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OPTIMAL DISCRETE NEUMANN ENERGY IN A BALL AND AN ANNULUS

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Abstract. In this paper, we prove some exact estimates for the discrete Neumann energy of a ball and an annulus in Euclidean space for points located on circles. The proofs are based on dissymmetrization and analysis of the asymptotic behavior of the Dirichlet integral of the potential function.

Keywords: discrete energy, Green function, Neumann function, dissymmetrization.

1. Introduction

In this paper, $\mathbb{R}^d$ will mean the d-dimensional Euclidean space of points $\mathbf{x}$ of the form $(x_1, \dots, x_d)$ with the usual length and distance, $d \geq 2$. In the case $d = 2$ we assume that $\mathbb{R}^2$ is the complex plane. The solution of the classical Neumann problem in the bounded domain $D \subset \mathbb{R}^d$ for the Poisson equation requires the construction of the Neumann function (this function is sometimes called the Green function for the Neumann problem or the Green function of the second kind). The classical Neumann function [1], [2] is defined as the function of $\mathbf{x} \in D$ in $D \setminus \{\mathbf{y}\}$ that satisfies the conditions

$$N(\mathbf{x}, \mathbf{y}, D) = \frac{\mu_d(|\mathbf{x} - \mathbf{y}|) + v(\mathbf{x}, \mathbf{y}, D)}{w_d}, \tag{1}$$

$$\frac{\partial N(\mathbf{x}, \mathbf{y}, D)}{\partial n_{\mathbf{x}}} = -\frac{1}{s_{d-1}(\partial D)},$$

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$$\int_{\partial D} N(\mathbf{x}, \mathbf{y}, D) d\sigma_{\mathbf{x}} = 0.$$

Here $\mu_d(\cdot)$ is the fundamental solution of the Laplace equation, ($\mu_2(\rho) = -\log \rho$, $\mu_d(\rho) = \rho^{2-d}/(d-2)$ for $d \geq 3$), $w_d = 2\pi^{d/2}/\Gamma(d/2)$ is area of the unit hypersphere, $v(\mathbf{x}, \mathbf{y}, D)$ is some harmonic function in D, s_{d-1} is the Lebesgue measure and differentiation is taken with respect to the outward normal.

There are many studies related to extremal problems for various types of discrete charge energies (see, for example, [3], [4], [5] and references therein). In [6], two estimates are obtained for the discrete energy of the Green function of an annulus on the plane in the case of points located on some circle. These results have been extended into Euclidean space in [7]. The purpose of this paper is to obtain results of a similar sort for the Neumann function.

Let us recall the definition of discrete Green energy [8]. Let $\Delta = \{\delta_k\}_{k=1}^n$ be an arbitrary discrete charge (the set of real numbers) that takes the value δ_k at the point $\mathbf{x}_k$, $k = 1, \dots, n$ of D . *The Green's energy* of this charge with respect to D is the quantity

$$E(X, \Delta, D) = \sum_{k=1}^{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} \delta_k \delta_l g_D(\mathbf{x}_k, \mathbf{x}_l),$$

where $g_D(\mathbf{x}_k, \mathbf{x}_l)$ is the Green's function of the domain D. In a similar way we define *Neumann energy*

$$En(X, \Delta, D) = \sum_{k=1}^{n} \sum_{\substack{l=1 \\ l \neq k}}^{n} \delta_k \delta_l N(\mathbf{x}_k, \mathbf{x}_l, D).$$

Everywhere below, D is either a ball of the form $\{|\mathbf{x}| < \tau\}$ or an annulus of the form $\{\tau_1 < |x| < \tau_2\}$. Let us take the following notation: $B(\mathbf{a}, r)$ is the open ball centered at point $\mathbf{a}$ of radius r, J is $(d-2)$–dimensional plane $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = (0, 0, x_3, \dots, x_d)\}$. We need the cylindrical coordinates $(r, \theta, \mathbf{x}')$ of the point $\mathbf{x} = (x_1, \dots, x_d)$ in $\mathbb{R}^d$, related to the Cartesian coordinates by the relations $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $\mathbf{x}' \in J$. Entries like $\{\theta = \varphi\}$ mean the set of points in $\mathbb{R}^d$ with polar coordinates (r, φ, x'), $r \geq 0$, $\mathbf{x}' \in J$, where φ is fixed.

Let $\Omega = \{S\}$ is a set consisting of a finite number of distinct circles S of the form $S = \{(r_0, \theta, \mathbf{x}_0') : 0 \leq \theta \leq 2\pi\}$ lying in D (here $r_0 > 0$ and $\mathbf{x}_0' \in J$ are assumed to be fixed). For arbitrary real numbers θ_j, $j = 0, \dots, m-1$,

$$0 \leq \theta_0 < \theta_1 < \ldots < \theta_{m-1} < 2\pi,$$

denote by $X = \{\mathbf{x}_k\}_{k=1}^n$ the set of intersection points of circles from Ω with half planes

$$L_j = \{(r, \theta, \mathbf{x}') : \theta = \theta_j\}, æ = 0, \dots, m-1.$$

We also denote the set of intersection points of circles from Ω with symmetrical half-planes

$$L_j^* = \{(r, \theta, \mathbf{x}') : \theta = 2\pi j/m\},\ j = 0, \dots, m-1.$$

as $X^* = \{\mathbf{x}_k^*\}_{k=1}^n$.

The following theorems show that for some choice of the charge Δ, the symmetric configuration can provide either the maximum or the minimum of the Neumann energy $En(X, \Delta, D)$.

Theorem 1. *Let D be the ball or the annulus, Ω, X and X^* are defined above, the charge $\Delta = \{\delta_k\}_{k=1}^n$ takes the same values $\delta_k = \delta_l$ at the points $\mathbf{x}_k \in X$ and $\mathbf{x}_l \in X$ located on the same circle from Ω and*

$$\sum_{k=1}^{n} \delta_k = 0.$$

In addition, let the points $\mathbf{x}_k \in X$ and $\mathbf{x}_k^* \in X^*$ lie on the same circle in Ω, $k = 1, \dots, n$. Then

$$En(X, \Delta, D) \geq En(X^*, \Delta, D).$$

Theorem 2. *Let D be the ball or the annulus, Ω, X, X^*, Δ are defined above, m is an even number and $\delta_k = -\delta_l$ at points $\mathbf{x}_k \in X$ and $\mathbf{x}_l \in X$ lying on the same circle from Ω and on neighboring half-planes from the collection $\{L_j\}_{j=0}^{m-1}$. Then*

$$En(X, \Delta, D) \leq En(X^*, \Delta, D),$$

where the points X^* are numbered as follows: if $\mathbf{x}_k^* \in X^*$ lies on the intersection of circle S from Ω with the half-plane L_j^*, then the corresponding point $\mathbf{x}_k \in X$ must lie at the intersection of S and the half-plane L_j, $k = 1, \dots, n$, $0 \leq j \leq m-1$.

Note that the theorems obtained in this paper are also valid when D is the domain of rotation (the domain $D \subset \mathbb{R}^d$ is called *domain of rotation* with respect to the J axis if the point $(r, \varphi, \mathbf{x}')$ belongs to D for any point $(r, \theta, \mathbf{x}') \in B$ and any φ).

Under the additional condition

$$\sum_{k=1}^{n} \delta_k = 0 \tag{2}$$

we define the function

$$u(\mathbf{x}) = u(\mathbf{x}; X, D, \Delta) = \sum_{k=1}^{n} \delta_k N(\mathbf{x}, \mathbf{x}_k, D),$$

which we call the potential Neumann function of the configuration X, Δ, D. The expansion of the potential function in a neighborhood of the point $\mathbf{x}_k$, $k = 1, \dots, n$, follows directly from the definition

$$u(\mathbf{x}) = \delta_k \frac{\mu_d(|\mathbf{x} - \mathbf{x}_k|)}{w_d} + a_k + o(1),\ \mathbf{x} \to \mathbf{x}_k, \tag{3}$$

where

$$a_k = \delta_k \frac{v(\mathbf{x}_k, \mathbf{x}_k, D)}{w_d} + \sum_{\substack{l=1 \\ l \neq k}}^{n} \delta_l N(\mathbf{x}_l, \mathbf{x}_k, D).$$

Sum

$$\sum_{k=1}^{n} \delta_k a_k = \sum_{k=1}^{n} \sum_{l=1}^{n} \eta_{kl}(D) \delta_k \delta_l = En(X, \Delta, D) + \sum_{k=1}^{n} \frac{\delta_k^2 v(\mathbf{x}_k,\ x_k, D)}{w_d} \tag{4}$$

is a quadratic form of variables Δ with coefficients $\eta_{kl}(D)$ depending on the Neumann function. Denote this quadratic form by

$$Qn(X, \Delta, D) = \sum_{k=1}^{n} \sum_{l=1}^{n} \eta_{kl}(D) \delta_k \delta_l, \tag{5}$$

where $\eta_{kl}(D) = N(\mathbf{x}_k, \mathbf{x}_l, D),\ k \neq l$, $\eta_{kk}(D) = v(\mathbf{x}_k, \mathbf{x}_k, D)/w_d$.

Quadratic forms of this kind, as well as forms with coefficients depending on the Green or Robin functions, play an important role in the geometric theory of functions. Various inequalities for such forms and their applications are found in the works of Alenitsin, Nehari, Duren, Schiffer, Dubinin and other mathematicians (see [9], [10], [11], [12]) . We prove that

$$Qn(X, \Delta, D) \geq Qn(X^*, \Delta, D)$$

under the conditions of Theorem 1 and

$$Qn(X, \Delta, D) \leq Qn(X^*, \Delta, D)$$

under the conditions of Theorem 2. To calculate the coefficients of the quadratic form Q_n under the additional condition (2), we can use the generalized Neumann function [13] instead of the classical one. The analytical expression of the quadratic form Q_n of a planar disk is known.The Neumann function of the unit disk U [2] is

$$N\left(z, z_0, U\right) = -\frac{\log|z - z_0|1 - z\overline{z_0}|}{2\pi},$$

so

$$\eta_{kl}(U) = -\frac{\log|z_k - z_l|1 - z_k\overline{z_l}|}{2\pi},\ k \neq l,$$

$$\eta_{kk}(U) = -\frac{\log(1 - |z_k|^2)}{2\pi}.$$

The coefficients $\eta_{kl}(K)$ of the quadratic form of the planar annulus $K = \{\mu < |z| < 1\}$ were given in [13]. Namely,

$$\eta_{kl}(K) = \begin{cases} -\frac{1}{2\pi}\log|\theta_1(i\log(z_k\overline{z_l})/2;\mu)\theta_1(i\log(z_k/z_l)/2;\mu)|, & k \neq l, \\ \frac{1}{2\pi}\log\frac{4|z_k|^2|\sin\left(i\log|z_k|\right)|}{(1-|z_k|^2)|\ theta_1(i\log|z_k|;\mu)\theta_1'(0;\mu)|}, & k = l, \end{cases}$$

where

$$\theta_1(z;\mu) = -i\sum_{n=-\infty}^{\infty}(-1)^n\mu^{(n+1/2)^2}e^{i(2n+1)z}.$$

In the space of dimension $d \geq 3$, we did not find in the literature an analytic expression for the Neumann function of an annulus. For the unit ball $U = B(0,1)$ the Neumann function was found in [2] and has the form

$$N(\mathbf{x}, \mathbf{y}, U) = \frac{1}{\omega_d}\left(\mu_d(|\mathbf{x} - \mathbf{y}|) + \mu_d\left(\left|x|\mathbf{y}| - \frac{\mathbf{y}}{|\mathbf{y}|}\right|\right) + \epsilon_1(\mathbf{x}, \mathbf{y})\right) + Const,$$

where $\epsilon_1(\mathbf{x}, \mathbf{y})$ is given by the formulas

$$\epsilon_1(\mathbf{x}, \mathbf{y}) = \log\frac{2}{\left|1 - (\mathbf{x}, \mathbf{y}) + \left|\mathbf{x}|\mathbf{y}| - \frac{\mathbf{y}}{|\mathbf{y}|}\right|\right|}, d = 3;$$

$$\epsilon_1(\mathbf{x}, \mathbf{y}) = \frac{(\mathbf{x}, \mathbf{y})}{\sqrt{|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}, \mathbf{y})^2}}\arctan\frac{\sqrt{|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}, \mathbf{y})^2}}{1 - (\mathbf{x}, \mathbf{y})} - \log\left|\mathbf{x}|\mathbf{y}| - \frac{\mathbf{y}}{|\mathbf{y}|}\right|, d = 4;$$

$$\epsilon_1(\mathbf{x},\mathbf{y}) = \log \frac{2}{\left|1-(\mathbf{x},\mathbf{y})+\left|\mathbf{x}|\mathbf{y}|-\frac{\mathbf{y}}{|\mathbf{y}|}\right|\right|} + \sum_{k=1}^{p-1}\frac{1}{(2k-1)}\left(\left|\mathbf{x}|\mathbf{y}|-\frac{\mathbf{y}}{|\mathbf{y}|}\right|^{1-2k}-1\right)$$

$$+\sum_{k=1}^{p-1}\sum_{i=0}^{p-k-1}\frac{2^i(k+i-1)!(2k-3)!!}{(k-1)!(2k+2i-1)!!}\frac{(\mathbf{x},\mathbf{y})|\mathbf{x}|^{2i}|\mathbf{y}|^{2i}}{(|\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x},\mathbf{y})^2)^{i+1}}\left(\frac{|\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x},\mathbf{y})}{\left|\mathbf{x}|\mathbf{y}|-\frac{\mathbf{y}}{|\mathbf{y}|}\right|^{2k-1}}+(\mathbf{x},\mathbf{y})\right),$$

$$d\geq 5, d=2p+1, p\geq 2;$$

$$\epsilon_1(\mathbf{x},\mathbf{y}) = -\log\left|\mathbf{x}|\mathbf{y}|-\frac{\mathbf{y}}{|\mathbf{y}|}\right| + \sum_{k=1}^{p-1}\frac{1}{2k}\left(\left|\mathbf{x}|\mathbf{y}|-\frac{\mathbf{y}}{|\mathbf{y}|}\right|^{-2k}-1\right)$$

$$+(\mathbf{x},\mathbf{y})\arctan\frac{\sqrt{|\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x},\mathbf{y})}^2}{(1-(\mathbf{x},\mathbf{y}))}\sum_{k=0}^{p-1}\frac{(2k-1)!!}{2^k k!}\frac{|\mathbf{x}|^{2k}|\mathbf{y}|^{2k}}{(|\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x},\mathbf{y})^2)^{k+\frac{1}{2}}}$$

$$+\sum_{k=1}^{p-1}\sum_{i=0}^{p-k-1}\frac{(2k+2i-1)!!(k+1)!}{2^{i+1}(2k-1)!!(k+i)!}\frac{(\mathbf{x},\mathbf{y})|\mathbf{x}|^{2i}|\mathbf{y}|^{2i}}{(|\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x},\mathbf{y})^2)^{i+1}}\left(\frac{|\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x},\mathbf{y})}{\left|\mathbf{x}|\mathbf{y}|-\frac{\mathbf{y}}{|\mathbf{y}|}\right|^{2k}}-(\mathbf{x},\mathbf{y})\right),$$

$$d\geq 6, d=2p+2, p\geq 2;$$

$0!=1$, $(-1)!!=1$.

2. Proof of Theorem 1.

Denote by D_r the domain obtained by removing from D balls with center $\mathbf{x}_k$ of radius r, $D_r = D\setminus(\cup_{k=1}^{n}\overline{B(\mathbf{x}_k,r)})$). Then the Dirichlet integral $I(u,D_r)=\int_{D_r}|\nabla u|^2 d\mathbf{x}$ of the potential function satisfies the asymptotic formula [14, Lemma 2.1], [15, Lemma 1]

$$(6)\quad I(u,D_r)=\left(\sum_{k=1}^{n}\delta_k^2\right)\frac{\mu_d(r)}{w_d}+En(X,\Delta,D)+\sum_{k=1}^{n}\frac{\delta_k^2 v(\mathbf{x}_k,\mathbf{x}_k,D)}{w_d}+o(1),\ \ r\to 0,$$

or

$$(7)\qquad I(u,D_r)=\left(\sum_{k=1}^{n}\delta_k^2\right)\frac{\mu_d(r)}{w_d}+\sum_{k=1}^{n}\delta_k a_k+o(1),\ \ r\to 0.$$

The function $v(\mathbf{x})$ is called admissible for D, X, Δ, if $v(\mathbf{x})\in\mathrm{Lip}$ in a neighborhood of every point D, except perhaps for a finite number of points, it is continuous in $\overline{D}\setminus\bigcup_{k=1}^{n}\{\mathbf{x}_k\}$, and in a neighborhood of $\mathbf{x}_k$ we have the asimptotic expansion

$$(8)\qquad v(\mathbf{x})=\delta_k\frac{\mu_d(|\mathbf{x}-\mathbf{x}_k|)}{w_d}+b_k+o(1),\ \mathbf{x}\to\mathbf{x}_k.$$

For an admissible function v and the potential function u we have the asymptotics [14, Lemma 2.2], [12, Lemma 2]

$$(9)\qquad I(v-u,D_r)=I(v,D_r)-I(u,D_r)-2\sum_{k=1}^{n}\delta_k(b_k-a_k)+o(1),\ \ r\to 0.$$

Let $u_1(\mathbf{x})$ be the potential Neumann function of X, Δ and let $u_2(\mathbf{x})$ be the potential Neumann function of X^*, Δ. We denote by Dis the dissymmetrization described in the proof of Theorem 1 in [7]. In the domain D we construct the function $v(\mathbf{x})$ according to the rule

$$v(\mathbf{x}) = u_2(Dis^{-1}(\mathbf{x})).$$

Since the configuration X^*, Δ, D is symmetric, the function $u_2(\mathbf{x})$ is invariant under any mapping from the symmetry group $\varphi \in \Phi$ involved in the definition of the dissymmetrization Dis. Therefore $v(\mathbf{x})$ is uniquely defined and it is admissible for X, Δ. Since dissymmetrization is, in fact, a special permutation of angles, then

$$I(v, D_r) = I(u_2, D_r^*),$$

where $D_r^* = D \setminus (\cup_{k=1}^n \overline{B(\mathbf{x}_k^*, r)})$. The relations (7), (9) imply

$$(10) \quad 0 \le I(v, D_r) - I(u_1, D_r) - 2\sum_{k=1}^n \delta_k(b_k - a_k) + o(1) =$$

$$I(u_2, D_r) - I(u_1, D_r) - 2\sum_{k=1}^n \delta_k(b_k - a_k) + o(1) = \sum_{k=1}^n \delta_k(a_k - b_k) + o(1), \quad r \to 0,$$

$$(11) \qquad \sum_{k=1}^n \delta_k b_k \le \sum_{k=1}^n \delta_k a_k.$$

Here b_k are the coefficients of the asymptotic expansion of the potential function of the symmetric configuration, and a_k correspond to the nonsymmetric configuration. Taking into account (4), we get

$$(12) \quad En(X^*, \Delta, D) + \sum_{k=1}^n \frac{\delta_k^2 v(\mathbf{x}_k^*, \mathbf{x}_k^*, D)}{w_d} \le En(X, \Delta, D) + \sum_{k=1}^n \frac{\delta_k^2 v(\mathbf{x}_k, \mathbf{x}_k, D)}{w_d}.$$

Since D is the ball or the annulus, $v(\mathbf{x}, \mathbf{x}, D) = v(\mathbf{y}, \mathbf{y}, D)$ for any two points $\mathbf{x}, \mathbf{y}$, on the same circle S from Ω. Therefore,

$$\sum_{k=1}^n \frac{\delta_k^2 v(\mathbf{x}_k^*, \mathbf{x}_k^*, D)}{w_d} = \sum_{k=1}^n \frac{\delta_k^2 v(\mathbf{x}_k, \mathbf{x}_k, D)}{w_d}.$$

Thus, the inequality (12) proves Theorem 1.

3. Proof of Theorem 2.

Let us first prove an auxiliary lemma.

Lemma 1. *Let $Y = \{\mathbf{y}_q\}_{q=1}^l$ be the set of points lying on the half-plane $\{\theta = 0\}$, $\Delta_0 = \{\sigma_q\}_{q=1}^l$ be some charge, $0 < \alpha < \pi$, $D(\alpha) = D \cap \{0 < \theta < \alpha\}$, $\Gamma(\alpha) = \partial D(\alpha) \cap \{\theta = \alpha\}$ or $D(\alpha) = D \cap \{-\alpha < \theta < 0\}$, $\Gamma(\alpha) = \partial D(\alpha) \cap \{\theta = -\alpha\}$. Consider the function $h_\alpha(\mathbf{x})$ that satisfies the following conditions: it is harmonic in $D(\alpha)$ except for the points of Y; it is continuous in $\overline{D(\alpha)} \setminus Y$; this function is zero on $\Gamma(\alpha)$; it has the zero derivative on the remainder of the boundary $\partial D(\alpha) \setminus Y$; and the decomposition is true*

$$(13) \qquad h_\alpha(\mathbf{x}) = \sigma_q \frac{\mu_d(|\mathbf{x} - \mathbf{y}_q|)}{w_d} + c_q(\alpha) + o(1), \, \mathbf{x} \to \mathbf{y}_q,$$

in a neighborhood of the points $\mathbf{y}_q$. *Then the function*

$$f(\alpha) = \sum_{q=1}^{l} \sigma_q c_q(\alpha)$$

is concave by $0 < \alpha < \pi$ *as a function of* α.

Proof. Outside the domain $\overline{D(\alpha)}$, we assume that the function h_α is extended by zero. In terms of [14], [15], the function $h_\alpha(\mathbf{x})$ is called the potential function of the collection $D(\alpha)$, $\Gamma(\alpha)$, Y, Δ_0. Repeating the proof of Lemma 2.1 in [14] we get the decomposition

$$I(h_\alpha, D(\alpha)_r) = \frac{1}{2}\left(\sum_{q=1}^{l}\sigma_q^2\right)\frac{\mu_d(r)}{w_d} + \frac{1}{2}\sum_{q=1}^{l}\sigma_q c_q(\alpha) + o(1),\ r \to 0. \tag{14}$$

For $0 < \alpha < \beta < \pi$ we construct in the domain $D((\alpha+\beta)/2)$ the function $v_{(\alpha+\beta)/2}(\mathbf{x})$ by rule

$$v_{(\alpha+\beta)/2}(\mathbf{x}) = \frac{h_\alpha(\mathbf{x}) + h_\beta(\mathbf{x}) - h_\beta(\mathbf{x}^*)}{2},$$

where $\mathbf{x}^*$ means the point symmetric to $\mathbf{x}$ with respect to the half-plane $\{\theta = (\alpha+\beta)/2\}$ (or $\{\theta = -(\alpha+\beta)/2\}$). The function $v_{(\alpha+\beta)/2}(\mathbf{x})$ is admissible for $D((\alpha+\beta)/2)$, $\Gamma((\alpha+\beta)/2))$, Y, Δ_0 and it has the decomposition

$$v_{(\alpha+\beta)/2}(\mathbf{x}) = \sigma_q \frac{\mu_d(|\mathbf{x}-\mathbf{y}_q|)}{w_d} + \frac{c_q(\alpha)+c_q(\beta)}{2} + o(1),\ \mathbf{x} \to \mathbf{y}_q. \tag{15}$$

Applying the analogue of the formula (9) (see the proof [15, Lemma 2], [14, Lemma 2.2]), we obtain

$$0 \le I(v_{(\alpha+\beta)/2}, D((\alpha+\beta)/2)_r) - I(h_{(\alpha+\beta)/2}, D((\alpha+\beta)/2)_r) - \sum_{q=1}^{l}\sigma_q\left(\frac{c_q(\alpha)+c_q(\beta)}{2} - c_q\left(\frac{\alpha+\beta}{2}\right)\right) + o(1),\ r \to 0. \tag{16}$$

From the definition of the function $v_{(\alpha+\beta)/2}(\mathbf{x})$ and the modulus property of the vector $|\mathbf{x}+\mathbf{y}|^2 \le 2(|\mathbf{x}|^2 + |\mathbf{y}|^2)$ follows

$$I(v_{(\alpha+\beta)/2}, D((\alpha+\beta)/2)_r) \le \frac{1}{2}\int\limits_{D(\alpha+\beta)/2}(|\nabla(h_\alpha(\mathbf{x}) - h_\beta(\mathbf{x}^*))|^2)d\mathbf{x} + \frac{1}{2}\int\limits_{D(\alpha+\beta)/2}|\nabla h_\beta(\mathbf{x})|^2 d\mathbf{x} = \frac{1}{2}\int\limits_{D(\alpha)}|\nabla h_\alpha(\mathbf{x})|^2 d\mathbf{x} + \frac{1}{2}\int\limits_{D(\beta)}|\nabla h_\beta(\mathbf{x})|^2 d\mathbf{x}. \tag{17}$$

The relations (14), (16), (17) imply

$$\sum_{q=1}^{l}\sigma_q c_q(\alpha) + \sum_{q=1}^{l}\sigma_q c_q(\beta) \le 2\sum_{q=1}^{l}\sigma_q c_q\left(\frac{\alpha+\beta}{2}\right),$$

or

$$\frac{f(\alpha)+f(\beta)}{2} \le f\left(\frac{\alpha+\beta}{2}\right).$$

The last inequality means that the function $f(\alpha)$ is concave. The lemma is proved. □

Let us now proceed to the proof of Theorem 2. Note that the conditions of the theorem guarantee that $\sum_{k=1}^{n}\delta_k=0$. We assume that $\theta_0=0$ and $\theta_m=2\pi$. Denote by

$$B_j=D\cap\{\theta_j\le\theta\le\theta_{j+1}\},$$
$$B_j^+=D\cap\{\theta_j\le\theta\le\frac{\theta_j+\theta_{j+1}}{2}\},\ B_j^-=D\cap\{\frac{\theta_j+\theta_{j+1}}{2}\le\theta\le\theta_{j+1}\},$$
$$\alpha_j=\frac{\theta_{j+1}-\theta_j}{2},$$

$j=0,\dots,m-1$. Let $Y=\{y_q\}_{q=1}^{l}$ are points from X lying on $\{\theta=0\}$ and $\Delta_0=\{\sigma_q\}_{q=1}^{l}$ are their corresponding charges ($\sigma_q=\delta_k$ if $\mathbf{y}_q=\mathbf{x}_k$). We denote by $h^1_\alpha(\mathbf{x})$ the function $h_\alpha(\mathbf{x})$ from Lemma 1 which defined by the set Y, the charge Δ_0, the domain $D(\alpha)=D\cap\{0<\theta<\alpha\}$. Similarly, let $h^2_\alpha(\mathbf{x})$ be defined by Y, $-\Delta_0=\{-\sigma_q\}_{q=1}^{l}$ and $D(\alpha)=D\cap\{0<\theta<\alpha\}$. The function $h^3_\alpha(\mathbf{x})$ corresponds to Y, $-\Delta_0$ and $D(\alpha)=D\cap\{-\alpha<\theta<0\}$, and the function $h^4_\alpha(\mathbf{x})$ corresponds to Y, Δ_0 and $D(\alpha)=D\cap\{-\alpha<\theta<0\}$. We will denote by $c^p_q(\alpha)$ constant from the expansion (13) of the function $h^p_\alpha(\mathbf{x})$, $p=1,2,3,4$.

We define the functions

$$\psi_j^+(\mathbf{x})=h^1_{\alpha_j}(\theta_j(\mathbf{x})),\mathbf{x}\in B_j^+,j=0,2,\dots,m-2,$$
$$\psi_j^+(\mathbf{x})=h^2_{\alpha_j}(\theta_j(\mathbf{x})),\mathbf{x}\in B_j^+,j=1,3,\dots,m-1,$$
$$\psi_j^-(\mathbf{x})=h^3_{\alpha_j}(\theta_{j+1}(\mathbf{x})),\mathbf{x}\in B_j^-,j=0,2,\dots,m-2,$$
$$\psi_j^-(\mathbf{x})=h^4_{\alpha_j}(\theta_{j+1}(\mathbf{x})),\mathbf{x}\in B_j^+,j=1,3,\dots,m-1,$$

where the notation $\varphi(\mathbf{x})$ means rotation through the angle φ (namely $\varphi(\mathbf{x})=(r,\theta-\varphi,\mathbf{x}')$ if $\mathbf{x}=(r,\theta,\mathbf{x}')$). In the domain B_j, $j=0,\dots,m-1$, we also consider the functions

$$\psi_j(\mathbf{x})=\begin{cases}\psi_j^+(\mathbf{x}),\mathbf{x}\in B_j^+,\\ \psi_j^-(\mathbf{x}),\mathbf{x}\in B_j^-,\\ 0,\ \mathbf{x}=(r,(\theta_j+\theta_{j+1})/2,x').\end{cases}$$

By construction, the function $\psi_j(\mathbf{x})$ is harmonic in B_j, has a zero normal derivative on the boundary of ∂B_j (except for the points of X), and a decomposition of type (8) in a neighborhood of points $X\cap\overline{B_j}$. Let $u(\mathbf{x})$ be the potential Neumann function of X, Δ, and $\sum^j\delta_k a_k$ means the sum of those terms $\delta_k a_k$ that correspond to the points $x_k\in\overline{B}_j$. Repeating the proof of Lemma 2.2 [14], we obtain

$$0\le I(u,(B_j)_r)-I(\psi_j,(B_j)_r)-\sum^{j}\delta_k a_k+\sum_{q=1}^{l}\sigma_q c_q^1(\alpha_j)+\sum_{q=1}^{l}(-\sigma_q)c_q^3(\alpha_j)+o(1),\ j=0,\dots,m-2,\tag{18}$$

$$0\le I(u,(B_j)_r)-I(\psi_j,(B_j)_r)-\sum^{j}\delta_k a_k+\sum_{q=1}^{l}(-\sigma_q)c_q^2(\alpha_j)+\sum_{q=1}^{l}\sigma_q c_q^4(\alpha_j)+o(1),\quad j=1,\dots,m-1.\tag{19}$$

To get inequality

$$\sum_{k=1}^{n}\delta_k a_k \le \frac{1}{2}\sum_{j=0,\dots,m-2}\sum_{q=1}^{l}\sigma_q c_q^1(\alpha_j)+\frac{1}{2}\sum_{j=0,\dots,m-2}\sum_{q=1}^{l}(-\sigma_q)c_q^3(\alpha_j)$$
$$+\frac{1}{2}\sum_{j=1,\dots,m-1}\sum_{q=1}^{l}(-\sigma_q)c_q^2(\alpha_j)+\frac{1}{2}\sum_{j=1,\dots,m-1}\sum_{q=1}^{l}\sigma_q c_q^4(\alpha_j), \tag{20}$$

we sum the inequalities (18), (19) over all $j=0,\dots,m-1$, apply the expansion (7) and equalities

$$I(\psi_j,(B_j)_r)=\sum_{q=1}^{l}\sigma_q^2\frac{\mu_d(r)}{w_d}+\frac{1}{2}\sum_{q=1}^{l}\sigma_q c_q^1(\alpha_j)+\frac{1}{2}\sum_{q=1}^{l}(-\sigma_q)c_q^3(\alpha_j)+o(1),$$
$$j=0,\dots,m-2,$$

$$I(\psi_j,(B_j)_r)=\sum_{q=1}^{l}\sigma_q^2\frac{\mu_d(r)}{w_d}+\frac{1}{2}\sum_{q=1}^{l}(-\sigma_q)c_q^2(\alpha_j)+\frac{1}{2}\sum_{q=1}^{l}\sigma_q c_q^4(\alpha_j)+o(1),$$
$$j=1,\dots,m-1,$$

$$m\sum_{q=1}^{l}\sigma_q^2=\sum_{k=1}^{n}\delta_k^2,$$

and also take into account the fact that each point $\mathbf{x}_k\in X$ belongs to two closed domains $\overline{B}_j$. Further note that the definition of $h_\alpha(\mathbf{x})$ given in Lemma 1 implies the equalities $h_\alpha^1(\mathbf{x})=-h_\alpha^2(\mathbf{x})$, $h_\alpha^3(\mathbf{x})=-h_\alpha^4(\mathbf{x})$, Therefore,

$$\sum_{q=1}^{l}\sigma_q c_q^1(\alpha_j)=\sum_{q=1}^{l}(-\sigma_q)c_q^2(\alpha_j),\ \sum_{q=1}^{l}\sigma_q c_q^4(\alpha_j)=\sum_{q=1}^{l}(-\sigma_q)c_q^3(\alpha_j).$$

In addition, there is the unique harmonic (except for points of Y) function in the domain $B(\alpha)=D\cap\{-\alpha<\theta<\alpha\}$ vanishing on $\partial B(\alpha)\cap(\{\theta=\alpha\}\cup\{\theta=-\alpha\})$, having zero normal derivative on the remaining part of the boundary $\partial B(\alpha)$ and with expansion (13) in the neighborhood of $\mathbf{y}_q$, $q=1,\dots,l$. This function coincides with $h_\alpha^1(\mathbf{x})$ in the $D\cap\{0<\theta<\alpha\}$, and it coincides with the function $h_\alpha^4(\mathbf{x})$ in the $D\cap\{-\alpha<\theta<0\}$. Therefore $c_q^1(\alpha)=c_q^4(\alpha)$ and the inequality (20) becomes

$$\sum_{k=1}^{n}\delta_k a_k\le\sum_{j=0}^{m-1}\sum_{q=1}^{l}\sigma_q c_q^1(\alpha_j)=\sum_{j=0}^{m-1}f(\alpha_j). \tag{21}$$

From (21), the concavity of the function $f(\alpha)$ due to Lemma 1, and the equality $\sum_{j=1}^{m}\alpha_j=\pi$, we obtain the inequality

$$\sum_{k=1}^{n}\delta_k a_k\le\sum_{j=0}^{m-1}f(\alpha_j)\le mf\left(\frac{\sum_{j=0}^{m-1}\alpha_j}{m}\right)=mf\left(\frac{\pi}{m}\right). \tag{22}$$

Now let $u^*(\mathbf{x})$ be the potential Neumann function of X^*, Δ and a_k^* denote the corresponding constants from the asymptotic expansion. Repeating the above proof with X replaced by X^*, it is easy to see that the equality sign holds in all inequalities and

$$\sum_{k=1}^{n} \delta_k a_k^* = mf\left(\frac{\pi}{m}\right). \tag{23}$$

So the inequality (23) means

$$\sum_{k=1}^{n} \delta_k a_k \leq \sum_{k=1}^{n} \delta_k a_k^*. \tag{24}$$

As noted in the proof of Theorem 1, (24) is equivalent to the required statement.

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