8. *If γ and $\widetilde{\gamma}$ are ordinary curves that have two similar first characteristic points and $\lambda_n = \widetilde{\lambda}_n$, $\mu_n = \widetilde{\mu}_n$, $n \geq 0$, then $W = \widetilde{W}$. That is, defining the spectra $\{\lambda_n, \mu_n\}_{n\geq 0}$ of the boundary value problems L and L_1 for equation* (2) *on some (not completely defined) ordinary curve γ, and also two first characteristic points of this curve (including its starting point), uniquely determines the set of basic data of equation* (2) *of class G on this curve.*

Proof. Due to Definitions 4 and 18, defining the spectra of the boundary value problems L and L_1 is equivalent to defining the zeroes of the elements of the transfer matrices $p_{12}(\lambda) := u_2(z_f, \lambda)$ and $p_{22}(\lambda) := u_2'(z_f, \lambda)$, which by Theorem 1 are the entire functions $\rho = \lambda^2$ of order $1/2$, and therefore, are defined by their zeroes up to constants multipliers. Hence, there relation is also determined up to the constant multiplier. Assume that

$$\frac{\widetilde{p}_{12}(\lambda)}{\widetilde{p}_{22}(\lambda)} = \eta \frac{p_{12}(\lambda)}{p_{22}(\lambda)}. \tag{51}$$

Then due to relation (51) and Lemma 13,

$$\frac{\widetilde{p}_{12}(\lambda)}{\widetilde{p}_{22}(\lambda)} = \frac{p_{12}^{(1)}(\lambda)}{p_{22}^{(1)}(\lambda)}, \tag{52}$$

and hence, since $z_0^{(1)} = z_0 = \widetilde{z}_0$, then $W^{(1)} = \widetilde{W}$ by Theorem 2, and, in particular, $z_1^{(1)} = \widetilde{z}_1$. But $z_1^{(1)} - z_0^{(1)} = \eta(z_1 - z_0)$ by the definition of the curve $\gamma^{(1)}$, provided after equation (46) , and $z_0 = \widetilde{z}_0$, $z_1 = \widetilde{z}_1$ by the condition of the theorem, moreover, $\widetilde{z}_1 - \widetilde{z}_0 \neq 0$, since the curve $\widetilde{\gamma}$ is ordinary by condition. Therefore, $\eta = 1$, which yields $W = \widetilde{W}$ due to relation (51) and Theorem 2. Theorem 8 is proved. □

Note that as it follows from the proof, in Theorem 8 the requirement of coinciding for the two curves of their first two characteristic points can be substituted with coinciding of any two corresponding characteristic points, given that these two points are distinct. In particular, for an a priory open curve we can define its starting and ending points. But in any case, it turns out that it is necessary to have at least minimal initial information about the curve on which the boundary value problems are considered. Moreover, this requirement is substantial since it fixes the scale on the complex plane z. The case is that as it follows from Lemma 13, for every $\eta \neq 1$ boundary value problems of the form L and L_1 for equations (2) and (46) will have similar spectra but distinct sets of basic data, if at least one of these curves possesses a non-empty set of basic data, that is, it does not degenerates into a point after successive removal of all "invisible loops".

That is why when considering inverse problems on curves that are not defined at all (except for defining their starting point) it is much more convenient to use the relation between the elements of the transfer matrix instead of the spectra of the boundary value problems, since in this case, the scale on the plane z is fixed automatically. Moreover, in many applied problems, the elements of the transfer matrix are connected by simple algebraic relations to the values measurable by experiment (for example, to the coefficients of reflection and transmission of light in spectroscopic problems for linear [20] and nonlinear [21] media with an arbitrary frequency dispersion), which also makes their usage in these cases more natural.

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