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A PROBLEM OF NORMAL OSCILLATIONS OF A SYSTEM OF BODIES PARTIALLY FILLED WITH IDEAL FLUIDS UNDER THE ACTION OF AN ELASTIC DAMPING DEVICE

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Abstract. We investigate a problem of normal oscillations of a system of bodies partially filled with ideal fluids under the action of an elastic damping device. We prove that the problem has a discrete spectrum localized in a vertical strip. The asymptotic behavior of the spectrum is investigated. The theorem on the Abel-Lidsky basis property of root elements of the problem is proved.

Keywords: system of bodies, ideal fluid, elastic damping device, basis of Abel-Lidsky, spectrum.

1. Introduction

The first work dedicated to the problem of small oscillations of a solid body with a cavity, completely filled with an ideal fluid, was the study by N. E. Zhukovskiy [30]. In his paper, by introducing auxiliary functions depending only on the shape of the cavity, the author showed that the motion of the body is performed in a way as if the liquid had been substituted by an equivalent solid body. In the 1950-60's due to the development of space technologies, a great interest reemerged to the problem of the motion of solid bodies with cavities filled with liquids; moreover, along with studies on the motion of bodies with cavities completely filled with a liquid, a new problem of the motion of a solid body with cavities, that are not completely filled with a liquid, was formulated. A large number of articles was devoted to different linear aspects of the theory, and among the first ones, the papers by N. N. Moiseev [20, 21], G. S. Narimanov [22], D. E. Ohotsimskiy [23], and L. N. Sretenskiy [26] are worth

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mentioning. Starting with the works by N. N. Moiseev and a joint work of S. G. Krein and N. N. Moiseev [16], in the studies of these problems the methods of functional analysis and the theory of linear operators acting in Hilbert space, the methods of spectral theory of operator pencils are applied. In studies by N. D. Kopachevsky et al. [8, 9, 10], evolutionary and spectral problems on the motion of a body with a cavity completely or partially filled with an ideal or a viscous fluid are investigated.

Further, in papers by N. D. Kopachevsky and his students, problems of oscillation of a system of bodies connected by spherical hinges filled with an ideal fluid [13, 14], problems of small movements of an ideal stratified liquid with a free surface completely covered with crushed or elastic ice [15, 27, 28], and also a problem of oscillation of a body filled by two ideal liquids with a free surface, partially covered by crushed ice [29], and others were studied.

In work [5], a theorem on solvability of an initial boundary value problem, describing small movements of a system of bodies, partially filled with ideal fluids, under the action of an elastic damping device, was proved. This initial boundary value problem is reduced to the following Cauchy problem for a differential operator equation of first order in some Hilbert space $\mathcal{H}$:

$$\mathcal{C}\frac{dz}{dt} + (\mathcal{P} + i\mathcal{B})z = f, \quad z(0) = z^0.$$

We will seek a solution of a homogeneous equation in the form $z(t) = ze^{-\lambda t}$, $\lambda \in \mathbb{C}$. As a result, we obtain a spectral problem

$$-\lambda\mathcal{C}z + (\mathcal{P} + i\mathcal{B})z = 0, \quad z \in \mathcal{D}(\mathcal{B}) \subset \mathcal{H}$$

of normal oscillations for a system of bodies partially filled with homogeneous ideal fluids under the action of an elastic damping device. In this paper, we study the properties of the obtained spectral problem. The main results, proved in Theorems 4, 5, and 6, can be formulated in the following way.

1. The spectrum of the studied problem is symmetrical with respect to the real axis, is located in the vertical strip $\{0 \le \operatorname{Re}\lambda \le \widetilde{\alpha}c^{-1}\}$, and consists of isolated eigenvalues of finite multiplicity with the following asymptotic behavior:

$$\lambda_k^{(\pm i)} = \pm i\left(\sum_{l=1}^{n}\frac{g\pi}{|\Gamma_l|}\right)^{1/2} k^{1/2}(1 + o(1)) \quad (k \to \infty).$$

2. The system of eigen- and associated elements of the studied spectral problem forms the basis of Abel-Lidsky with brackets in the Hilbert space $\mathcal{H}$ of order $\beta > 1$.

2. The statement of the initial boundary value problem

Consider a hydromechanical system consisting of n bodies which are fixed between two supports such that the l-th body is connected to the adjacent $(l-1)$-th and $(l+1)$-th bodies by springs with stiffnesses k_{l-1}^2, k_l^2 respectively, and the first and last bodies are attached to springs with stiffnesses k_0^2, k_n^2, whose ends are fixed motionlessly on the supports.

Every l-th body represents an open vessel, partially filled with a homogeneous ideal fluid of density $\rho_l > 0$, which at rest occupies the area $\Omega_l \subset \mathbb{R}^2$ with a free boundary Γ_l and a solid wall S_l, $l = \overline{1,n}$. Suppose that $m_l := m_{\mathrm{b},l} + m_{\mathrm{f},l}$, where $m_{\mathrm{b},l}$ is the mass of the body, $m_{\mathrm{f},l}$ is the mass of the fluid. While at rest, we assume that the boundaries Γ_l are horizontal lines. By an elastic damping device we mean the presence of springs attached to solid walls of the vessels, as shown on Figure 1, and the friction of the bottoms of the vessels against the motionless horizontal support. By $\alpha_l > 0$ we denote the friction coefficient of the bottom of the l-the body against the support.

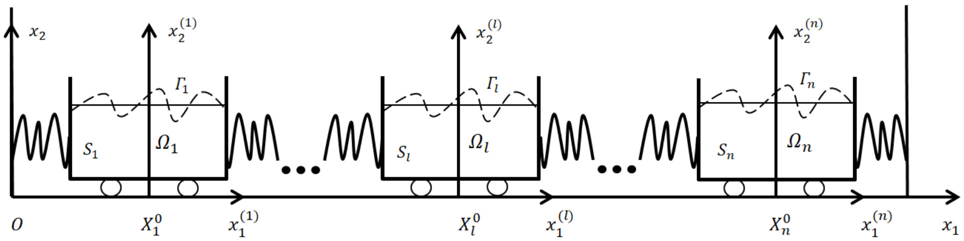

Figure 1. The scheme of the hydromechanical system

We introduce a motionless coordinate system Ox_1x_2 with unit vectors $\mathbf{e}_j$, $j = 1, 2$, such that the bodies perform movements along the axis Ox_1. Moreover, we introduce moving coordinate systems $X_l^0 x_1^{(l)} x_2^{(l)}$, rigidly fixed with the l-th bodies. We denote the unit vectors of the moving systems by $\mathbf{e}_j^{(l)}$, $j = 1, 2$, $l = \overline{1, n}$. Moreover, $\mathbf{e}_j = \mathbf{e}_j^{(l)}$.

During the process of small motions of the body, we consider the displacement of the l-the body $X_l(t)$ along the axis Ox_1, we denote by X_l^0 the position of the l-th body in equilibrium state. Then $X_l(t) = X_l^0 + x_l(t)$, where $x_l(t)$ is a small displacement of the l-th body with respect to the equilibrium state.

We denote by $\mathbf{u}_l(t, x^{(l)})$ the field of relative velocities of the l-th fluid in the l-th moving coordinate system. Then the full velocity of the l-th fluid in the motionless system will be represented by the formula $\dot{x}_l(t)\mathbf{e}_1 + \mathbf{u}_l(t, x^{(l)})$.

The change of the length of the spring, connecting the bodies with the numbers l and $l+1$, compared to its length at rest equals

$$X_{l+1}(t) - X_l(t) - \big(X_{l+1}^0 - X_l^0\big) = \big(X_{l+1}(t) - X_{l+1}^0\big) - \big(X_l(t) - X_l^0\big) = x_{l+1}(t) - x_l(t).$$

Applying Newton's second law, we write the equations of the movement of the bodies with fluids in the motionless coordinate system Ox_1x_2. The equation for the movement of the first body with a fluid will have the form

$$m_1\ddot{x}_1\mathbf{e}_1 + \rho_1 \int\limits_{\Omega_1} \frac{\partial \mathbf{u}_1}{\partial t}\, d\Omega_1 = -k_0^2 x_1 \mathbf{e}_1 + k_1^2(x_2 - x_1)\mathbf{e}_1 - \alpha_1 \dot{x}_1 \mathbf{e}_1 + \tag{1}$$
$$+ k_0^2 x_0 \mathbf{e}_1 - g m_1 \mathbf{e}_2 + \mathbf{f}_{\mathrm{b},1} + N_1 \mathbf{e}_2,$$

where $x_0 = x_0(t)$ is a given law of motion of the left wall, $\mathbf{f}_{\mathrm{b},1} = \mathbf{f}_{\mathrm{b},1}(t)$ is a small external force acting on the first body, $N_1 = N_1(t)$ is the reaction of the support, g is the gravitational acceleration.

We write the motion equation of the l-th body with the fluid ($l = \overline{2, n-1}$):

$$m_l\ddot{x}_l\mathbf{e}_1 + \rho_l \int\limits_{\Omega_l} \frac{\partial \mathbf{u}_l}{\partial t}\, d\Omega_l = -k_{l-1}^2 (x_l - x_{l-1})\mathbf{e}_1 + k_l^2(x_{l+1} - x_l)\mathbf{e}_1 - \alpha_l \dot{x}_l \mathbf{e}_1 - \tag{2}$$
$$- g m_l \mathbf{e}_2 + \mathbf{f}_{\mathrm{b},l} + N_l \mathbf{e}_2,$$

where $\mathbf{f}_{\mathrm{b},l} = \mathbf{f}_{\mathrm{b},l}(t)$ is a small external force acting on the l-th body, $N_l = N_l(t)$ is the reaction of the support.

We write the motion equation of the n-th body with a fluid:

$$m_n\ddot{x}_n\mathbf{e}_1+\rho_n\int\limits_{\Omega_n}\frac{\partial\mathbf{u}_n}{\partial t}\,d\Omega_n=-k_{n-1}^2(x_n-x_{n-1})\mathbf{e}_1-k_n^2x_n\mathbf{e}_1+k_n^2x_{n+1}\mathbf{e}_1-$$
$$-\alpha_n\dot{x}_n\mathbf{e}_1-gm_n\mathbf{e}_2+\mathbf{f}_{\mathrm{b},n}+N_n\mathbf{e}_2, \tag{3}$$

where $x_{n+1}=x_{n+1}(t)$ is a given law of motion of the right wall, $\mathbf{f}_{\mathrm{b},n}=\mathbf{f}_{\mathrm{b},n}(t)$ is a small external force acting on the n-th body, $N_n=N_n(t)$ is the reaction of the support.

The small motions of the ideal homogeneous fluid in the area Ω_l are expressed by a linearized Euler equation:

$$\rho_l\left(\frac{\partial\mathbf{u}_l}{\partial t}+\ddot{x}_l\mathbf{e}_1^{(l)}\right)+\nabla p_l=\rho_l\mathbf{f}_{\mathrm{f},l},\quad \operatorname{div}\mathbf{u}_l=0\quad(\text{in }\Omega_l),\quad l=\overline{1,n}, \tag{4}$$

where $p_l=p_l(t,x^{(l)})$ is a pressure deviation in the fluid during the movement from the equilibrium state, and $\mathbf{f}_{\mathrm{f},l}=\mathbf{f}_{\mathrm{f},l}(t,x^{(l)})$ is a small force acting on the fluid in the domain Ω_l.

Let $\zeta_l(t,x_1^{(l)})$ $(x_1^{(l)}\in\Gamma_l)$ be a function defining small deviations of the free boundary $\Gamma_l(t)$ along $\mathbf{e}_2^{(l)}$ with respect to the equilibrium line Γ_l, which is defined by the equation $x_2^{(l)}=b_l$, by the formula:

$$x_2^{(l)}=b_l+\zeta_l(t,x_1^{(l)}),\quad |\zeta_l|\ll 1.$$

The boundary conditions for the considered problem include the condition of non-leaking of the ideal fluid on the solid wall S_l, and also dynamic, cinematic conditions on the boundary Γ_l and the condition of the preservation of the volume of the fluid of every body respectively:

$$\mathbf{u}_l\cdot\mathbf{n}_l=0\quad(\text{on }S_l),\quad l=\overline{1,n}, \tag{5}$$
$$p_l=\rho_lg\zeta_l\quad(\text{on }\Gamma_l),\quad l=\overline{1,n}, \tag{6}$$
$$\frac{\partial\zeta_l}{\partial t}=\mathbf{u}_l\cdot\mathbf{e}_2^{(l)}\quad(\text{on }\Gamma_l),\quad l=\overline{1,n}, \tag{7}$$
$$\int\limits_{\Gamma_l}\zeta_l\,d\Gamma_l=0,\quad l=\overline{1,n}. \tag{8}$$

Here by $\mathbf{n}_l$ we denote the unit vector pointed outside the domain Ω_l and perpendicular to the boundary $\partial\Omega_l=\Gamma_l\cup S_l$. On the boundary Γ_l, obviously, the relation $\mathbf{n}_l=\mathbf{e}_2^{(l)}$ will be true.

The initial conditions have the form

$$x_l(0)=x_l^0,\quad \dot{x}_l(0)=x_l^1,\quad \mathbf{u}_l(0,x^{(l)})=\mathbf{u}_l^0(x^{(l)}),\quad \zeta_l(0,x_1^{(l)})=\zeta_l^0(x_1^{(l)}). \tag{9}$$

A complete statement of the studied initial boundary problem consists of solving the equations (1)-(4) with boundary and initial conditions (5)-(9).

Theorem 1. *We assume that the stated problem (1)-(9) has a classical solution, that is, such that all the functions in the equations, boundary and initial conditions* *are continuous with respect to their variables. Then the identity*

$$\frac{1}{2}\frac{d}{dt}\Big\{\sum_{l=1}^{n}\Big(m_{b,l}|\dot{x}_l|^2+\rho_l\int\limits_{\Omega_l}\big|\mathbf{u}_l+\dot{x}_l\mathbf{e}_1^{(l)}\big|^2\,d\Omega_l+\rho_l g\int\limits_{\Gamma_l}|\zeta_l|^2 d\Gamma_l\Big)+ + k_0^2x_1^2+k_n^2x_n^2+\sum_{l=1}^{n-1}k_l^2|x_{l+1}-x_l|^2\Big\}=$$

$$=\sum_{l=1}^{n}\Big(-\alpha_l|\dot{x}_l|^2+(\mathbf{f}_{b,l}\cdot\mathbf{e}_1)\dot{x}_l+\rho_l\int\limits_{\Omega_l}\mathbf{f}_{f,l}\cdot\mathbf{u}_l\,d\Omega_l\Big)+k_0^2x_0\dot{x}_1+k_n^2x_{n+1}\dot{x}_n,$$

represents the law of balance of the full energy of the studied hydromechanical system, written in the differential form.

Proof. See the proof of the Theorem in [5, p. 891, Theorem 1]. □

3. Choosing the functional spaces and projecting the motion equations of fluids. Deriving the basic operator equation

We introduce the Hilbert space $\mathbf{L}_2(\Omega):=\bigoplus_{l=1}^{n}\mathbf{L}_2(\Omega_l)$, where $\mathbf{L}_2(\Omega_l)$ is a Hilbert space with a scalar product and a square of a norm of the form

$$(\mathbf{u}_l,\mathbf{v}_l)_{\mathbf{L}_2(\Omega_l)}=\int\limits_{\Omega_l}\mathbf{u}_l(x^{(l)})\cdot\overline{\mathbf{v}_l(x^{(l)})}\,d\Omega_l,\quad \|\mathbf{u}_l\|^2_{\mathbf{L}_2(\Omega_l)}:=\int\limits_{\Omega_l}|\mathbf{u}_l(x^{(l)})|^2\,d\Omega_l.$$

It is well known that the space $\mathbf{L}_2(\Omega_l)$ has an orthogonal decomposition (see, for example, [8, Ch. 2, § 1, S. 10, decomposition (1.24)])

$$\mathbf{L}_2(\Omega_l)=\mathbf{J}_0(\Omega_l)\oplus\mathbf{G}_{h,S_l}(\Omega_l)\oplus\mathbf{G}_{0,\Gamma_l}(\Omega_l), \tag{10}$$

where

$$\mathbf{J}_0(\Omega_l):=\big\{\mathbf{u}_l\in\mathbf{L}_2(\Omega_l):\,\operatorname{div}\mathbf{u}_l=0\ (\text{in }\Omega_l),\quad \mathbf{u}_l\cdot\mathbf{n}_l=0\ (\text{on }\partial\Omega_l)\big\},$$

$$\mathbf{G}_{h,S_l}(\Omega_l):=\big\{\mathbf{u}_l=\nabla\Phi_l\in\mathbf{L}_2(\Omega_l):\ \Delta\Phi_l=0\ (\text{in }\Omega_l),\ \frac{\partial\Phi_l}{\partial n_l}=0\ (\text{on }S_l),\ \int\limits_{\Gamma_l}\Phi_l d\Gamma_l=0\big\},$$

$$\mathbf{G}_{0,\Gamma_l}(\Omega_l):=\big\{\mathbf{u}_l=\nabla\Psi_l\in\mathbf{L}_2(\Omega_l):\Psi_l=0\ (\text{on }\Gamma_l)\big\}.$$

Therefore, taking into account decomposition (10), we can assume that

$$\mathbf{L}_2(\Omega)=\mathbf{J}_0(\Omega)\oplus\mathbf{G}_{h,S}(\Omega)\oplus\mathbf{G}_{0,\Gamma}(\Omega),$$

where

$$\mathbf{J}_0(\Omega):=\bigoplus_{l=1}^{n}\mathbf{J}_0(\Omega_l),\quad \mathbf{G}_{h,S}(\Omega):=\bigoplus_{l=1}^{n}\mathbf{G}_{h,S_l}(\Omega_l),\quad \mathbf{G}_{0,\Gamma}(\Omega):=\bigoplus_{l=1}^{n}\mathbf{G}_{0,\Gamma_l}(\Omega_l).$$

We introduce the Hilbert space

$$L_{2,\Gamma}:=\bigoplus_{l=1}^{n}L_{2,\Gamma_l},\quad L_{2,\Gamma_l}:=\Big\{\zeta_l\in L_2(\Gamma_l):\int\limits_{\Gamma_l}\zeta_l\,d\Gamma_l=0\Big\},$$

where the Hilbert space L_{2,Γ_l} is a subspace in $L_2(\Gamma_l)$, orthogonal to the unit function $1_{\Gamma_l}:=1|_{\Gamma_l}$.

In further studying the problem, we will assume that the required vector and scalar fields are the functions of the variable t with values in the corresponding Hilbert spaces and subspaces introduced above.

Due to the statement of the problem, the velocity field $\mathbf{u}_l(t)$ has to belong for every $t \geq 0$ to the subspace $\mathbf{J}_0(\Omega_l) \oplus \mathbf{G}_{h,S_l}(\Omega_l)$, and the field $\nabla p_l(t)$ of pressure gradients to the subspace $\mathbf{G}_{h,S_l}(\Omega_l) \oplus \mathbf{G}_{0,\Gamma_l}(\Omega_l)$ respectively.

We will find the velocity field of the fluid $\mathbf{u}_l$ in the form

$$\mathbf{u}_l = \mathbf{v}_l + \nabla\Phi_l, \quad \mathbf{v}_l \in \mathbf{J}_0(\Omega_l), \quad \nabla\Phi_l \in \mathbf{G}_{h,S_l}(\Omega_l),$$

and the gradient of the pressure field in the following form:

$$\nabla p_l = \nabla\tilde{p}_l + \nabla\Psi_l, \quad \nabla\tilde{p}_l \in \mathbf{G}_{h,S_l}(\Omega_l), \quad \nabla\Psi_l \in \mathbf{G}_{0,\Gamma_l}(\Omega_l). \tag{11}$$

The potential $\tilde{p}_l$ due to (11) and the dynamic relation (6) is the solution of the Zaremba's problem for the Laplace equation:

$$\Delta\tilde{p}_l = 0 \quad (\text{in } \Omega_l), \quad \frac{\partial\tilde{p}_l}{\partial n_l} = 0 \quad (\text{on } S_l), \quad \tilde{p}_l = \rho_l g \zeta_l \quad (\text{on } \Gamma_l).$$

We will assume that the boundary of the domain Ω_l is a Lipschitz one. It is known (see [8, Ch. 1, § 3, S. 6, problem (P.37)]) that such problem has a unique solution $\tilde{p}_l \in H^1_{\Gamma_l}(\Omega_l)$, if the condition $\zeta_l \in H^{1/2}_{\Gamma_l} := H^{1/2}(\Gamma_l) \cap L_{2,\Gamma_l}$ is satisfied. Here $H^{1/2}(\Gamma_l)$ is a Sobolev–Slobodecki space with a fractional index (see [1, Ch. 1, § 5, S. 5.1]). Thus, we can assume that $\nabla\tilde{p}_l = \rho_l g Q_l \zeta_l \in \mathbf{G}_{h,S_l}(\Omega_l)$, where $Q_l \in \mathcal{L}(H^{1/2}_{\Gamma_l}, \mathbf{G}_{h,S_l}(\Omega_l))$.

We denote a conjugate space by $\tilde{H}^{-1/2}_{\Gamma_l} := (H^{1/2}_{\Gamma_l})^*$ (see [1, Ch. 1, S. 5.1, Theorem 5.1.12]). Then $H^{1/2}_{\Gamma_l} \subsetneq\subsetneq L_{2,\Gamma_l} \subsetneq\subsetneq \tilde{H}^{-1/2}_{\Gamma_l}$ (see [1, Ch. 1, S. 5.1]).

According to decomposition (10), we introduce the orthoprojectors $P_{0,l}$, P_{h,S_l}, P_{0,Γ_l} of the space $\mathbf{L}_2(\Omega_l)$ on its subspaces $\mathbf{J}_0(\Omega_l)$, $\mathbf{G}_{h,S_l}(\Omega_l)$, $\mathbf{G}_{0,\Gamma_l}(\Omega_l)$ respectively.

We apply the orthoprojectors $P_{0,l}, P_{h,S_l}, P_{0,\Gamma_l}$ to the left- and right-hand sides of the motion equation of the fluid (4). We obtain for every $l = \overline{1,n}$ three relations

$$\rho_l\left(\frac{d\mathbf{v}_l}{dt} + \ddot{x}_l P_{0,l}\mathbf{e}_1^{(l)}\right) = \rho_l P_{0,l}\mathbf{f}_{\mathrm{f},l}, \tag{12}$$

$$\rho_l\left(\frac{d\nabla\Phi_l}{dt} + \ddot{x}_l P_{h,S_l}\mathbf{e}_1^{(l)}\right) + \rho_l g Q_l \zeta_l = \rho_l P_{h,S_l}\mathbf{f}_{\mathrm{f},l}, \tag{13}$$

$$\rho_l \ddot{x}_l P_{0,\Gamma_l}\mathbf{e}_1^{(l)} + \nabla\Psi_l = \rho_l P_{0,\Gamma_l}\mathbf{f}_{\mathrm{f},l}. \tag{14}$$

From relations (12) and (14), we will find the vortex component of the velocity field $\mathbf{v}_l$ and a part of the dynamical pressure Ψ_l, if the displacement x_l is known. Hence, further we will only consider the relation (13).

We project the motion equations of the bodies with fluids (1)-(3) on the unit vector $\mathbf{e}_1$. We obtain the following relations:

(15)
$$m_1\ddot{x}_1 + \rho_1\int\limits_{\Omega_1}\frac{d\mathbf{u}_1}{dt}\cdot\mathbf{e}_1\,d\Omega_1 + k_0^2x_1 - k_1^2(x_2-x_1) + \alpha_1\dot{x}_1 = k_0^2x_0 + \mathbf{f}_{\mathrm{b},1}\cdot\mathbf{e}_1,$$

(16)
$$m_l\ddot{x}_l + \rho_l\int\limits_{\Omega_l}\frac{d\mathbf{u}_l}{dt}\cdot\mathbf{e}_1\,d\Omega_l + k_{l-1}^2(x_l-x_{l-1}) - k_l^2(x_{l+1}-x_l) + \alpha_l\dot{x}_l = \mathbf{f}_{\mathrm{b},l}\cdot\mathbf{e}_1,$$
$$l=\overline{2,n-1},$$

(17)
$$m_n\ddot{x}_n + \rho_n\int\limits_{\Omega_n}\frac{d\mathbf{u}_n}{dt}\cdot\mathbf{e}_1\,d\Omega_n + k_{n-1}^2(x_n-x_{n-1}) + k_n^2x_n + \alpha_n\dot{x}_n = k_n^2x_{n+1} + \mathbf{f}_{\mathrm{b},n}\cdot\mathbf{e}_1.$$

We project the equations of motion of bodies with fluids (1)-(3) on the unit vector $\mathbf{e}_2$. We obtain

$$\rho_l\int\limits_{\Omega_l}\frac{d\mathbf{u}_l}{dt}\cdot\mathbf{e}_2\,d\Omega_l = -gm_l + \mathbf{f}_{\mathrm{b},l}\cdot\mathbf{e}_2 + N_l(t),\quad l=\overline{1,n}.$$

This relation yields the formula for finding the reaction of the support $N_l(t)$, if the velocity field of the fluid $\mathbf{u}_l$ is known.

Now we transform the integral summand from (15)-(17). We obtain that

$$\begin{aligned}\int\limits_{\Omega_l}\frac{d\mathbf{u}_l}{dt}\cdot\mathbf{e}_1\,d\Omega_l &= \frac{d}{dt}\int\limits_{\Omega_l}\nabla\Phi_l\cdot\mathbf{e}_1^{(l)}\,d\Omega_l + \frac{d}{dt}\int\limits_{\Omega_l}\mathbf{v}_l\cdot\mathbf{e}_1^{(l)}\,d\Omega_l = \\ &= \frac{d}{dt}\int\limits_{\Omega_l}\nabla\Phi_l\cdot\mathbf{e}_1^{(l)}\,d\Omega_l + \frac{d}{dt}\int\limits_{\Omega_l}\mathbf{v}_l\cdot\nabla x_1^{(l)}\,d\Omega_l = \frac{d}{dt}\int\limits_{\Omega_l}\nabla\Phi_l\cdot\mathbf{e}_1^{(l)}\,d\Omega_l + \\ &\quad + \frac{d}{dt}\int\limits_{\Omega_l}\Big(\mathrm{div}\big(x_1^{(l)}\mathbf{v}_l\big) - x_1^{(l)}\mathrm{div}\mathbf{v}_l\Big)\,d\Omega_l = \frac{d}{dt}\int\limits_{\Omega_l}\nabla\Phi_l\cdot\mathbf{e}_1^{(l)}\,d\Omega_l.\end{aligned}\tag{18}$$

We introduce the operators P_{ρ_l} and $\gamma_{n,l}$ in the following way:

$$P_{\rho_l}\nabla\Phi_l := \rho_l\int\limits_{\Omega_l}\nabla\Phi_l\cdot\mathbf{e}_1^{(l)}d\Omega_l,\qquad P_{\rho_l}:\quad \mathbf{G}_{h,S_l}(\Omega_l)\to\mathbb{C},\tag{19}$$

$$\gamma_{n,l}\nabla\Phi_l := \nabla\Phi\cdot\mathbf{e}_2^{(l)} = \frac{\partial\Phi_l}{\partial n_l}\bigg|_{\Gamma_l},\quad \gamma_{n,l}:\ \mathcal{D}(\gamma_{n,l})\subset\mathbf{G}_{h,S_l}(\Omega_l)\to L_{2,\Gamma_l}.\tag{20}$$

It is known that $\gamma_{n,l}\in\mathcal{L}(\mathbf{G}_{h,S_l}(\Omega_l),\tilde{H}_{\Gamma_l}^{-1/2})$ and the operator $\gamma_{n,l}$ is conjugate to the operator Q_l introduced above. Note that the operator $(Q_l\gamma_{n,l})^{-1}=\gamma_{n,l}^{-1}Q_l^{-1}$ is compact due to the embeddings $H_{\Gamma_l}^{1/2}\subsetneq\subsetneq L_{2,\Gamma_l}\subsetneq\subsetneq\tilde{H}_{\Gamma_l}^{-1/2}$.

Using relations (18) and operators (19), (20), we rewrite the system of equations (13), (15)-(17) in the following way:

(21)
$$\begin{cases} \rho_l \dfrac{d\nabla\Phi_l}{dt} + \ddot{x}_l \rho_l P_{h,S_l}\mathbf{e}_1^{(l)} + \rho_l g Q_l \zeta_l = \rho_l P_{h,S_l}\mathbf{f}_{\mathrm{f},l}, \quad l=\overline{1,n}, \\ m_1\ddot{x}_1 + \dfrac{d}{dt}P_{\rho_1}\nabla\Phi_1 + k_0^2 x_1 - k_1^2(x_2 - x_1) + \alpha_1\dot{x}_1 = k_0^2 x_0 + \mathbf{f}_{\mathrm{b},1}\cdot\mathbf{e}_1, \\ m_l\ddot{x}_l + \dfrac{d}{dt}P_{\rho_l}\nabla\Phi_l + k_{l-1}^2(x_l - x_{l-1}) - k_l^2(x_{l+1} - x_l) + \alpha_l\dot{x}_l = \mathbf{f}_{\mathrm{b},l}\cdot\mathbf{e}_1, \quad l=\overline{2,n-1}, \\ m_n\ddot{x}_n + \dfrac{d}{dt}P_{\rho_n}\nabla\Phi_n + k_{n-1}^2(x_n - x_{n-1}) + k_n^2 x_n + \alpha_n\dot{x}_n = k_n^2 x_{n+1} + \mathbf{f}_{\mathrm{b},n}\cdot\mathbf{e}_1. \end{cases}$$

We will write the system (21) in the form of a differential operator equation of first order in the Hilbert space $\mathbf{G}_{h,S}(\Omega)\oplus\mathbb{C}^n$:

$$\begin{pmatrix} R & E_{h,S} \\ P_\rho & M \end{pmatrix} \frac{d}{dt}\begin{pmatrix} \nabla\Phi \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix}\begin{pmatrix} \nabla\Phi \\ \dot{x} \end{pmatrix} + $$
$$+ \begin{pmatrix} gRQ & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} \zeta \\ x \end{pmatrix} = \begin{pmatrix} RP_{h,S}\mathbf{f}_{\mathrm{f}} \\ \mathbf{f}_{\mathrm{b}}\cdot\mathbf{e}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ k \end{pmatrix}. \tag{22}$$

Here the following notations are used:

$$R := \mathrm{diag}(\rho_1 I, \dots, \rho_n I), \quad M := \mathrm{diag}(m_1 I, \dots, m_n I), \quad \Lambda := \mathrm{diag}(\alpha_1 I, \dots, \alpha_n I),$$
$$P_\rho := \mathrm{diag}(P_{\rho_1}, \dots, P_{\rho_n}), \quad Q := \mathrm{diag}(Q_1, \dots, Q_n), \quad \gamma_n := \mathrm{diag}(\gamma_{n,1}, \dots, \gamma_{n,n}),$$
$$P_{h,S} := \mathrm{diag}(P_{h,S_1}, \dots, P_{h,S_n}), \quad E_{h,S} := \mathrm{diag}(\rho_1 P_{h,S_1}\mathbf{e}_1^{(1)}, \dots, \rho_n P_{h,S_n}\mathbf{e}_1^{(n)}),$$
$$\nabla\Phi := (\nabla\Phi_1; \dots; \nabla\Phi_n)^\tau, \quad x := (x_1; \dots; x_n)^\tau, \quad \dot{x} := (\dot{x}_1; \dots; \dot{x}_n)^\tau,$$
$$\zeta := (\zeta_1; \dots; \zeta_n)^\tau, \quad k := (k_0^2 x_0; 0; \dots; 0; k_n^2 x_{n+1})^\tau,$$
$$\mathbf{f}_{\mathrm{f}} := (\mathbf{f}_{\mathrm{f},1}; \dots; \mathbf{f}_{\mathrm{f},n})^\tau, \quad \mathbf{f}_{\mathrm{b}}\cdot\mathbf{e}_1 := (\mathbf{f}_{\mathrm{b},1}\cdot\mathbf{e}_1; \dots; \mathbf{f}_{\mathrm{b},n}\cdot\mathbf{e}_1)^\tau,$$

where the symbol τ marks the transpose operation. The operator K is defined by the formula

$$K := \begin{pmatrix} k_0^2 + k_1^2 & -k_1^2 & 0 & \cdots & 0 \\ -k_1^2 & k_1^2 + k_2^2 & -k_2^2 & \cdots & 0 \\ 0 & -k_2^2 & k_2^2 + k_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -k_{n-1}^2 \\ 0 & 0 & 0 & -k_{n-1}^2 & k_{n-1}^2 + k_n^2 \end{pmatrix}.$$

Taking into account the following notations

$$B_{12} := \begin{pmatrix} gRQ & 0 \\ 0 & K \end{pmatrix}, \quad \mathcal{D}(B_{12}) = \mathcal{D}(Q)\oplus\mathbb{C}^n,$$
$$C_1 := \begin{pmatrix} R & E_{h,S} \\ P_\rho & M \end{pmatrix}, \quad P_\alpha := \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix},$$
$$f_1 := \begin{pmatrix} RP_{h,S}\mathbf{f}_{\mathrm{f}} \\ \mathbf{f}_{\mathrm{b}}\cdot\mathbf{e}_1 \end{pmatrix}, \quad f_2 := \begin{pmatrix} 0 \\ k \end{pmatrix}, \quad z_1 := \begin{pmatrix} \nabla\Phi \\ \dot{x} \end{pmatrix}, \quad z_2 := \begin{pmatrix} \zeta \\ x \end{pmatrix},$$

equation (22) takes the form

$$C_1\frac{dz_1}{dt} + P_\alpha z_1 + B_{12}z_2 = f_1 + f_2. \tag{23}$$

Consider the system of two obvious connections

$$\begin{cases} \rho_l g \dfrac{\partial \zeta_l}{\partial t} = \rho_l g \gamma_{n,l} \nabla \Phi_l, \quad l = \overline{1,n}, \\ K \dfrac{dx}{dt} = K \dot{x}. \end{cases} \tag{24}$$

We write the system (24) in the form of a differential operator equation of first order in the Hilbert space $L_{2,\Gamma} \oplus \mathbb{C}^n$:

$$\begin{pmatrix} gR & 0 \\ 0 & K \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \zeta \\ x \end{pmatrix} + \begin{pmatrix} -gR\gamma_n & 0 \\ 0 & -K \end{pmatrix} \begin{pmatrix} \nabla \Phi \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which, taking into account the notations introduced above and the notations

$$C_2 := \begin{pmatrix} gR & 0 \\ 0 & K \end{pmatrix}, \quad B_{21} := \begin{pmatrix} -gR\gamma_n & 0 \\ 0 & -K \end{pmatrix}, \quad \mathcal{D}(B_{21}) = \mathcal{D}(\gamma_n) \oplus \mathbb{C}^n,$$

takes the form

$$C_2 \frac{dz_2}{dt} + B_{21} z_1 = 0. \tag{25}$$

Therefore, the original initial boundary value problem (1)-(9) is reduced to differential operator equations (23), (25) with corresponding initial conditions. Thus, we have the following Cauchy problem:

$$\begin{cases} C_1 \dfrac{dz_1}{dt} + P_\alpha z_1 + B_{12} z_2 = f_1 + f_2, \\ C_2 \dfrac{dz_2}{dt} + B_{21} z_1 = 0, \end{cases} \tag{26}$$

$$z_1(0) = (P_{h,S}\mathbf{u}^0; x^1)^\tau, \;\; z_2(0) = (\zeta^0; x^0)^\tau. \tag{27}$$

The system of differential equations (26) and initial conditions (27) can be neatly written in the form of a Cauchy problem for a differential operator equation of first order in the Hilbert space $\mathcal{H}$:

$$\mathcal{C} \frac{dz}{dt} + (\mathcal{P} + i\mathcal{B}) z = f, \quad z(0) = (z_1(0); z_2(0))^\tau, \tag{28}$$

where

$$z := (z_1; z_2)^\tau \in \mathcal{H} := \big(\mathbf{G}_{h,S}(\Omega) \oplus \mathbb{C}^n\big) \oplus \big(L_{2,\Gamma} \oplus \mathbb{C}^n\big), \quad f := (f_1 + f_2; 0)^\tau,$$

$$\mathcal{C} := \operatorname{diag}\big(C_1, C_2\big), \quad \mathcal{B} := \begin{pmatrix} 0 & -iB_{12} \\ -iB_{21} & 0 \end{pmatrix}, \quad \mathcal{P} := \operatorname{diag}\big(P_\alpha, 0\big).$$

We present without a proof a lemma on the properties of operator coefficients in (28).

Lemma 1. *The following properties of the operator coefficients are true:*
1. The operator $\mathcal{B}$ is self-adjoint on $\mathcal{D}(\mathcal{B}) = \mathcal{D}(B_{21}) \oplus \mathcal{D}(B_{12})$.
2. The operator $\mathcal{P}$ is bounded and non-negative in the Hilbert space $\mathcal{H}$.
3. The operator matrix $\mathcal{C}$ is a bounded positive definite operator in the Hilbert space $\mathcal{H}$.

Proof. For the proof, see [5, pp. 897-899]. □

We will formulate a definition of a strong solution of problem (1)-(9) and present without proofs the theorems on solvability of an operator equation and original initial boundary value problem.

Definition 1. *We will refer as a strong solution* of the initial boundary value problem (1)-(9) to fields $\mathbf{u}_l$ and functions p_l, ζ_l, such that the function

$$z = ((\nabla\Phi; \dot{x})^\tau; (\zeta; x)^\tau)^\tau$$

is the solution of Cauchy problem (28).

We will say that z is a solution of Cauchy problem (28), if all summands in the equation from (28) are continuous functions on $\mathbb{R}_+ := [0; +\infty)$ with values in $\mathcal{H}$, the equation from (28) is true for every $t \in \mathbb{R}_+$, and the initial condition holds.

Theorem 2. *Suppose that the following conditions are fulfilled:*

$$z(0) \in \mathcal{D}(\mathcal{B}), \quad f \in C^1(\mathbb{R}_+; \mathcal{H}).$$

Then problem (28) *has a unique solution.*

Using Theorem 2, we obtained a statement on unique solvability of the original initial boundary value problem (1)-(9) on the semi-axis $\mathbb{R}_+$.

Theorem 3. *Suppose that the following conditions are fulfilled:*

$$P_{h,S}\mathbf{u}^0 \in \mathcal{D}(\gamma_n), \quad \zeta^0 \in \mathcal{D}(Q) = H_\Gamma^{1/2}, \quad x^0, x^1 \in \mathbb{C}^n,$$
$$\mathbf{f}_b \cdot \mathbf{e}_1 \in C^1(\mathbb{R}_+; \mathbb{C}^n), \quad \mathbf{f}_f \in C^1(\mathbb{R}_+; \mathbf{L}_2(\Omega)), \quad k \in C^1(\mathbb{R}_+; \mathbb{C}^n).$$

Then problem (1)-(9) has a unique strong solution.

See the proofs of Theorems 2 and 3 in [5, p. 899, Theorem 2].

4. A problem of normal oscillations, the main properties of the spectrum

We will seek a solution of a homogeneous equation from (28) in the form $z(t) = ze^{-\lambda t}$, $\lambda \in \mathbb{C}$. As a result, we obtain a spectral problem

$$-\lambda\mathcal{C}z + (\mathcal{P} + i\mathcal{B})z = 0, \quad z = (z_1; z_2)^\tau \in \mathcal{D}(\mathcal{B}) \subset \mathcal{H} \tag{29}$$

on normal oscillations of a system of bodies partially filled with homogeneous ideal fluids under the action of an elastic damping device.

We write the spectral problem (29), taking into account the notations introduced above, in the form of a system of two equations in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, where $\mathcal{H}_1 := \mathbf{G}_{h,S}(\Omega) \oplus \mathbb{C}^n$ and $\mathcal{H}_2 := L_{2,\Gamma} \oplus \mathbb{C}^n$:

$$\begin{cases} -\lambda C_1 z_1 + P_\alpha z_1 + B_{12} z_2 = 0, \\ -\lambda C_2 z_2 + B_{21} z_1 = 0. \end{cases} \tag{30}$$

Lemma 2. *The number $\lambda = 0$ is not an eigenvalue of the spectral problem* (30).

Proof. We substitute the value $\lambda = 0$ into system (30). From the second equation of the system, we obtain that $B_{21}z_1 = 0$. Hence,

$$\begin{cases} gR\gamma_n\nabla\Phi = 0, \\ K\dot{x} = 0, \end{cases} \implies \begin{cases} \nabla\Phi = 0, \\ \dot{x} = 0, \end{cases} \implies z_1 = (\nabla\Phi; \dot{x})^\tau = 0. \tag{31}$$

Taking into account (31), from the first equation of system (30) we obtain $B_{12}z_2 = 0$, which yields

$$\begin{cases} gRQ\zeta = 0, \\ Kx = 0, \end{cases} \implies \begin{cases} \zeta = 0, \\ x = 0, \end{cases} \implies z_2 = (\zeta; x)^\tau = 0.$$

Therefore, when $\lambda = 0$, the spectral problem (30) only has a trivial solution $z = (z_1; z_2)^\tau = 0$. Hence, the point $\lambda = 0$ is not an eigenvalue of the spectral problem (30). □

Due to Lemma 2, from the second equation of system (30) we express the element z_2 and substitute it into the first equation of the system. We have that

$$\lambda^2 C_1 z_1 - \lambda P_\alpha z_1 - B_{12} C_2^{-1} B_{21} z_1 = 0. \tag{32}$$

We define the operator C_B by the formula

$$C_B := -B_{12} C_2^{-1} B_{21}, \quad \mathcal{D}(C_B) = \{z_1 \in \mathcal{D}(B_{21}) : C_2^{-1} B_{21} z_1 \in \mathcal{D}(B_{12})\}.$$

Lemma 3. *The operator C_B is self-adjoint and positive definite in $\mathcal{H}_1$, and the operator C_B^{-1} is compact.*

Proof. By direct calculations, we can verify that

$$C_B^{-1} = \operatorname{diag}(g^{-1} R^{-1} (Q\gamma_n)^{-1}, K^{-1}).$$

The compactness of the operator C_B^{-1} follows from the compactness of the operator $(Q\gamma_n)^{-1} = \operatorname{diag}((Q_1\gamma_{n,1})^{-1}, \ldots, (Q_n\gamma_{n,n})^{-1})$ and finite-dimensionality of the operator K^{-1}. This fact and the positivity of the operator C_B^{-1} yield the first statement of the Lemma. □

We perform in the spectral problem (32) a substitution $C_B^{1/2} z_1 =: u$ and apply to left-hand side of the relation (32) the operator $C_B^{-1/2}$, as a result, we arrive at the following basic spectral problem in the Hilbert space $\mathcal{H}_1$:

$$L(\lambda)u := (I - \lambda V_1 + \lambda^2 V_2)u = 0, \tag{33}$$

where $V_1 := C_B^{-1/2} P_\alpha C_B^{-1/2}$, $V_2 := C_B^{-1/2} C_1 C_B^{-1/2}$.

The following theorem on localization and discreetness of the spectrum of problem (33) is true.

Theorem 4. *The following statements hold:*

1. The spectrum of problem (33) *is symmetrical with respect to the real axis.*

2. Problem (33) *has a discreet spectrum with a possible limit point in the infinity.*

3. The spectrum of problem (33) *lies in the strip*

$$0 \le \operatorname{Re} \lambda \le \frac{\widetilde{\alpha}}{c},$$

where $c > 0$ is the lower bound of the operator C_1, $\widetilde{\alpha} := \max\limits_{l=\overline{1,n}} \{\alpha_l\}$.

Proof. We give the proof in several steps.

1. To prove the first statement, it suffices to show (see [18, Ch. 4, § 30, S. 1]), that the pencil $L(\lambda)$ is self-adjoint, that is,

$$\big(L(\bar{\lambda})\big)^* = L(\lambda).$$

Due to the fact that the operators $C_1, C_B^{-1/2}, P_\alpha$ are self-adjoint, we have that

$$\big(L(\bar{\lambda})\big)^* = I - \lambda V_1^* + \lambda^2 V_2^* = I - \lambda V_1 + \lambda^2 V_2 = L(\lambda).$$

2. To prove the discreetness of the spectrum, it suffices to verify that the Fredholm pencil (33) is continuously reversible at least at one point (see [8, Ch. 1, § 7, S. 3]). Indeed, given $\lambda = -1$, the operator

$$L(-1) = I + V_1 + V_2$$

is positive definite, and therefore, it has a bounded inverse. Since the Fredholm operator function $L(\lambda)$ has a singularity only at an infinitely remote point, its spectrum consists of isolated eigenvalues of finite multiplicity (that is, it is discrete) with a possible limit point in the infinity.

3. We will prove that the spectrum of problem (33) belongs to the mentioned strip. Since the spectrum is discrete, the property that we are proving needs to be verified for the eigenvalues of problem (33). Suppose that λ, u is the eigenvalue and the associated eigenelement. We scalar multiply the pencil (33) by u in the space $\mathcal{H}_1$, we will have that

$$(L(\lambda)u,u)_{\mathcal{H}_1} = (u,u)_{\mathcal{H}_1} - \lambda(V_1u,u)_{\mathcal{H}_1} + \lambda^2(V_2u,u)_{\mathcal{H}_1} = 0.$$

We divide the obtained expression by λ, now we have

$$\frac{\bar\lambda}{\bar\lambda}\cdot\frac{\|u\|^2_{\mathcal{H}_1}}{\lambda} - \|V_1^{1/2}u\|^2_{\mathcal{H}_1} + \lambda\|V_2^{1/2}u\|^2_{\mathcal{H}_1} = 0. \tag{34}$$

Separating in (34) the real part, we find that

$$\operatorname{Re}\lambda\cdot\left(\frac{\|u\|^2_{\mathcal{H}_1}}{|\lambda|^2} + \|V_2^{1/2}u\|^2_{\mathcal{H}_1}\right) = \|V_1^{1/2}u\|^2_{\mathcal{H}_1}.$$

From here follows the estimate

$$0 \le \operatorname{Re}\lambda = \frac{\|V_1^{1/2}u\|^2_{\mathcal{H}_1}}{\dfrac{\|u\|^2_{\mathcal{H}_1}}{|\lambda|^2} + \|V_2^{1/2}u\|^2_{\mathcal{H}_1}} \le \frac{\|V_1^{1/2}u\|^2_{\mathcal{H}_1}}{\|V_2^{1/2}u\|^2_{\mathcal{H}_1}} =$$

$$= \frac{(P_\alpha C_B^{-1/2}u, C_B^{-1/2}u)_{\mathcal{H}_1}}{(C_1C_B^{-1/2}u, C_B^{-1/2}u)_{\mathcal{H}_1}} \le \frac{\widetilde{\alpha}\|C_B^{-1/2}u\|^2_{\mathcal{H}_1}}{c\|C_B^{-1/2}u\|^2_{\mathcal{H}_1}} = \frac{\widetilde{\alpha}}{c},$$

where $c > 0$ is the exact lower bound of the operator C_1, $\widetilde{\alpha} := \max\limits_{l=\overline{1,n}}\{\alpha_l\}$. □

5. On asymptotics of the spectrum and basicity of the system of root elements by Abel-Lidsky

Theorem 5. *The spectral problem* (33) *has in the area* $\{0 \le \operatorname{Re}\lambda \le \widetilde{\alpha}c^{-1}\}$ *two branches of eigenvalues with the asymptotics*

$$\lambda_k^{(\pm i)} = \pm i\left(\sum_{l=1}^n \frac{g\pi}{|\Gamma_l|}\right)^{1/2} k^{1/2}(1+o(1)) \quad (k\to\infty).$$

Proof. We rewrite problem (33) in the form

$$L(\lambda)u := \left[\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} - \lambda\begin{pmatrix} 0 & 0 \\ 0 & K^{-1/2}\Lambda K^{-1/2}\end{pmatrix} + \right. \tag{35}$$
$$\left. + \lambda^2\begin{pmatrix} g^{-1}A & A^{1/2}C_{12}K^{-1/2} \\ K^{-1/2}C_{21}A^{1/2} & K^{-1/2}MK^{-1/2}\end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix},$$

where

$$A := (Q\gamma_n)^{-1}, \quad C_{12} := (gR)^{-1/2}E_{h,S}, \quad C_{21} := P_\rho(gR)^{-1/2}.$$

We write the spectral problem (35) in the form of a system

$$\begin{cases} u_1 + \lambda^2 g^{-1} A u_1 + \lambda^2 A^{1/2} C_{12} K^{-1/2} u_2 = 0, \\ u_2 - \lambda K^{-1/2} \Lambda K^{-1/2} u_2 + \lambda^2 K^{-1/2} C_{21} A^{1/2} u_1 + \lambda^2 K^{-1/2} M K^{-1/2} u_2 = 0 \end{cases} \tag{36}$$

and from the second equation given that λ is great enough by absolute value, we find u_2:

$$u_2 = -K^{1/2}\Big(\frac{1}{\lambda^2}K - \frac{1}{\lambda}\Lambda + M\Big)^{-1} C_{21} A^{1/2} u_1.$$

Substituting this expression for u_2 into the first equation of system (36), we obtain the following spectral problem

$$\Big[I + \lambda^2 g^{-1} A - \lambda^2 A^{1/2} C_{12} M^{-1/2}\Big(I - \frac{1}{\lambda} M^{-1/2} \Lambda M^{-1/2} + \\ + \frac{1}{\lambda^2} M^{-1/2} K M^{-1/2}\Big)^{-1} M^{-1/2} C_{21} A^{1/2}\Big] u_1 = 0. \tag{37}$$

From Neumann's theorem on inversion of an operator close to the unit one (see [7, Ch. 4, § 5, S. 4, Theorem 5]), it follows that problems (36) and (37) are equivalent if

$$\Big\|\frac{1}{\lambda} M^{-1/2} \Lambda M^{-1/2} - \frac{1}{\lambda^2} M^{-1/2} K M^{-1/2}\Big\| \le \big\|M^{-1/2}\big\|^2 \cdot \Big(\frac{\|\Lambda\|}{|\lambda|} + \frac{\|K\|}{|\lambda|^2}\Big) < 1,$$

or

$$|\lambda|^2 - |\lambda| \cdot \big\|M^{-1/2}\big\|^2 \cdot \|\Lambda\| - \big\|M^{-1/2}\big\|^2 \cdot \|K\| > 0,$$

or

$$|\lambda| > \frac{\big\|M^{-1/2}\big\|^2 \cdot \|\Lambda\| + \sqrt{\big\|M^{-1/2}\big\|^4 \cdot \|\Lambda\|^2 + 4\big\|M^{-1/2}\big\|^2 \cdot \|K\|}}{2} =: R. \tag{38}$$

In the area $\{\lambda : |\lambda| > R\}$ taking into account [7, Ch. 4, § 5, S. 4, Theorem 5], the spectral problem (37) can be transformed in the following way:

$$\Big[I + \lambda^2 (g^{-1} A - A^{1/2} C_{12} M^{-1} C_{21} A^{1/2}) - \lambda A^{1/2} C_{12} M^{-1} \Lambda M^{-1} C_{21} A^{1/2} + \\ + A^{1/2} C_{12} M^{-1} K M^{-1} C_{21} A^{1/2} - \sum_{n=2}^{\infty} \frac{1}{\lambda^{n-2}} A^{1/2} C_{12} M^{-1/2} \Big(M^{-1/2} \Lambda M^{-1/2} - \\ - \frac{1}{\lambda} M^{-1/2} K M^{-1/2}\Big)^n M^{-1/2} C_{21} A^{1/2}\Big] u_1 = 0. \tag{39}$$

We rewrite problem (39) in the form

$$l(\lambda) u_1 = 0, \quad l(\lambda) := I + \lambda^2 C + A^{1/2} F(\lambda) A^{1/2}, \tag{40}$$

$$C := g^{-1} A - A^{1/2} C_{12} M^{-1} C_{21} A^{1/2} = g^{-1} A^{1/2} \big(I - g C_{12} M^{-1} C_{21}\big) A^{1/2},$$

$$F(\lambda) := -\lambda C_{12} M^{-1} \Lambda M^{-1} C_{21} + C_{12} M^{-1} K M^{-1} C_{21} - \\ - \sum_{n=2}^{\infty} \frac{1}{\lambda^{n-2}} C_{12} M^{-1/2} \Big(M^{-1/2} \Lambda M^{-1/2} - \frac{1}{\lambda} M^{-1/2} K M^{-1/2}\Big)^n M^{-1/2} C_{21}.$$

We will prove that the operator $I-gC_{12}M^{-1}C_{21}$ is positive definite. Taking into account the definition of the operators C_{12}, C_{21} (see (35)), the studied operator can be written in the following way:

$$I-gC_{12}M^{-1}C_{21}=I-R^{-1}M^{-1}E_{h,S}P_{\rho}= \\ =\operatorname{diag}\big(I-m_1^{-1}P_{h,S_1}\mathbf{e}_1^{(1)}P_{\rho_1},\ldots,I-m_n^{-1}P_{h,S_n}\mathbf{e}_1^{(n)}P_{\rho_n}\big).$$

To prove that the operator $I-gC_{12}M^{-1}C_{21}$ is positive definite, it suffices to prove that the operators $I-m_l^{-1}P_{h,S_l}\mathbf{e}_1^{(l)}P_{\rho_l}$, $l=\overline{1,n}$ are positive definite. We write these operators in the following way:

$$(41)\quad I-m_l^{-1}P_{h,S_l}\mathbf{e}_1^{(l)}P_{\rho_l}=I-\frac{\rho_l\big(\cdot,\mathbf{e}_1^{(l)}\big)_{\mathbf{G}_{h,S_l}(\Omega_l)}P_{h,S_l}\mathbf{e}_1^{(l)}}{m_l}=$$

$$=I-\frac{\rho_l\|P_{h,S_l}\mathbf{e}_1^{(l)}\|^2_{\mathbf{G}_{h,S_l}(\Omega_l)}}{m_l}\left(\cdot,\frac{P_{h,S_l}\mathbf{e}_1^{(l)}}{\|P_{h,S_l}\mathbf{e}_1^{(l)}\|_{\mathbf{G}_{h,S_l}(\Omega_l)}}\right)_{\mathbf{G}_{h,S_l}(\Omega_l)}\frac{P_{h,S_l}\mathbf{e}_1^{(l)}}{\|P_{h,S_l}\mathbf{e}_1^{(l)}\|_{\mathbf{G}_{h,S_l}(\Omega_l)}}=: \\ =:I-\frac{\rho_l\|P_{h,S_l}\mathbf{e}_1^{(l)}\|^2_{\mathbf{G}_{h,S_l}(\Omega_l)}}{m_l}P_l.$$

Note that the following estimate is true:

$$\|P_{h,S_l}\mathbf{e}_1^{(l)}\|^2_{\mathbf{G}_{h,S_l}(\Omega_l)}<\|\mathbf{e}_1^{(l)}\|^2_{\mathbf{L}_2(\Omega_l)}=\int\limits_{\Omega_l}|\mathbf{e}_1^{(l)}|^2\,d\Omega_l=|\Omega_l|,\quad l=\overline{1,n}.$$

Due to this estimate and (41), we find that for every $\nabla\Phi_l\in\mathbf{G}_{h,S_l}(\Omega_l)$,

$$\big((I-m_l^{-1}P_{h,S_l}\mathbf{e}_1^{(l)}P_{\rho_l})\nabla\Phi_l,\nabla\Phi_l\big)_{\mathbf{G}_{h,S_l}(\Omega_l)}=((I-P_l)\nabla\Phi_l,\nabla\Phi_l)_{\mathbf{G}_{h,S_l}(\Omega_l)}+ \\ +\left(1-\frac{\rho_l}{m_l}\|P_{h,S_l}\mathbf{e}_1^{(l)}\|^2_{\mathbf{G}_{h,S_l}(\Omega_l)}\right)(P_l\nabla\Phi_l,\nabla\Phi_l)_{\mathbf{G}_{h,S_l}(\Omega_l)}> \\ >\left(1-\frac{\rho_l}{m_l}|\Omega_l|\right)(P_l\nabla\Phi_l,\nabla\Phi_l)_{\mathbf{G}_{h,S_l}(\Omega_l)}+((I-P_l)\nabla\Phi_l,\nabla\Phi_l)_{\mathbf{G}_{h,S_l}(\Omega_l)}> \\ >\left(1-\frac{m_{\mathrm{f},l}}{m_l}\right)\big\{(P_l\nabla\Phi_l,\nabla\Phi_l)_{\mathbf{G}_{h,S_l}(\Omega_l)}+((I-P_l)\nabla\Phi_l,\nabla\Phi_l)_{\mathbf{G}_{h,S_l}(\Omega_l)}\big\}= \\ =\left(1-\frac{m_{\mathrm{f},l}}{m_l}\right)(\nabla\Phi_l,\nabla\Phi_l)_{\mathbf{G}_{h,S_l}(\Omega_l)}.$$

This and the inequality $m_l>m_{\mathrm{f},l}$ imply the required statement.

We write the operator C in the following way:

$$C=g^{-1}A^{1/2}\big(I-gC_{12}M^{-1}C_{21}\big)A^{1/2}= \\ =\big(g^{-1/2}(I-gC_{12}M^{-1}C_{21})^{1/2}A^{1/2}\big)^*\big(g^{-1/2}(I-gC_{12}M^{-1}C_{21})^{1/2}A^{1/2}\big).$$

By the theorem on polar decomposition (see, for example, [6, Ch. 6, § 2, S. 7]), there exists a partially isometric operator U such that

$$g^{-1/2}(I-gC_{12}M^{-1}C_{21})^{1/2}A^{1/2}=UC^{1/2}, \\ g^{-1/2}A^{1/2}(I-gC_{12}M^{-1}C_{21})^{1/2}=C^{1/2}U^*.$$

From here we obtain the following representations for the operator $A^{1/2}$:

$$A^{1/2} = g^{1/2} C^{1/2} U^* (I - gC_{12} M^{-1} C_{21})^{-1/2} = \\ = g^{1/2} (I - gC_{12} M^{-1} C_{21})^{-1/2} U C^{1/2}. \tag{42}$$

For every arbitrarily small $\varepsilon > 0$ and number R defined in (38), we define the sectors $\Lambda^{\pm}_{R,\varepsilon} := \{\lambda : |\lambda| > R, \ \ |\arg\lambda \pm \pi/2| < \varepsilon, \ \ -\pi < \arg\lambda < \pi\}$.

For the problem (40), we verify that the conditions of the theorem by M. B. Orazov (see [24, Ch. 7, § 7.3, Lemma 7.7] or [25, p. 412, Lemma 3]) are fulfilled. We prove that if $\lambda \in \Lambda^{\pm}_{R,\varepsilon}$, then

$$T(\lambda) := (I - \lambda C^{1/2})^{-1} A^{1/2} F(\lambda) A^{1/2} (I + \lambda C^{1/2})^{-1} \longrightarrow 0 \quad \text{as } \lambda \to \infty. \tag{43}$$

We preliminary estimate the norm of the bordering operators for $\lambda \in \Lambda^{\pm}_{R,\varepsilon}$:

$$\big\|(I \pm \lambda C^{1/2})^{-1}\big\| = \Big\| \pm \lambda^{-1} \Big(C^{1/2} \pm \frac{1}{\lambda}\Big)^{-1} \Big\| = \frac{|\lambda^{-1}|}{\operatorname{dist}\big(\mp\lambda^{-1}, \sigma(C^{1/2})\big)} \leq \\ \leq \frac{|\lambda^{-1}|}{|\operatorname{Im}(\mp\lambda^{-1})|} = \frac{|\lambda^{-1}|}{|\lambda^{-1}|\cos(\arg\lambda)} \leq \frac{1}{\cos\varepsilon}. \tag{44}$$

Using the representation (42), estimate (44), obvious relation $\|F(\lambda)\| = O(|\lambda|)$ $(\lambda \to \infty)$ (see (40)), and Lemma 3.3 from the paper by A. S. Markus, V. I. Matsaev (see [17, § 3, Lemma 3.3, estimate (3.1)]), we find that

$$\begin{aligned}\|T(\lambda)\| &= \|(I - \lambda C^{1/2})^{-1} A^{1/2} F(\lambda) A^{1/2} (I + \lambda C^{1/2})^{-1}\| = \\ &= \|(I - \lambda C^{1/2})^{-1} C^{1/2} \cdot g^{1/2} U^* (I - gC_{12} M^{-1} C_{21})^{-1/2} \cdot F(\lambda) \times \\ &\qquad \times g^{1/2} (I - gC_{12} M^{-1} C_{21})^{-1/2} U \cdot (I + \lambda C^{1/2})^{-1} C^{1/2}\| \leq \\ &\leq 4g|\lambda|^{-2} \cdot \|F(\lambda)\| \cdot \max\big\{1; \big\|(I + \lambda C^{1/2})^{-1}\big\|\big\} \cdot \max\big\{1; \big\|(I - \lambda C^{1/2})^{-1}\big\|\big\} \times \\ &\qquad \times \|(I - gC_{12} M^{-1} C_{21})^{-1/2}\|^2 \leq \\ &\leq 4g|\lambda|^{-2} \cdot \|F(\lambda)\| \cdot \left(\frac{1}{\cos\varepsilon}\right)^2 \|(I - gC_{12} M^{-1} C_{21})^{-1/2}\|^2 = O(|\lambda|^{-1})\end{aligned}$$

given $\lambda \in \Lambda^{\pm}_{R,\varepsilon}$, $\lambda \to \infty$. From here follows (43).

To apply the mentioned theorem by M. B. Orazov, we only need to show that the operator C has a power asymptotics of eigenvalues. We write this operator in the form of a difference of two operators:

$$C = g^{-1} A - A^{1/2} C_{12} M^{-1} C_{21} A^{1/2} = (gQ\gamma_n)^{-1} - \frac{M^{-1} (Q\gamma_n)^{-1/2} P_{h,S} P_\rho (Q\gamma_n)^{-1/2}}{g}.$$

From the review by M. Sh. Birman, M. Z. Solomjak (see [3, § 7]), it follows that the asymptotics of eigenvalues of the operator $(gQ_l\gamma_{n,l})^{-1}$ has the form

$$\lambda_k\big((gQ_l\gamma_{n,l})^{-1}\big) = \frac{|\Gamma_l|}{g\pi}\, k^{-1}(1 + o(1)) \quad (k \to \infty).$$

Then from the representation $Q\gamma_n = \operatorname{diag}(Q_1\gamma_{n,1}, \dots, Q_n\gamma_{n,n})$ we obtain that the eigenvalues of the operator $(gQ\gamma_n)^{-1}$ have the following asymptotics

$$\lambda_k\big((gQ\gamma_n)^{-1}\big) = \left(\sum_{l=1}^{n} \frac{g\pi}{|\Gamma_l|}\right)^{-1} k^{-1}(1 + o(1)) \quad (k \to \infty).$$

We introduce the operators

$$T_1 := (gQ\gamma_n)^{-1}, \quad T_2 := (gM)^{-1}(Q\gamma_n)^{-1/2}P_{h,S}P_\rho(Q\gamma_n)^{-1/2}.$$

These operators are non-negative, since their s-numbers coincide with their eigenvalues. The operator T_2 is finite dimensional, hence all its eigenvalues, except for a finite number, equal zero. Therefore, we have

$$s_k(T_1) = \lambda_k(T_1) = \left(\sum_{l=1}^{n}\frac{g\pi}{|\Gamma_l|}\right)^{-1} k^{-1}(1+o(1)) \quad (k\to\infty),$$
$$s_k(T_2) = \lambda_k(T_2) = o(k^{-1}) \quad (k\to\infty).$$

Then from the theorem by Ky Fan (see [4, Ch. 2, § 2, S. 5, Theorem 2.3]) it follows that

$$\lambda_k(C) = \lambda_k(T_1 - T_2) = \left(\sum_{l=1}^{n}\frac{g\pi}{|\Gamma_l|}\right)^{-1} k^{-1}(1+o(1)) \quad (k\to\infty). \tag{45}$$

Thus, by the theorem by M. B. Orazov, we obtain that the studied spectral problem (33) has in the strip $\{0 \le \operatorname{Re}\lambda \le \widetilde{\alpha}c^{-1}\}$ two branches of eigenvalues with the following asymptotic behavior:

$$\lambda_k^{(\pm i)} = \pm i\lambda_k^{-1/2}(C)(1+o(1)) \quad (k\to\infty).$$

From here and (45) follows the formula from the formulation of the theorem. □

Theorem 6. *The system of root elements of problem* (29) *forms the Abel-Lidsky basis with brackets in the Hilbert space $\mathcal{H}$ of order $\beta > 1$.*

Proof. We transform the spectral problem (29) into the form

$$\big(\mathcal{C}^{-1}\mathcal{B} - i\mathcal{C}^{-1}\mathcal{P} - (-i\lambda)I\big)z = 0.$$

We perform a substitution of the spectral parameter $-i\lambda =: \mu$ and obtain the problem

$$\big(\mathcal{C}^{-1}\mathcal{B} - i\mathcal{C}^{-1}\mathcal{P} - \mu I\big)z = 0. \tag{46}$$

We introduce the operators

$$\mathcal{A} := \mathcal{A}_0 + \mathcal{A}_1, \quad \mathcal{A}_0 := \mathcal{C}^{-1}\mathcal{B}, \quad \mathcal{A}_1 := -i\mathcal{C}^{-1}\mathcal{P}.$$

The operator $\mathcal{A}_0$ is self-adjoint in the energy space $\mathcal{H}_\mathcal{C}$ of the operator $\mathcal{C}$. Using the reasoning similar to the one performed in Theorem 5, we show that the spectrum of the operator $\mathcal{A}_0$ is discreet and has the following asymptotic distribution:

$$\mu_k^{\pm}(\mathcal{C}^{-1}\mathcal{B}) = \pm\left(\sum_{l=1}^{n}\frac{g\pi}{|\Gamma_l|}\right)^{1/2} k^{1/2}(1+o(1)) \quad (k\to\infty). \tag{47}$$

The operator $\mathcal{A}_1$ is bounded in the space $\mathcal{H}_\mathcal{C}$. From here it follows that the operator $\mathcal{A}_1\mathcal{A}_0^{-q}$ is bounded in $\mathcal{H}_\mathcal{C}$ given $q = 0$:

$$\|\mathcal{A}_1\mathcal{A}_0^{-q}\|_{\mathcal{L}(\mathcal{H}_\mathcal{C})} = \|\mathcal{A}_1\|_{\mathcal{L}(\mathcal{H}_\mathcal{C})} < \infty.$$

From (47) and Theorem 6.2.4 (see [2, p. 106]), that applies for the case when the eigenvalues of the operator $\mathcal{A}_0$ have two accumulation points $\pm\infty$, we obtain that the system of root elements of the spectral problem (29) forms the Abel-Lidsky basis with brackets of order β, where

$$\beta > \beta_0 = \frac{1}{p} - (1-q) = 2 - (1-0) = 1.$$

The definition of the notion of the Abel-Lidsky basis is quite enormous and is not provided here. One can learn about this method of summing by root elements in detail, for example, in [2, p. 106]. □

References

[1] M.S. Agranovich, *Sobolev spaces, their generalizations and elliptic problems in smooth and Lipschitz domains*, Springer, Cham, 2015. Zbl 1322.46002

[2] M.S. Agranovich, *Elliptic operators on closed manifolds*, in *Partial differential equations. VI. Elliptic operators on closed manifolds. Encycl. Math. Sci.*, **63**, 1994, 1–130. Zbl 0802.58050

[3] M.Sh. Birman, M.Z. Solomyak, *Asymptotic behavior of the spectrum of differential equations*, J. Sov. Math., **12**:3 (1979), 247–283. Zbl 0424.35069

[4] I.Ts. Gohberg, M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs, **18**, AMS, Providence, 1969. Zbl 0181.13504

[5] K.V. Forduk, D.A. Zakora, *Problem on small motions of a system of bodies filled with ideal fluids under the action of an elastic damping device*, Lobachevskii J. Math., **42**:5 (2021), 889–900. Zbl 7357839

[6] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin etc., 1980. Zbl 0435.47001

[7] A.N. Kolmogorov, S.V. Fomin, *Elements of the theory of functions and functional analysis*, Mir, Moscow, 1982. Zbl 0501.46001

[8] N.D. Kopachevsky, S.G. Krein, Ngo Zuy Kan, *Operator methods in hydrodynamics. Evolutionary and spectral problems*, Nauka, Moscow, 1989. Zbl 0681.76001

[9] N.D. Kopachevsky, S.G. Krein, *Operator approach to linear problems of hydrodynamics. Vol. 1: Self–adjoint problems for an ideal fluid.* Operator Theory: Advances and Applications, **128**, Birkhäuser, Basel, 2001. Zbl 0979.76002

[10] N.D. Kopachevsky, S.G. Krein, *Operator approach to linear problems of hydrodynamics. Vol. 2: Nonself-adjoint problems for viscous fluids*, Operator Theory: Advances and Applications, **146**, Birkhäuser, Basel, 2003. Zbl 1048.76001

[11] N.D. Kopachevsky, E.V. Syomkina, K.V. Forduk, *On oscillations of a body partially filled with a fluid under the action of an elastic damping device*, in *Mathematics informatics computer science modeling education,* **1** *(MICME-2019_1)* (2019), 35–39.

[12] N.D. Kopachevsky, E.V. Plohaya, K.V. Forduk, *On oscillations of a body partially filled with a fluid under the action of an elastic damping device*, The International Conference "Crimea Autumn Mathematical School–2019, **1** (2019), 176–179.

[13] N.D. Kopachevsky, V.I. Voytitsky, *On small oscillations of three joined pendulums with cavities filled with homogeneous ideal fluids*, Sib. Élektron. Mat. Izv., **17** (2020), 260–299. Zbl 1434.70027

[14] N.D. Kopachevsky, V.I. Voytitsky, Z.Z. Sitshaeva, *On oscillations of two connected pendulums containing cavities partially filled with incompressible fluid*, Sovrem. Mat. Fundam. Napravl., **63**:4 (2017), 627–677. MR3806419

[15] N.D. Kopachevsky, D.O. Tsvetkov, *Small motions of ideal stratified liquid with a free surface totally covered by a crumbled ice*, Ufa Math. J., **10**:3 (2018), 43–58. Zbl 1463.35172

[16] S.G. Krein, N.N. Moiseev, *On oscillations of a vessel containing a liquid with a free surface*, Prikl. Mat. Mekh., **21**:2 (1957), 169–174. MR0089572

[17] A.S. Markus, V.I. Matsaev, *A theorem on comparison of spectra and spectral asymptotics for a Keldysh pencil*, Mat. Sb., N. Ser., **123(165)**:3 (1984), 391–406. Zbl 0562.47014

[18] A.S. Markus, *Introduction to the spectral theory of polynomial operator pencils*, Translations of Mathematical Monographs, **71**, AMS, Providence, 1988. Zbl 0678.47005

[19] N.N. Moiseev, V.V. Rumyantsev, *Dynamics of bodies with fluid-filled cavities*, Nauka, Moscow, 1965. Zbl 0145.22205

[20] N.N. Moiseev, *Motion of a rigid body having a hollow partly filled by an ideal liquid*, Dokl. Akad. Nauk SSSR, **85**:4 (1952), 719–722. Zbl 0049.40607
[21] N.N. Moiseev, *The problem of the motion of a rigid body containing a liquid mass having a free surface*, Mat. Sb. (N.S.), **32(74)**:1 (1953), 61–96. Zbl 0051.17103
[22] G.S. Narimanov, *On the motion of a rigid body of which a cavity is partly filled with a fluid*, Prikl. Mat. Mekh., **20**:1 (1956), 21–38. Zbl 0075.32602
[23] D.E. Okhochimskij, *Theory of the motion of a body with cavities partly filled with a liquid*, Prikl. Mat. Mekh., **20**:1 (1956), 3–20. Zbl 0075.32601
[24] M.B. Orazov, *Some questions of the spectral theory of non-self-adjoint operators and related problems from mechanics*, Dissert. D-ra Fiz.-Mat. Nauk, Ashgabat, 1982.
[25] M.B. Orazov, *Localization of the spectrum in the problem of normal oscillations of an elastic shell filled with a viscous incompressible fluid*, U.S.S.R. Comput. Math. Math. Phys., **25**:2 (1985), 52–58. Zbl 0622.73056
[26] L.N. Sretenskiy, *Oscillation of liquid in a moving vessel*, Dokl. Akad. Nauk SSSR, **10** (1951), 1483–1494.
[27] D.O. Tsvetkov, *Small motions of an ideal stratified fluid partially covered with elastic ice*, Vestn. Udmurt. Univ. Mat. Mekh. Komp. Nauki, **28**:3 (2018), 328–347. Zbl 1421.35293
[28] D.O. Tsvetkov, *Small motions of an ideal stratified fluid with a free surface completely covered with the elastic ice*, Sib. Élektron. Mat. Izv., **15** (2018), 422–435. Zbl 1398.35182
[29] D.O. Tsvetkov, *Crumbled ice on the surface of a multilayered fluid*, Sib. Élektron. Mat. Izv., **17** (2020), 777–801. Zbl 1446.35143
[30] N.E. Zhukovskii, *On motion of solid body with cavities filled with homogeneous drip fluid*, in Izbrannye socineniya. Vol. 1, OGIZ, Moscow-Leningrad, 1948, 31–52. MR0034357

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