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ONE NECESSARY CONDITION FOR THE REGULARITY OF A p-GROUP AND ITS APPLICATION TO WEHRFRITZ'S PROBLEM

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ABSTRACT. We obtain a necessary condition for the regularity of a p-group in terms of segments of P. Hall's collection formula. For any prime number p such that $(p+2)/3$ is an integer, we prove that a Sylow p-subgroup of the group $GL_n(\mathbb{Z}_{p^m})$ is not regular if $n \geqslant (p+2)/3$ and $m \geqslant 3$. We also list all regular Sylow p-subgroups of the Chevalley group of type G_2 over the ring $\mathbb{Z}_{p^m}$.

Keywords: regular p-group, linear group, Chevalley group.

1. Introduction

In 1982, B. Wehrfritz posed a question in the Kourovka Notebook [1]: find n, m, p for which a Sylow p-subgroup $P_n(\mathbb{Z}_{p^m})$ of the group $GL_n(\mathbb{Z}_{p^m})$ over the ring $\mathbb{Z}_{p^m}$ of integers modulo p^m is regular. Recall that a finite p-group G is said to be regular if for every two elements $a, b \in G$ and every $n = p^k$ the equality $(ab)^n = a^n b^n S_1^n \dots S_t^n$ holds, where $S_1, \dots, S_t$ are suitable elements of the derived subgroup of a group generated by the elements a and b [2, p. 205]. The answer to the question is known for the following cases: $nm-1 < p$ (follows from the work by Yu. I. Merzlyakov [3]), $n \geqslant p+1$ (A. V. Yagzhev [4]), $n \geqslant (p+1)/2$ or $n^2 < p$ (S. G. Kolesnikov [5], [6]). In this work, a necessary condition of regularity is obtained, which allows to partially study the case $n \geqslant (p+1)/3$, and also to obtain a complete solution of the analogue of this question for a Sylow p-subgroup $P\Phi(\mathbb{Z}_{p^m})$ of the Chevalley group $\Phi(\mathbb{Z}_{p^m})$ for Φ of type G_2.

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Theorem 1. *If a finite p-group G is regular, then for every $a, b \in G$ there exists an element $d \in \langle a, b\rangle'$ such that*

$$d^p = \prod R_i^{f_i(p)},$$

where the product is taken over all commutators R_i of weight $w(R_i) \geqslant p$ from the P. Hall's collection formula. In particular, for every non-negative integer j

$$d^p \equiv \prod_{p \leqslant w(R_i) \leqslant p+j} R_i^{f_i(p)} \pmod{G^{(p+j+1)}},$$

where $G^{(p+j+1)}$ is the $(p+j+1)$-th term of the lower central series of the group G.

Theorem 1 and Theorem 3 in [7] imply

Corollary 1. *Let G be a regular p-group, $p > 2$, and $a, b \in G$. Suppose that every commutator of a and b :*

1) *that has more than two entries of b, equals 1,*
2) *weighting more than $p-1$, has an order 1 or p.*

Then there exists an element $d \in \langle a, b\rangle'$ such that

$$d^p \equiv [b, {}_{p-1}a]\,[b, {}_{p-2}a, b]^{-1} \prod_{v=1}^{(p-3)/2} [[b, {}_{p-2-v}a], [b, {}_{v}a]]^{(-1)^{v+1}} \pmod{G^{(p+1)}}.$$

Theorem 2. *Let p be such an odd prime number that the number $(p+2)/3$ is an integer. Then the group $P_n(\mathbb{Z}_{p^m})$ is not regular if $n \geqslant (p+2)/3$ and $m \geqslant 3$.*

Moreover, in [8] it is shown that the groups $P_n(\mathbb{Z}_{p^m})$ for $m \geqslant 2$, $p \geqslant 5$, $n \leqslant (p-1)/2$ satisfy the conditions of regularity listed in [2].

Theorem 3. *A group $PG_2(\mathbb{Z}_{p^m})$ is regular if and only if $p \geqslant 17$ or $(m, p) \in \{(1,7), (1,11), (1,13), (2,13)\}$.*

2. Proof of Theorem 1 and Corollary 1. Auxiliary statements

We will now prove Theorem 1. Suppose that $a, b \in G$. By Theorem 12.4.2 from [2], there exists an element $c \in \langle a, b\rangle'$ such that $(ab)^p = a^p b^p c^p$. On the other hand, according to [9, Theorem 3.1], there exists a sequence of commutators R_i of a and b, ordered by weight, and a sequence of integers $f_i(p)$, such that

$$(ab)^p = a^p b^p \prod_{2 \leqslant w(R_i) < p} R_i^{f_i(p)} \prod_{w(R_i) \geqslant p} R_i^{f_i(p)}.$$

Since the group G is regular and the exponents $f_i(p)$ are multiples of p, when $2 \leqslant w(R_i) < p$, then by Corollary 12.4.1 from [2] there exists an element $u \in \langle a, b\rangle'$ such that

$$u^p = \prod_{2 \leqslant w(R_i) < p} R_i^{f_i(p)}.$$

Therefore,

$$a^p b^p c^p = a^p b^p u^p \prod_{2 \leqslant w(R_i) < p} R_i^{f_i(p)}$$

or

$$\prod_{2 \leqslant w(R_i) < p} R_i^{f_i(p)} = (u^{-1})^p c^p = d^p$$

for some $d \in \langle a, b\rangle'$. Theorem 1 is proved.

Remark. *For every integer m and every non-negative integer n, we use a classic definition of a binomial coefficient:*

$$\binom{n}{m} = \begin{cases} \frac{1}{m!}\prod\limits_{i=0}^{m-1}(n-i), & \text{if } m \geqslant 0; \\ 0, & \text{if } m < 0. \end{cases}$$

For such definition, the following relation holds: $\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}$.

We will prove Corollary 1. According to [7, Theorem 3], the P. Hall's collection formula given the condition 1) of the corollary is reduced to the form

$$(ab)^n = a^p b^p \prod_{u=1}^{p-1}[b, {}_u a]^{\binom{p}{u+1}} \prod_{u=1}^{p-1}[b, {}_u a, b]^{p\binom{p}{u+1}-\binom{p+1}{u+2}} \prod_{1\leqslant v<u\leqslant p-1} [[b, {}_u a], [b, {}_v a]]^{g_p(u,v)},$$

where

$$g_p(u, v) = \sum_{m=1}^{p-1}\sum_{k=1}^{v}\sum_{i=v-k}^{p-m-k}\binom{i}{u-k+1}\binom{p-m-i-1}{k-1}\binom{i}{v-k} + \sum_{m=1}^{p-2}\sum_{k=m+1}^{p-1}\binom{m}{v}\binom{k}{u}.$$

By condition 2) of the corollary, all commutators weighting more than $p-1$ have an order 1 or p. We will calculate modulo p the exponents of all commutators of weight p, occurring in the collection formula. The first and the second products contain one commutator of weight p each. That is $[b, {}_{p-1}a]$ and $[b, {}_{p-2}a, b]$ respectively. The exponent of the first commutator equals $\binom{p}{p} = 1$. For the second one, we have

$$p\binom{p}{p-1} - \binom{p+1}{p} = p^2 - p - 1 \equiv -1 \pmod{p}.$$

Consider the latter product. The commutators of weight p, occurring in it, are as follows: $[[b, {}_{p-2-v}a], [b, {}_v a]]$, where $v = 1, \dots, (p-3)/2$. According to [10, Theorem 1], the function $g_p(u, v)$ admits representation

$$g_p(u, v) = \sum_{k=1}^{v}\sum_{s=0}^{v-k}\binom{u-k+1+s}{v-k}\binom{v-k}{s}\binom{p}{u+s+2} + \sum_{i=0}^{v+1}(-1)^i\binom{p+i}{u+i+1}\binom{p}{v+1-i}.$$

Therefore,

$$g_p(p-2-v, v) = \sum_{k=1}^{v}\sum_{s=0}^{v-k}\binom{t+s-k-1}{v-k}\binom{v-k}{s}\binom{p}{t+s} + \sum_{i=0}^{v+1}(-1)^i\binom{p+i}{t+i-1}\binom{p}{v+1-i},$$

where $t = p-v$. The inequality $1 \leqslant v \leqslant (p-3)/2$ yields that $(p+3)/2 \leqslant p+s-v \leqslant p-1$ and $0 \leqslant v+1-i \leqslant (p-1)/2$. Hence, in the double sum the binomial coefficient $\binom{p}{t+s} = \binom{p}{p+s-v}$ is always a multiple of p. In the single sum, the binomial coefficient

$\binom{p}{v+1-i}$ is not a multiple of p only in the case when $v+1-i=0$. Therefore,

$$g_p(p-2-v,v) \equiv (-1)^{v+1}\binom{p+v+1}{p}\binom{p}{0} \equiv$$
$$\equiv (-1)^{v+1}\frac{(p+1)\dots(p+v+1)}{(v+1)!} \equiv (-1)^{v+1}\frac{(v+1)!}{(v+1)!} \equiv (-1)^{v+1} \pmod p.$$

Now, Corollary 1 follows from Theorem 1.

To prove Theorems 2 and 3, we will need the following statements.

Lemma 1. *Let G be a group, $y_1,\dots,y_s \in G$, $s \geqslant 2$. Assume that the nilpotency class of the subgroup $H = \langle y_1,\dots,y_s\rangle$ equals 2. Then for every natural number n, the following equality holds:*

$$(y_1\dots y_s)^n = y_1^n \dots y_s^n \prod_{1\leqslant i<j\leqslant s} [y_j,y_i]^{\binom{n}{2}}.$$

Proof. Induction on n. Since the nilpotency class of H equals 2, we have

$$(y_1\dots y_s)^{n+1} = y_1^n \dots y_s^n \cdot \left(\prod_{1\leqslant i<j\leqslant s} [y_j,y_i]^{\binom{n}{2}}\right) \cdot y_1\dots y_s =$$
$$= y_1^n \dots y_s^n \cdot y_1\dots y_s \cdot \prod_{1\leqslant i<j\leqslant s} [y_j,y_i]^{\binom{n}{2}}.$$

Using the relation $y_j^n y_i = y_i y_j^n [y_j,y_i]^n$ which follows from the condition of the lemma, we collect the terms in the right-hand side in order $y_1,\dots,y_s$. Then, taking into account the permutability of the commutators, we convert the obtained expression into the form

$$(y_1\dots y_s)^{n+1} = y_1^{n+1} \dots y_s^{n+1} \prod_{1\leqslant i<j\leqslant s} [y_j,y_i]^{\binom{n}{2}+n}.$$

The equalities $\binom{n}{2}+n = \binom{n}{2}+\binom{n}{1} = \binom{n+1}{2}$ complete the proof. ☐

Corollary 2. *Let the subgroup H from Lemma 1 be a p-group, $p>2$, with an elementary Abelian derived subgroup, and number n is a multiple of p. Then for any integer $\alpha_1,\dots,\alpha_s$ and every permutation π on the set $\{1,\dots,s\}$, we have*

$$(y_1^{\alpha_1}\dots y_s^{\alpha_s})^n = \left(y_{\pi(1)}^{\alpha_{\pi(1)}}\dots y_{\pi(s)}^{\alpha_{\pi(s)}}\right)^n = y_1^{n\alpha_1}\dots y_s^{n\alpha_s}.$$

If for some i additionally $y_i^n = 1$ or $y_i^p \in H'$ and α_i is a multiple of p, then

$$(y_1^{\alpha_1}\dots y_i^{\alpha_i}\dots y_s^{\alpha_s})^n = (y_1^{\alpha_1}\dots \hat{y}_i^{\alpha_i}\dots y_s^{\alpha_s})^n.$$

Here the superscript ˆ marks the absence of this element in the product.

3. Sylow p-subgroups of the groups $GL_n(\mathbb{Z}_{p^m})$ and $\Phi(\mathbb{Z}_{p^m})$

Following [3], we define the sequence of functions f_n, $n = 1, 2, \dots$, of natural arguments i,j,k, setting that

$$f_n(i,j,k) = -\left[\frac{i-j-k}{n}\right],$$

here $[x]$ is the integer part of number x (the closest integer to x on the left). By J we denote the ideal of the ring $\mathbb{Z}_{p^m}$, generated by the element p, and by E the identity matrix of order n. We select in $GL_n(\mathbb{Z}_{p^m})$ the subgroups

$$G^{(k)} = \langle E + A \mid A = (a_{ij}),\ a_{ij} \in J^{f_n(i,j,k)},\ 1 \leqslant i,j \leqslant n\rangle, \qquad k = 1,2,\ldots.$$

From here on we set by definition $J^0 = \mathbb{Z}_{p^m}$. According to [3], $G^{(1)}$ is isomorphic to the group

$$P_n(\mathbb{Z}_{p^m}) = \{E + (a_{ij}) \mid a_{ij} \in \mathbb{Z}_{p^m} \text{ for } i > j;\ a_{ij} \in J, \text{ for } i \leqslant j\},$$

and the sequence

$$G^{(1)} \supset G^{(2)} \supset \ldots \supset G^{(mn-1)} \supset \langle E\rangle$$

is its lower central series, if $p > 2$. Moreover, for every prime p and every natural k, l, the following relation holds:

$$\left[G^{(k)}, G^{(l)}\right] \subseteq G^{(k+l)}. \tag{1}$$

In [8, Lemma 2], it was shown that if $A = (a_{ij})$, $B = (b_{ij})$ are such matrices that $a_{ij} \in J^{f_n(i,j,k)}$ and $b_{ij} \in J^{f_n(i,j,l)}$, then the elements c_{ij} of the matrix $C = AB$ belong to the ideals $J^{f_n(i,j,k+l)}$. This and the properties of divisibility of binomial coefficients easily yield

Lemma 2. *If $p > n$, then for every natural k the following inclusion holds: $\left[G^{(k)}\right]^p \subseteq G^{(k+n)}$.*

Let Φ be an arbitrary reduced indecomposable root system. In the Chevalley group $\Phi(\mathbb{Z}_{p^m})$, we select a sequence of subgroups ($k = 1,2,\ldots$)

$$S^{(k)} = \langle x_r(t), h_s(1+u) \mid r,s \in \Phi,\ t \in J^{f(r,k)}, u \in J^{f(0,k)}\rangle,$$

where the function $f(r,k)$ is defined on the set $\Phi_0 \times \mathbb{N}$, $\Phi_0 = \Phi \cup \{0\}$, by the equality $f(r,k) = -[(\mathrm{ht}(r) - k)/h]$. Here $\mathrm{ht}(r)$ is theroot height function, $\mathrm{ht}(0) = 0$, h is the Coxeter number of the root system Φ. According to [11], the group $S^{(1)}$ is isomorphic to the Sylow p-subgroup

$$P\Phi(\mathbb{Z}_{p^m}) = \langle x_r(\mathbb{Z}_{p^m}), x_{-r}(J), h_r(1+J) \mid r \in \Phi^+\rangle,$$

of the Chevalley group $\Phi(\mathbb{Z}_{p^m})$, and the sequence

$$S^{(1)} \supset S^{(2)} \ldots \supset S^{(mh)} = \langle 1\rangle.$$

is its lower central series, if p does not divide $p(\Phi)!$, where

$$p(\Phi) = \max\{(r,r)/(s,s) \mid r,s \in \Phi\}.$$

As above, for every prime p and every natural k, l, the following relation holds:

$$\left[S^{(k)}, S^{(l)}\right] \subseteq S^{(k+l)}. \tag{2}$$

Recall that in the group $P\Phi(\mathbb{Z}_{p^m})$ the following relations are satisfied: $(r, s \in \Phi)$

1) *addititve property of root elements:*

$$x_r(t)x_r(u) = x_r(t+u), \qquad t,u \in J^{f(r,1)};$$

2) *multiplicative property of diagonal elements:*

$$h_r(t)h_r(u) = h_r(tu), \qquad t,u \in 1 + J^{f(0,1)};$$

3) *relations between root and diagonal elements:*

$$[x_r(t), h_s(u)] = x_r(t(u^{(r,h_s)} - 1)), \qquad t \in J^{f(r,1)}, u \in J^{f(0,1)};$$

4) *Chevalley's commutator formula:*

$$[x_s(u), x_r(t)] = \prod_{\substack{ir+js \in \Phi,\\ i,j>0}} x_{ir+js\in\Phi}(C_{ij,rs}(-t)^i u^j), \quad t \in J^{f(r,1)}, u \in J^{f(s,1)};$$

5) *relations between the opposite root elements:*

$$[x_r(t), x_{-r}(u)] = x_r(c^{-1}t^2u)h_r(c)x_{-r}(-c^{-1}tu^2), \quad c = 1-tu, t \in J^{f(r,1)}, u \in J^{f(-r,1)}.$$

The analogue of Lemma 2 for Sylow p-subgroups of Chevalley groups is

Lemma 3. *If $p > h$, then for every natural number k the following inclusion holds: $\left[S^{(k)}\right]^p \subseteq S^{(k+h)}$.*

Proof. Induction on k. We have

$$\left[S^{(mh)}\right]^p \subseteq [\langle 1\rangle]^p = \langle 1\rangle = S^{(mh+h)}.$$

Suppose that $1 \leqslant k < mh$ and $y \in S^{(k)}$. Then

$$y = \prod_{i=1}^{l} x_{r_i}(t_i) \prod_{j=1}^{q} h_{s_j}(1+z_j),$$

where $r_i \in \Phi$, $t_i \in J^{f(r_i,k)}$ and $s_j \in \Pi(\Phi)$, $z_j \in J^{f(0,k)}$. We will show that $y^p \in S^{(k+h)}$. According to [2, Theorem 12.3.1],

$$y^p = \prod_{i=1}^{l} x_{r_i}(t_i)^p \prod_{j=1}^{q} h_{s_j}(1+z_j)^p \cdot c_1^{\alpha_1} \dots c_u^{\alpha_u} \cdot c_{u+1}^{\alpha_{u+1}} \dots c_{u+v}^{\alpha_{u+v}},$$

where the wights of the commutators $c_1, \dots, c_u$ are from 2 to $p-1$, and the numbers $\alpha_1, \dots, \alpha_u$ are multiplies of p, the weights of the commutators $c_{u+1}, \dots, c_{u+v}$ exceed $p-1$. Since $f(r_i,k)+1 = f(r_i, k+h)$, we have that

$$pt_i \in pJ^{f(r_i,k)} = J^{f(r_i,k)+1} = J^{f(r_i,k+h)}$$

and therefore $x_{r_i}(t_i)^p = x_{r_i}(pt_i) \in S^{(k+h)}$.

Next, the function $f(0,k)$ satisfies the following inequalities:

1) $f(0,k_1) + f(0,k_2) \geqslant f(0, k_1+k_2)$ for every $k_1, k_2 \in \mathbb{N}$;

2) $f(0,k_1) \geqslant f(0,k_2)$, if $k_1 \geqslant k_2$.

Also from the condition $p > h$, it follows that $kp \geqslant k+h$. Hence, $f(0,k)p \geqslant f(0,kp) \geqslant f(0,k+h)$. From here,

$$(1+z_j)^p = 1 + \sum_{w=1}^{p-1} \binom{w}{p} z_j^w + z_j^p \in 1 + \sum_{w=1}^{p-1} pJ^{f(0,k)w} + J^{f(0,k)p} \subseteq 1 + J^{f(0,k+h)}$$

and therefore, $h_{s_j}(1+z_j)^p \in S^{(k+h)}$.

The commutators $c_1, \dots, c_u$ belong to $S^{(2k)} \subseteq S^{(k+1)}$, hence, $c_1^{\alpha_1}, \dots, c_u^{\alpha_u} \in S^{(k+1+h)} \subseteq S^{(k+h)}$ by the induction assumption. Finally, from relation (2) it follows that $c_{u+1}, \dots c_{u+v} \in S^{(kp)} \subseteq S^{(k+h)}$. □

We denote by $K_n(J^k)$ and $\Phi(J^k)$ respectively the congruence subgroups of the groups $GL_n(\mathbb{Z}_{p^m})$ and $\Phi(\mathbb{Z}_{p^m})$, which are defined as the kernels of the homomorphisms $GL_n(\mathbb{Z}_{p^m}) \to GL_n(\mathbb{Z}_{p^{m-k}})$ and $\Phi(\mathbb{Z}_{p^m}) \to \Phi(\mathbb{Z}_{p^{m-k}})$, induced by the ring homomorphism $\mathbb{Z}_{p^m} \to \mathbb{Z}_{p^{m-k}}$. Relations 1) of the following lemma can be found in [3] and [11], relations 2) are easily established by the methods used in the proofs of Lemmas 2 and 3.

Lemma 4. *The following inclusions hold:*

1) $[K_n(J^k), K_n(J^l)] \subseteq K_n(J^{k+l})$, $[\Phi(J^k), \Phi(J^l)] \subseteq \Phi(J^{k+l})$;

2) $\left[K_n(J^k)\right]^p \subseteq K_n(J^{k+1})$, $\left[\Phi(J^k)\right]^p \subseteq \Phi(J^{k+1})$.

4. Proof of Theorem 2.

Since the regularity property is inherited by subgroups and quotient groups, to prove Theorem 2, it suffices to establish the nonregularity of the group $P_{(p+2)/3}(\mathbb{Z}_{p^3})$. We will need a number of auxiliary statements.

Lemma 5. *For every* $E+X, E+Y \in P_n(\mathbb{Z}_{p^m})$, *the following identity holds*

$$[E+X, E+Y] = E + \sum_{k=2}^{\infty} \sum_{t=0}^{k-2} (-1)^k X^t Y^{k-t-2} (X, Y), \tag{3}$$

where $(X, Y) = XY - YX$.

Proof. Due to nilpotency of the matrices X and Y, we have

$$[E+X, E+Y] = \left(\sum_{i=0}^{\infty} (-X)^i \right) \left(\sum_{j=0}^{\infty} (-Y)^j \right) (E+X)(E+Y).$$

Opening the brackets, we obtain a sum of homogeneous polynomials $f_k(X, Y)$ of degree k, moreover, it is obvious that $f_0(X, Y) = E$ and $f_1(X, Y) = O$, where O is a zero matrix. We fix $k \geqslant 2$. Then

$$\begin{aligned} f_k(X, Y) &= (-X)^k + (-X)^{k-1}(-Y) + (-X)^{k-1}X + (-X)^{k-1}Y \\ &+ \sum_{t=0}^{k-2} (-X)^t \left((-Y)^{k-t} + (-Y)^{k-t-1}X + (-Y)^{k-t-1}Y + (-Y)^{k-t-2}XY \right) \\ &= \sum_{t=0}^{k-2} (-1)^k X^t Y^{k-t-2} (X, Y). \end{aligned}$$

□

Substituting X with pX in (3), the commutator $[E+pX, E+Y]$ can be represented in the form of a series by exponents of p with coefficients depending on X and Y. Next, we will be interested in the coefficient of the term p in the decomposition of the complex commutators.

Lemma 6. *Suppose that* $E+pB, E+A \in P_n(\mathbb{Z}_{p^m})$. *The coefficient of* p *in the expansion of the commutator* $[E+pB, {}_sE+A]$, $s \in \mathbb{N}$, *in powers of* p *equals*

$$F(s) = \sum_{j=0}^{s-1} \sum_{k=j}^{\infty} (-1)^k \binom{s-1}{j} \binom{s+k-j-1}{s-1} A^k (B, A) A^{s-j-1}. \tag{4}$$

Proof. Induction on s. By Lemma 5,

$$[E+pB, E+A] = E + \sum_{k=2}^{\infty}\sum_{t=0}^{k-2}(-1)^k (pB)^t A^{k-t-2}(pB, A)$$
$$= E + p\sum_{k=2}^{\infty}(-1)^k A^{k-2}(B, A) + \ldots = E + p\sum_{k=0}^{\infty}(-1)^k A^k(B, A) + \ldots.$$

The obtained coefficient of p, obviously equals the expression (4), if in the latter one we take $s = 1$.

Suppose that $s \geqslant 1$. To calculate the coefficient at p in the expansion of the commutator $[E+pB,_{s+1} E+A]$ in powers of p, we will use the inductive assumption and substitute B in the sum

$$\sum_{k=0}^{\infty}(-1)^k A^k(B, A)$$

with the expression (4). We will transform the obtained multiple sum by opening the outer Lie commutator:

$$\sum_{t=0}^{\infty}\sum_{j=0}^{s-1}\sum_{k=j}^{\infty}(-1)^k\binom{s-1}{j}\binom{s+k-j-1}{s-1}A^t(A^k(B,A)A^{s-j-1}, A)$$
$$= \sum_{t=0}^{\infty}\sum_{j=0}^{s-1}\sum_{k=j}^{\infty}(-1)^{t+k}\binom{s-1}{j}\binom{s+k-j-1}{s-1}A^{t+k}(B,A)A^{s-j}$$
$$+ \sum_{t=0}^{\infty}\sum_{j=0}^{s-1}\sum_{k=j}^{\infty}(-1)^{t+k+1}\binom{s-1}{j}\binom{s+k-j-1}{s-1}A^{t+k+1}(B,A)A^{s-j-1}.$$

We fix the number j', $0 \leqslant j' \leqslant s$, and the number k', $k' \geqslant j'$. The coefficient at $A^{k'}(B, A)A^{s-j'}$ equals:

$$\sum_{t=0}^{k'}(-1)^{k'}\binom{s-1}{0}\binom{s+k'-t-1}{s-1} = (-1)^{k'}\binom{s}{0}\sum_{t=0}^{k'}\binom{s-1+k'-t}{s-1}$$
$$= (-1)^{k'}\binom{s}{0}\binom{s+k'}{s-1} = (-1)^{k'}\binom{s}{j'}\binom{s+k'-j'}{s},$$

if $j' = 0$;

$$\sum_{t=0}^{k'-s}(-1)^{k'}\binom{s-1}{s-1}\binom{s+k'-s-t-1}{s-1} = \binom{s}{s}\sum_{t=0}^{k'-s}(-1)^{k'}\binom{k'-t-1}{s-1}$$
$$= (-1)^{k'}\binom{s}{s}\binom{k'}{s} = (-1)^{k'}\binom{s}{j'}\binom{s+k'-j'}{s},$$

if $j'=s$;

$$\sum_{t=0}^{k'-j'}(-1)^{k'}\binom{s-1}{j'}\binom{s+k'-j'-t-1}{s-1}\left(\binom{s-1}{j'}+\binom{s-1}{j'-1}\right)$$

$$=(-1)^{k'}\binom{s}{j'}\sum_{t=0}^{k'-j'}\binom{s-1+k'-j'-t}{s-1}=(-1)^{k'}\binom{s}{j'}\binom{s+k'-j'}{s},$$

where $0<j'<s$. Therefore, the coefficient at $A^{k'}(B,A)A^{s-j'}$ in all cases equals

$$(-1)^{k'}\binom{s}{j'}\binom{s+k'-j'}{s}.$$

□

Next, we assume that $n\geqslant 3$. The matrix of order n with one in position (i,j), where $1\leqslant i,j\leqslant n$, and zeros elsewhere will be denoted by e_{ij} and referred to as an matrix unit. Recall that the following formula of multiplication of matrix units is true:

$$e_{ij}e_{ts}=\delta_{jt}e_{is},$$

where δ_{ij} is the Kronecker delta. We also agree to consider: $e_{ij}=O$, if $i\notin\{1,\dots,n\}$ or $j\notin\{1,\dots,n\}$. A sum with a lower limit exceeding the upper one is considered to be zero (is a zero matrix).

We fix the following matrices till the end of the paragraph

$$A=e_{21}+e_{32}+\ldots+e_{n,n-1},\qquad B=e_{1n}.$$

In the following lemmas, we calculate the product from the Corollary 1 and study the derived subgroup of the group generated by the elements $E+pB$ and $E+A$.

Lemma 7. *For every natural s, the following equality holds:*

$$F(s)=\sum_{j=0}^{s}\sum_{k=j}^{n-1}(-1)^k\binom{s}{j}\binom{s+k-j-1}{s-1}e_{k+1,n-s+j}.$$

Proof. Taking into account the above-mentioned agreements, for every non-negative integer k we have

$$A^k=\sum_{t=k+1}^{n}e_{t,t-k}.$$

Therefore,

$$A^k(B,A)A^{s-j-1}$$

$$=\left(\sum_{t=k+1}^{n}e_{t,t-k}\right)(e_{1,n-1}-e_{2n})\left(\sum_{t=s-j}^{n}e_{t,t-s+j+1}\right)=e_{k+1,n-s+j}-e_{k+2,n-s+j+1}.$$

Substitute into (4) and break into two sums

$$F(s)=\sum_{j=0}^{s-1}\sum_{k=j}^{n-1}(-1)^k\binom{s-1}{j}\binom{s-1+k-j}{s-1}e_{k+1,n-s+j}$$

$$-\sum_{j=0}^{s-1}\sum_{k=j}^{n-1}(-1)^k\binom{s-1}{j}\binom{s-1+k-j}{s-1}e_{k+2,n-s+j+1}.$$

In the second double sum, we substitute k with $k-1$ and j with $j-1$:

$$F(s)=\sum_{j=0}^{s-1}\sum_{k=j}^{n-1}(-1)^k\binom{s-1}{j}\binom{s-1+k-j}{s-1}e_{k+1,n-s+j}$$
$$+\sum_{j=1}^{s}\sum_{k=j}^{n-1}(-1)^k\binom{s-1}{j-1}\binom{s-1+k-j}{s-1}e_{k+1,n-s+j}.$$

Since $\binom{s-1}{s}=\binom{s-1}{-1}=0$, we extend the summation over j in the first sum to $j=s$ and the summation over j in the second one to $j=0$ and summarize them

$$F(s)=\sum_{j=0}^{s}\sum_{k=j}^{n-1}(-1)^k\left[\binom{s-1}{j}+\binom{s-1}{j-1}\right]\binom{s-1+k-j}{s-1}e_{k+1,n-s+j}$$
$$=\sum_{j=0}^{s}\sum_{k=j}^{n-1}(-1)^k\binom{s}{j}\binom{s+k-j-1}{s-1}e_{k+1,n-s+j}.$$

□

Note that the arbitrary element $E+C$ from the group $P_n(\mathbb{Z}_{p^m})$ can be represented (of course, not in a unique way) in the form

$$E+C=E+p^0C_0+pC_1+\ldots+p^{m-1}C_{m-1}.$$

The following simple lemma often simplifies the calculations in $P_n(\mathbb{Z}_{p^3})$.

Lemma 8. *Suppose that* $E+pX+p^2Y, E+pU+p^2V\in P_n(\mathbb{Z}_{p^3})$. *Then*

1) $(E+pX+p^2Y)^{-1}=E-pX-p^2Y+p^2X^2$;
2) $(E+pX+p^2Y)^p=E+p^2X$;
3) $[E+pX+p^2Y,E+pU+p^2V]=E+p^2(X,U)$.

Lemma 9. *Suppose that a prime* p *is such that* $n=(p+2)/3$ *is an integer. In the group* $P_n(\mathbb{Z}_{p^3})$, *the following equality holds*

$$[E+pB,{}_{p-1}E+A]\,[E+pB,{}_{p-2}E+A,E+pB]^{-1}$$
$$\times\prod_{s=1}^{(p-3)/2}[[E+pB,{}_{p-2-s}E+A],[E+pB,{}_sE+A]]^{(-1)^{s+1}}$$
$$=\alpha e_{n,1}-p^2e_{n,2}-p^2e_{n-1,1},$$

where $\alpha\in\mathbb{Z}_{p^3}$.

Proof. First, we will show that $[E+pB,{}_sE+A]=E$, if $s\geqslant 2n$. In particular, $[E+pB,{}_{p-1}E+A]=E$, since $p\geqslant 7$, and therefore $n\geqslant 3$, hence,

$$p-1=3n-3=2n+(n-3)\geqslant 2n.$$

We introduce the following notation:

$$\phi(x,{}^ry)=[x,y],\quad \phi(x,{}^ly)=[y,x],$$

by induction for $q\geqslant 2$, we put

$$\phi(x,{}^{\alpha_1}y_1,\ldots,{}^{\alpha_q}y_q)=\phi(\phi(x,{}^{\alpha_1}y_1,\ldots,{}^{\alpha_{q-1}}y_{q-1}),{}^{\alpha_q}y_q),$$

where $\alpha_i=r$ or $\alpha_i=l$.

We decompose the matrices $E+A$ and $E+pB$ into the product of transvections:

$$E+A = t_{21}(1)t_{32}(1)\dots t_{n,n-1}(1), \quad E+pB = t_{1n}(p).$$

Commutator identities

$$[x,yz] = [x,y][x,y,z][x,z], \quad [yz,x] = [z,x][z,[x,y]][y,x],$$

yield that $[E+pB, {}_sE+A]$ can be factorized into a commutator product of the form

$$\phi(t_{1n}(p), {}^{\alpha_1}t_{i_1,i_1-1}(1), \dots, {}^{\alpha_q}t_{i_q,i_q-1}(1)), \qquad q \geqslant s. \tag{5}$$

We study in detail the expression (5). The relation

$$[t_{ij}(\alpha), t_{km}(\beta)] = \\ = (1-\delta_{jk})(1-\delta_{im})E + \delta_{jk}(1-\delta_{im})t_{im}(\alpha\beta) + \delta_{im}(1-\delta_{jk})t_{kj}(-\alpha\beta), \tag{6}$$

that holds when $j \neq k$ or $i \neq m$, implies that if we commute transvections with differences between the first and second indices being equal to s_1 and s_2, we obtain the identity matrix, or a transvection with index difference s_1+s_2. Therefore, the expression (5) for $q = n-2$ is an identity matrix, or a transvection with index difference that equals -1, that is,

$$\phi(t_{1n}(p), {}^{\alpha_1}t_{i_1,i_1-1}(1), \dots, {}^{\alpha_{n-2}}t_{i_{n-2},i_{n-2}-1}(1)) = t_{i,i+1}(\epsilon_i p),$$

where $\epsilon_i = 0$ or $\epsilon_i = \pm 1$ for some i. Next, using the relation

$$[t_{ij}(\alpha), t_{ji}(\beta)] = t_{ij}(c^{-1}\alpha^2\beta)d_{ij}(c)t_{ji}(-c^{-1}\alpha\beta^2), \tag{7}$$

where $d_{ij}(c) = E + (c-1)e_{ii} + (c^{-1}-1)e_{jj}$, that holds if the element $c = 1-\alpha\beta$ is invertible (in this case that is true, since $c \equiv 1 \pmod p$), we obtain

$$[t_{i,i-1}(\epsilon_i p), t_{i_{n-1},i_{n-1}-1}(1)] = \\ = (1-\delta_{i,i_{n-1}})E + \delta_{i,i_{n-1}}t_{i,i+1}(\epsilon_i^2p^2)d_{i,i+1}(1-\epsilon_i p)t_{i+1,i}(-\epsilon_i p(1-\epsilon_i p)^{-1}). \tag{8}$$

Here we used the fact that $p^3 = 0$ in $\mathbb{Z}_{p^3}$ and therefore, $\epsilon_i^2p^2(1-\epsilon_i p)^{-1} = \epsilon_i^2 p^2$. Finally, the relations (6), (7) and

$$[t_{km}(\alpha), \mathrm{diag}(\beta_1, \dots, \beta_n)] = t_{km}(\alpha(\beta_m\beta_k^{-1}-1)) \tag{9}$$

show that commuting (8) with the transvection $t_{i_n,i_n-1}(1)$ yields: the identity matrix if $i_n \neq i-1, i, i+1$; the matrix from the unitriangular group $UT_n(\mathbb{Z}_{p^3})$ if $i_n = i-1, i+1$; and finally, the matrix $d_{i,i+1}(1-\epsilon_i^2p^2)\theta$, where $\theta \in UT_n(\mathbb{Z}_{p^3})$, if $i_n = i$. This and (9) yield that the expression (5) given $q = n+1$ belongs to $UT_n(\mathbb{Z}_{p^3})$ and equals E given $q = 2n$, since the group $UT_n(\mathbb{Z}_{p^3})$ is of nilpotency class $n-1$.

Note that if $p > 7$, we have $p-2 \geqslant 2n$, and hence,

$$[E+pB, {}_{p-2}E+A, E+pB] = E.$$

When $p = 7$, we have $n = 3$, and direct calculations show that

$$[E+pB, {}_{p-2}E+A] = t_{31}(-3p^2), \quad [t_{31}(-3p^2), t_{13}(p)] = E.$$

Therefore, in all cases,

$$[E+pB, {}_{p-1}E+A]\,[E+pB, {}_{p-2}E+A, E+pB]^{-1} = E.$$

We will calculate the remaining product. According to item 3) of Lemma 8, we have

$$[[E+pB,{}_{p-2-s}E+A],[E+pB,{}_{s}E+A]]=E+p^2F(p-2-s)F(s). \tag{10}$$

Suppose that $s\geqslant 2n-1$. Then the expression

$$A^k(B,A)A^{s-j-1}=A^kBA^{s-j}+A^{k+1}BA^{s-j-1}$$

equals zero matrix, if $0\leqslant j\leqslant s$ and $k\geqslant j$, since $A^n=O$. Indeed, when $0\leqslant j\leqslant n-2$, we have $s-j-1\geqslant n$; if $j=n-1$, then $s-j\geqslant n$ and $k+1\geqslant n$; finally, if $j\geqslant n$, then $k\geqslant n$. This and Lemma 6 yield that $F(s)=O$, when $s\geqslant 2n-1$, and $F(p-2-s)=O$, when $p-2-s\geqslant 2n-1$. Therefore, commutator (10) is distinct from E when

$$n-2\leqslant s\leqslant \min\{2n-2,(p-3)/2\}=(3n-5)/2.$$

We put $s=n-2+\alpha$. Then

$$\begin{aligned}(11)\qquad &\prod_{s=1}^{(p-3)/2}[[E+pB,{}_{p-2-s}E+A],[E+pB,{}_{s}E+A]]^{(-1)^{s+1}}\\ &=\prod_{\alpha=0}^{(n-1)/2}[[E+pB,{}_{2n-2-\alpha}E+A],[E+pB,{}_{n-2+\alpha}E+A]]^{(-1)^{\alpha+1}}\\ &=E+p^2\sum_{\alpha=0}^{(n-1)/2}(-1)^{\alpha+1}(F(2n-2-\alpha),F(n-2+\alpha)).\end{aligned}$$

The nilpotency class of the group $P_n(p^3)$ equals $3n-1=p+1$. Hence, the product (11), consisting of commutators of weight p, belongs to the hypercenter and equals to

$$E+p^2\theta e_{n,1}+p^2\beta e_{n-1,1}+p^2\gamma e_{n,2}$$

for the suitable θ,β,γ.

We will calculate the coefficient β for $e_{n-1,1}$. To do that, using Lemma 6, we choose in the factorization of $F(2n-2-\alpha)$ into matrix units the summands with the first index that equals $n-1$, and in $F(n-2-\alpha)$ the summands with the second index that equals 1:

$$\sum_{j=0}^{n-2}(-1)^{n-2}\binom{2n-2-\alpha}{j}\binom{3n-5-\alpha-j}{2n-3-\alpha}e_{n-1,-n+2+\alpha+j}, \tag{12}$$

$$\sum_{k=\alpha-1}^{n-1}(-1)^{k}\binom{n-2+\alpha}{\alpha-1}\binom{n-2+k}{n-3+\alpha}e_{k+1,1}. \tag{13}$$

Note that in (13) for $\alpha=0$ and $k=-1$, not only e_{01} equals zero matrix, but the binomial coefficient $\binom{n-2+\alpha}{\alpha-1}=\binom{n-2}{-1}$ also equals zero. Taking into account that $-n+2+\alpha+j\leqslant\alpha\leqslant k+1$, the coefficient at $e_{n-1,1}$ in the product of (12) and (13) equals

$$\begin{aligned}&(-1)^{n-2+\alpha-1}\binom{2n-2-\alpha}{n-2}\binom{3n-5-\alpha-n+2}{2n-3-\alpha}\binom{n-2+\alpha}{\alpha-1}\binom{n-2+\alpha-1}{n-3+\alpha}\\ (14)\qquad &=(-1)^{\alpha}\binom{2n-2-\alpha}{n-2}\binom{n-2+\alpha}{n-1}.\end{aligned}$$

(When $\alpha = 0$, the expression (14), obviously, equals zero). Next, selecting in the expansion of $F(n-2+\alpha)$ the summands with the first index equal to $n-1$, and in $F(2n-2-\alpha)$ the summands with the second index equal to 1, we obtain

$$\sum_{j=0}^{n-2}(-1)^{n-2}\binom{n-2+\alpha}{j}\binom{2n-5+\alpha+j}{n-3+\alpha}e_{n-1,2-\alpha+j}, \tag{15}$$

$$\sum_{k=n-1-\alpha}^{n-1}(-1)^{k}\binom{2n-2-\alpha}{n-1-\alpha}\binom{n-2+k}{2n-3-\alpha}e_{k+1,1}. \tag{16}$$

Taking into account that $2-\alpha+j \leqslant n-\alpha \leqslant k+1$, the coefficient at $e_{n-1,1}$ in the product of (15) and (16) equals

$$(-1)^{n-2+n-1-\alpha}\binom{n-2+\alpha}{n-2}\binom{2n-5+\alpha-n+2}{n-3+\alpha}\binom{2n-2-\alpha}{n-1-\alpha}\binom{2n-3-\alpha}{2n-3-\alpha}$$

$$= (-1)^{\alpha+1}\binom{n-2+\alpha}{n-2}\binom{2n-2-\alpha}{n-1}. \tag{17}$$

Multiplying the difference between (14) and (17) by $(-1)^{\alpha+1}$, and then summing over α, we can see that

$$\beta = -\sum_{\alpha=0}^{(n-1/2)}\left[\binom{2n-2-\alpha}{n-2}\binom{n-2+\alpha}{n-1}+\binom{n-2+\alpha}{n-2}\binom{2n-2-\alpha}{n-1}\right]. \tag{18}$$

We calculate the obtained sum. We have

$$\sum_{\alpha=0}^{(n-1)/2}\binom{2n-2-\alpha}{n-2}\binom{n-2+\alpha}{n-1} = \sum_{\alpha=-(n-1)/2}^{0}\binom{2n-2+\alpha}{n-2}\binom{n-2-\alpha}{n-1}$$

$$= \sum_{\alpha=n-(n-1)/2}^{n}\binom{n-2+\alpha}{n-2}\binom{2n-2-\alpha}{n-1} = \sum_{\alpha=(n+1)/2}^{n}\binom{n-2+\alpha}{n-2}\binom{2n-2-\alpha}{n-1}.$$

Therefore,

$$-\beta = \sum_{\alpha=0}^{n}\binom{n-2+\alpha}{n-2}\binom{2n-2-\alpha}{n-1} = \sum_{\alpha=n-2}^{2n-2}\binom{\alpha}{n-2}\binom{3n-4-\alpha}{n-1}$$

$$= \binom{3n-3}{2n-2} = \binom{p-1}{2n-2} \equiv (-1)^{2n-2} \equiv 1 \pmod{p}.$$

Here we used the fact that $\binom{3n-4-(2n-2)}{n-1} = 0$, and the proved in [10] identity

$$\sum_{i=b}^{n-a}\binom{n-i}{a}\binom{i}{b} = \binom{n+1}{a+b+1}.$$

We calculate the coefficient γ. We select in the expansion of $F(2n-2-\alpha)$ the summands with the first index equal to n, and in $F(n-2-\alpha)$ the summands with the second index equal to 2:

$$\sum_{j=0}^{n-1}(-1)^{n-1}\binom{2n-2-\alpha}{j}\binom{3n-4-\alpha-j}{2n-3-\alpha}e_{n,-n+2+\alpha+j}, \tag{19}$$

$$\sum_{k=\alpha}^{n-1}(-1)^k\binom{n-2+\alpha}{\alpha}\binom{n-3+k}{n-3+\alpha}e_{k+1,2}. \tag{20}$$

Taking into account that $-n+2+\alpha+j \leqslant \alpha+1 \leqslant k+1$, the coefficient at $e_{n,2}$ in the product of (19) and (20) equals

$$(-1)^{n-1+\alpha}\binom{2n-2-\alpha}{n-1}\binom{2n-3-\alpha}{2n-3-\alpha}\binom{n-2+\alpha}{\alpha}\binom{n-3+\alpha}{n-3+\alpha}$$
$$= (-1)^{\alpha}\binom{2n-2-\alpha}{n-1}\binom{n-2+\alpha}{n-2}. \tag{21}$$

Next, selecting in the expansion of $F(n-2+\alpha)$ the summands with the first index equal to n, and of $F(2n-2-\alpha)$ the ones with the second index equal to 2, we obtain

$$\sum_{j=0}^{n-1}(-1)^{n-1}\binom{n-2+\alpha}{j}\binom{2n-4+\alpha-j}{n-3+\alpha}e_{n,2-\alpha+j}, \tag{22}$$

$$\sum_{k=n-\alpha}^{n-1}(-1)^k\binom{2n-2-\alpha}{n-\alpha}\binom{n-3+k}{2n-3-\alpha}e_{k+1,2}. \tag{23}$$

Note that in (22) when $\alpha=0$ and $j=n-1$, the matrix $e_{n,n+1}$ is zero and its binomial coefficient $\binom{n-2}{n-1}$ equals zero. The sum (23) for $\alpha=0$ also by definition equals zero. Taking into account that $2-\alpha+j \leqslant n-\alpha+1 \leqslant k+1$, the coefficient at $e_{n,2}$ in the product of (22) and (23) equals

$$(-1)^{n-1+n-\alpha}\binom{n-2+\alpha}{n-1}\binom{n-3+\alpha}{n-3+\alpha}\binom{2n-2-\alpha}{n-\alpha}\binom{2n-3-\alpha}{2n-3-\alpha}$$
$$= (-1)^{\alpha+1}\binom{n-2+\alpha}{n-1}\binom{2n-2-\alpha}{n-2}. \tag{24}$$

(When $\alpha=0$, the expression (24), obviously, equals zero). Summing over α the difference between (21) and (24), multiplied by $(-1)^{\alpha+1}$, we obtain again the expression (18). Therefore, $\gamma=-1$. ☐

Lemma 10. *The derived subgroup of the subgroup $H=\langle E+A, E+pB\rangle$ of the group $P_n(\mathbb{Z}_{p^3})$ is generated by the commutators*

$$y_i = [E+pB, {}_iE+A],\ i=1,2,\dots,$$
$$y_{jl} = [[E+pB, {}_jE+A],[E+pB, {}_lE+A]],\ j,l=0,1,\dots.$$

If $d\in H'$ and $d^p\in G^{(p)}$ (the p-th element of the lower central series of the group $P_n(\mathbb{Z}_{p^3})$), then

$$d^p = E+\lambda p^2 e_{n,1} - \tau p^2\binom{2n-3}{n-2}e_{n-1,1} + \tau p^2\binom{2n-3}{n-2}e_{n,2}.$$

Proof. Note that $y_i\in K_n(J)$ and $y_{jl}\in K_n(J^2)$ for every i,j,l. Since $K_n(J^3)=\langle E\rangle$ and by Lemma 4, the following inclusions are true: $[K_n(J),K_n(J^2)]\subseteq K_n(J^3)$ and $\left[K_n(J^2)\right]^p\subseteq K_n(J^3)$, then $[y_i,y_{jl}]=E$ and $y_{jl}^p=E$. Therefore, due to Corollary 2, we can assume that

$$d = y_1^{\tau_1}\dots y_{3n-1}^{\tau_{3n-1}}.$$

Next, if $i \geqslant 2n-1$, then $y_i \in G^{(2n)} \subseteq K_n(J^2)$ and hence, $y_i^p = E$. Therefore, the product $y_{2n-1}^{\tau_{2n-1}} \dots y_{3n-1}^{\tau_{3n-1}}$ in the expansion of d can be dropped as well.

Suppose that $1 \leqslant i \leqslant 2n-4$. Then $y_i \in G^{(i+1)}$ and by Lemma 4, we have $y_i^p \in G^{(i+n+1)}$. We will show that $y_i^p \notin G^{(i+n+2)}$. Indeed, if $1 \leqslant i \leqslant n-1$, then using Lemmas 4 and 8, we obtain

$$\begin{aligned} y_i^p &= (E + pF(i) + p^2Q(i))^p = E + p^2F(i) \\ &= E + \dots + (-1)^0 \binom{i}{0} \binom{i-1}{i-1} p^2 e_{1,n-i} + \dots = E + \dots + p^2 e_{1,n-i} + \dots . \end{aligned}$$

At the same time,

$$f_n(1, n-i, i+n+2) = -\left[\frac{1-(n-i)-(i+n+2)}{n}\right] = 3,$$

which means that the element located at the position $(i, n-i)$, of the arbitrary matrix from $G^{(i+n+2)}$ belongs to the ideal J^3. Similarly, if $n \leqslant i \leqslant 2n-4$, then

$$y_i^p = E + \dots + (-1)^{i+1-n} \binom{i}{i+1-n} \binom{i-1}{i-1} p^2 e_{i+2-n,1} + \dots$$

and $\binom{i}{i+1-n} \not\equiv 0 \pmod p$, and on the other hand,

$$f_n(i+2-n, 1, i+n+2) = -\left[\frac{i+2-n-1-(i+2+n)}{n}\right] = 3.$$

Therefore, $y_i^p \notin G^{(i+n+2)}$. This and the inclusion $d^p \in G^p = G^{(3n-2)}$ yield that the commutators $y_1, \dots, y_{2n-4}$ must be included into the expansion of d with exponents divisible by p, and hence, they can also be dropped. Therefore, we can assume that $d = y_{2n-3}^{\tau} y_{2n-2}^{\mu}$. Using again Corollary 2 and Lemma 7, we find

$$\begin{aligned} d^p &= y_{2n-3}^{p\tau} y_{2n-2}^{p\mu} = \left(E + p^2F(2n-3)\right)^{\tau} \left(E + p^2F(2n-2)\right)^{\mu} \\ &= \left(E + p^2(2n-3)\binom{2n-3}{n-2} e_{n,1} - p^2 \binom{2n-3}{n-2} e_{n-1,1} + p^2 \binom{2n-3}{n-2} e_{n,2}\right)^{\tau} \\ &\times \left(E + p^2 \binom{2n-2}{n-1} e_{n,1}\right)^{\mu} = E + \gamma p^2 e_{n,1} - \tau p^2 \binom{2n-3}{n-2} e_{n-1,1} + \tau p^2 \binom{2n-3}{n-2} e_{n,2}. \end{aligned}$$

□

We will now prove Theorem 2. From Lemmas 9 and 10 it follows that in the case when the subgroup H is regular, the following equivalences have to be solvable simultaneously

$$-\tau \binom{2n-3}{n-2} \equiv -1 \pmod p, \qquad \tau \binom{2n-3}{n-2} \equiv -1 \pmod p$$

with respect to τ. Summing them, we obtain $0 \equiv -2 \pmod p$, which is a contradiction. Therefore, the group $P_{\frac{p+2}{3}}(\mathbb{Z}_{p^3})$ is not regular. Theorem 2 is proved.

5. Proof of Theorem 3

Hereinafter, a, b are fundamental roots of a root system of type G_2, moreover, $|a| < |b|$. As above, we will split the calculations necessary for the proof of the theorem into separate statements. The list of all nontrivial Chevalley commutator formulas is provided in Appendix 2. The numbers $C_{ij,rs}$ in the formulas are defined with respect to the structural constants $\epsilon_1 = N_{a,b}$, $\epsilon_2 = N_{a,a+b}$, $\epsilon_3 = N_{a,2a+b}$, $\epsilon_4 = N_{b,3a+b}$, that correspond to the extraspecial pairs (a,b), $(a,a+b)$, $(a,2a+b)$, $(b,3a+b)$. In the calculations, we everywhere assume that $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 1$.

Lemma 11. *Suppose that*

$$g = x_b(\alpha)\, x_{a+b}(\beta)\, x_{2a+b}(\gamma)\, x_{3a+b}(\delta)\, x_{3a+2b}(\epsilon) \in PG_2(\mathbb{Z}_p).$$

The following equalities hold:

$$g^{x_a(1)} =$$
$$x_b(\alpha)\, x_{a+b}(\beta-\alpha)\, x_{2a+b}(\gamma+\alpha-2\beta)\, x_{3a+b}(\delta-\alpha+3\beta-3\gamma)\, x_{3a+2b}(\epsilon-\alpha^2-3\beta^2+3\alpha\beta),$$

$$g^{x_b(1)} = x_b(\alpha)\, x_{a+b}(\beta)\, x_{2a+b}(\gamma)\, x_{3a+b}(\delta)\, x_{3a+2b}(\epsilon-\delta).$$

Proof. Using the identity $xy = yx[x,y]$ and the Chevalley commutator formula (formulas (29), (28), (27), (31) from Appendix 2), we will switch places the element $x_a(1)$ and every factor from g in the product $g\, x_a(1)$:

$$\begin{aligned}
g\, x_a(1) &= x_b(\alpha)\, x_{a+b}(\beta)\, x_{2a+b}(\gamma)\, x_{3a+b}(\delta)\, x_{3a+2b}(\epsilon)\, x_a(1)\\
&= x_b(\alpha)\, x_{a+b}(\beta)\, x_{2a+b}(\gamma)\, x_a(1) x_{3a+b}(\delta)\, x_{3a+2b}(\epsilon)\\
&= x_b(\alpha)\, x_{a+b}(\beta)\, x_a(1)\, x_{2a+b}(\gamma)\, [x_{2a+b}(\gamma), x_a(1)]\, x_{3a+b}(\delta)\, x_{3a+2b}(\epsilon)\\
&= x_b(\alpha)\, x_{a+b}(\beta)\, x_a(1) x_{2a+b}(\gamma)\, x_{3a+b}(-3\gamma)\, x_{3a+b}(\delta)\, x_{3a+2b}(\epsilon)\\
&= x_b(\alpha)\, x_a(1)\, x_{a+b}(\beta)\, [x_{a+b}(\beta), x_a(1)]\, x_{2a+b}(\gamma)\, x_{3a+b}(\delta-3\gamma)\, x_{3a+2b}(\epsilon)\\
&= x_a(1)\, x_b(\alpha)\, [x_b(\alpha), x_a(1)]\, x_{a+b}(\beta)\, x_{2a+b}(-2\beta)\, x_{3a+b}(3\beta)\times\\
&\quad x_{3a+2b}(-3\beta^2)\, x_{2a+b}(\gamma),\ \ x_{3a+b}(\delta-3\gamma)\, x_{3a+2b}(\epsilon)\\
&= x_a(1) x_b(\alpha)\, x_{a+b}(-\alpha)\, x_{2a+b}(\alpha)\, x_{3a+b}(-\alpha)\, x_{3a+2b}(-\alpha^2)\, x_{a+b}(\beta)\times\\
&\quad x_{2a+b}(\gamma-2\beta)\, x_{3a+b}(\delta+3\beta-3\gamma)\, x_{3a+2b}(\epsilon-3\beta^2)\\
&= x_a(1)\, x_{a+b}(\beta-\alpha)\, x_{2a+b}(\gamma+\alpha-2\beta)\, x_{3a+b}(\delta-\alpha+3\beta-3\gamma)\times\\
&\quad x_{3a+2b}(\epsilon-\alpha^2-3\beta^2+3\alpha\beta).
\end{aligned}$$

Therefore,

$$g\, x_a(1) = x_a(1)\, x_b(\alpha) x_{a+b}(\beta-\alpha)\times$$
$$x_{2a+b}(\gamma+\alpha-2\beta)\, x_{3a+b}(\delta-\alpha-3\beta-3\gamma)\, x_{3a+2b}(\epsilon-\alpha^2-3\beta^2+3\alpha\beta),$$

which yields the required equality.

Similarly,

$$\begin{aligned}
g\, x_b(1) &= x_b(\alpha)\, x_{a+b}(\beta)\, x_{2a+b}(\gamma)\, x_{3a+b}(\delta)\, x_{3a+2b}(\epsilon)\, x_b(1)\\
&= x_b(\alpha)\, x_{a+b}(\beta)\, x_{2a+b}(\gamma)\, x_b(1)\, x_{3a+b}(\delta)\, [x_{3a+b}(\delta), x_b(1)]\, x_{3a+2b}(\epsilon)\\
&= x_b(\alpha)\, x_{a+b}(\beta)\, x_{2a+b}(\gamma)\, x_b(1)\, x_{3a+b}(\delta)\, x_{3a+2b}(\epsilon-\delta)\\
&= x_b(1)\, x_b(\alpha)\, x_{a+b}(\beta)\, x_{2a+b}(\gamma)\, x_{3a+b}(\delta)\, x_{3a+2b}(\epsilon-\delta).
\end{aligned}$$

□

Lemma 12. *Let*

$$g = x_{a+b}(\alpha)\, x_{2a+b}(\beta)\, x_{3a+b}(\gamma)\, x_{3a+2b}(\delta) \in PG_2(\mathbb{Z}_p).$$

For every $n \in \mathbb{N}$, the following equality holds:

$$g^n = x_{a+b}(n\alpha)\, x_{2a+b}(n\beta)\, x_{3a+b}(n\gamma)\, x_{3a+2b}(n\delta + 3\tbinom{n}{2}\alpha\beta).$$

Proof. Induction on n and formula (29) from Appendix 2.

$$\begin{aligned} g^{n+1} &= g^n g \\ &= x_{a+b}(n\alpha)\, x_{2a+b}(n\beta)\, x_{3a+b}(n\gamma)\, x_{3a+2b}(n\delta + 3\tbinom{n}{2}\alpha\beta)\times \\ &\quad x_{a+b}(\alpha)\, x_{2a+b}(\beta)\, x_{3a+b}(\gamma)\, x_{3a+2b}(\delta) \\ &= x_{a+b}((n+1)\alpha)\, x_{2a+b}((n+1)\beta)\, x_{3a+b}((n+1)\gamma)\times \\ &\quad x_{3a+2b}((n+1)\delta + 3\tbinom{n}{2}\alpha\beta + 3n\alpha\beta) \\ &= x_{a+b}((n+1)\alpha)\, x_{2a+b}((n+1)\beta)\, x_{3a+b}((n+1)\gamma)\times \\ &\quad x_{3a+2b}((n+1)\delta + 3\tbinom{n+1}{2}\alpha\beta). \end{aligned}$$

□

Lemma 13. *In the group $PG_2(\mathbb{Z}_p)$, the following relations hold:*

$$(x_a(1)\, x_b(1))^2 = x_a(2)\, x_b(2)\, x_{a+b}(-1)\, x_{2a+b}(1)\, x_{3a+b}(-1),$$
$$(x_a(1)\, x_b(1))^3 = x_a(3)\, x_b(3)\, x_{a+b}(-3)\, x_{2a+b}(5)\, x_{3a+b}(-9)\, x_{3a+2b}(-4),$$
$$(x_a(1)\, x_b(1))^4 = x_a(4)\, x_b(4)\, x_{a+b}(-6)\, x_{2a+b}(14)\, x_{3a+b}(-36)\, x_{3a+2b}(-31),$$
$$(x_a(1)\, x_b(1))^5 = x_a(5)\, x_b(5)\, x_{a+b}(-10)\, x_{2a+b}(30)\, x_{3a+b}(-100)\, x_{3a+2b}(-127).$$

Proof. We put $D = x_a(1)\, x_b(1)$ and use Lemma 11.

$$\begin{aligned} (D)^2 &= x_a(1)\, x_b(1)\, x_a(1)\, x_b(1) \\ &= x_a(2)\, x_b(1)\, [x_b(1), x_a(1)]\, x_b(1) \\ &= x_a(2)\, x_b(1)\, x_{a+b}(-1)\, x_{2a+b}(1)\, x_{3a+b}(-1)\, x_{3a+2b}(-1)\, x_b(1) \\ &= x_a(2)\, x_b(2)\, x_{a+b}(-1)\, x_{2a+b}(1)\, x_{3a+b}(-1). \\ (D)^3 &= x_a(2)\, x_b(2)\, x_{a+b}(-1)\, x_{2a+b}(1)\, x_{3a+b}(-1)\, x_a(1)\, x_b(1) \\ &= x_a(3)\, x_b(2)\, x_{a+b}(-3)\, x_{2a+b}(5)\, x_{3a+b}(-9)\, x_{3a+2b}(-13)\, x_b(1) \\ &= x_a(3)\, x_b(3)\, x_{a+b}(-3)\, x_{2a+b}(5)\, x_{3a+b}(-9)\, x_{3a+2b}(-4). \\ (D)^4 &= x_a(3)\, x_b(3)\, x_{a+b}(-3)\, x_{2a+b}(5)\, x_{3a+b}(-9)\, x_{3a+2b}(-4)\, x_a(1)\, x_b(1) \\ &= x_a(4)\, x_b(3)\, x_{a+b}(-6)\, x_{2a+b}(14)\, x_{3a+b}(-36)\, x_{3a+2b}(-67)\, x_b(1) \\ &= x_a(4)\, x_b(4)\, x_{a+b}(-6)\, x_{2a+b}(14)\, x_{3a+b}(-36)\, x_{3a+2b}(-31). \end{aligned}$$

$$\begin{aligned} (D)^5 &= x_a(4)\, x_b(4)\, x_{a+b}(-6)\, x_{2a+b}(14)\, x_{3a+b}(-36)\, x_{3a+2b}(-31)\, x_a(1)\, x_b(1) \\ &= x_a(5)\, x_b(4)\, x_{a+b}(-10)\, x_{2a+b}(30)\, x_{3a+b}(-100)\, x_{3a+2b}(-227)\, x_b(1) \\ &= x_a(5)\, x_b(5)\, x_{a+b}(-10)\, x_{2a+b}(30)\, x_{3a+b}(-100)\, x_{3a+2b}(-127). \end{aligned}$$

□

Lemma 14. *In the group $PG_2(\mathbb{Z}_{p^m})$ for $i = 1, \dots, 12$ the following equalities hold:*

$$V_i = [x_{-3a-2b}(p),\ {}_i x_a(1)\, x_b(1)] = W_{i+1}Y_{i+2}, \qquad Y_{i+2} \in S^{(i+2)},$$

where

i	W_i	i	W_i	i	W_i	i	W_i
2	$x_{-3a-b}(p)$	*5*	$x_{-a}(-2p)\, x_{-b}(6p)$	*8*	$x_{a+b}(28p)$	*11*	$x_{3a+2b}(-168p)$
3	$x_{-2a-b}(p)$	*6*	$h_{-a}(1+2p)\, h_{-b}(1-6p)$	*9*	$x_{2a+b}(-56p)$	*12*	*1*
4	$x_{-a-b}(2p)$	*7*	$x_a(10p)\, x_b(-18p)$	*10*	$x_{3a+b}(168p)$	*13*	*1*

Proof. The case when $i = 1$. Formula (44) from Appendix 2:

$$V_1 = [x_{-3a-2b}(p),\, x_b(1)] = x_{-3a-b}(p).$$

The case when $i = 2$. Formula (40) from Appendix 2:

$$V_2 = [x_{-3a-b}(p),\, x_a(1)x_b(1)] \equiv [x_{-3a-b}(p),\, x_a(1)] \equiv x_{-2a-b}(p) \pmod{S^{(4)}}.$$

The case when $i = 3$. Formula (39) from Appendix 2:

$$V_3 = [x_{-2a-b}(p)\, Y_4,\, x_a(1)x_b(1)] \equiv [x_{-2a-b}(p),\, x_a(1)] \equiv x_{-a-b}(2p) \pmod{S^{(5)}}.$$

The case when $i = 4$. Formulas (13) and (18) from Appendix 2:

$$V_4 = [x_{-a-b}(2p)\, Y_5,\, x_a(1)x_b(1)] \equiv \\ \equiv [x_{-a-b}(2p),\, x_b(1)][x_{-a-b}(2p),\, x_a(1)] \equiv x_{-a}(-2p)\, x_{-b}(6p) \pmod{S^{(6)}}.$$

The case when $i = 5$. Relations between the opposite root elements:

$$V_5 = [x_{-a}(-2p)\, x_{-b}(6p) Y_6,\, x_a(1)x_b(1)] \equiv \\ \equiv [x_{-a}(-2p), x_a(1)]\, [x_{-b}(6p), x_b(1)] \equiv h_{-a}(1+2p)\, h_{-b}(1-6p) \pmod{S^{(7)}}.$$

The case when $i = 6$. Relations between root and diagonal elements and Table 2 from Appendix 1:

$$V_6 = [h_{-a}(1+2p)\, h_{-b}(1-6p) Y_7,\, x_a(1)x_b(1)] \equiv \\ \equiv [h_{-a}(1+2p)\, h_{-b}(1-6p), x_b(1)]\, [h_{-a}(1+2p)\, h_{-b}(1-6p), x_a(1)] \equiv \\ \equiv x_a(10p)\, x_b(-18p) \pmod{S^{(8)}}.$$

The case when $i = 7$. Formulas (26) and (27) from Appendix 2:

$$V_7 = [x_a(10p)\, x_b(-18p) Y_8,\, x_a(1)x_b(1)] \equiv \\ \equiv [x_a(10p), x_b(1)]\, [x_b(-18p), x_a(1)] \equiv x_{a+b}(28p) \pmod{S^{(9)}}.$$

The case when $i = 8$. Formula (28) from Appendix 2:

$$V_8 = [x_{a+b}(28p) Y_9,\, x_a(1)x_b(1)] \equiv \\ \equiv [x_{a+b}(28p), x_a(1)] \equiv x_{2a+b}(-56p) \pmod{S^{(10)}}.$$

The case when $i = 9$. Formula (29) from Appendix 2:

$$V_9 = [x_{2a+b}(-56p) Y_{10},\, x_a(1)x_b(1)] \equiv \\ \equiv [x_{2a+b}(-56p), x_a(1)] \equiv x_{3a+b}(168p) \pmod{S^{(11)}}.$$

The case when $i = 10$. Formula (30) from Appendix 2:

$$V_{10} = [x_{3a+b}(168p) Y_{11},\, x_a(1)x_b(1)] \equiv \\ \equiv [x_{3a+b}(168p), x_b(1)] \equiv x_{3a+2b}(-168p) \pmod{S^{(12)}}.$$

The case when $i = 11$. We have

$$V_{11} = [x_{3a+2b}(-168p) Y_{12},\, x_a(1)x_b(1)] \equiv [Y_{12},\, x_a(1)x_b(1)] \in S^{(13)}.$$

The case when $i = 12$ is trivial. □

Lemma 15. *In the group $PG_2(\mathbb{Z}_{p^m})$ for $i = 1, \dots, 5$ the following equalities hold*

$$U_i = [[x_{-3a-2b}(p),\ _{11-i}\, x_a(1)\, x_b(1)], [x_{-3a-2b}(p),\ _i\, x_a(1)\, x_b(1)]] = P_i Q_i,$$

where $Q_i \in S^{(14)}$, $P_1 = x_b(-168p^2)$ and

i	*2*	*3*	*4*	*5*
P_i	$x_a(168p^2)$	$x_a(-224p^2)$	$x_a(168p^2)\, x_b(168p^2)$	$x_a(-100p^2)\, x_b(-324p^2)$

Proof. We use Lemma 14, noting that $W_i \in S^{(i)}$.

The case when $i = 1$. Formula (66) from Appendix 2:

$$U_1 = [x_{3a+2b}(-168p)Y_{12}, x_{-3a-b}(p)Y_3] \equiv$$
$$\equiv [x_{3a+2b}(-168p), x_{-3a-b}(p)] = x_b(-168p^2) \pmod{S^{(14)}}.$$

The case when $i = 2$. Formula (60) from Appendix 2:

$$U_2 = [x_{3a+b}(168p)Y_{11}, x_{-2a-b}(p)Y_4] \equiv$$
$$\equiv [x_{3a+b}(168p), x_{-2a-b}(p)] \equiv x_a(168p^2) \pmod{S^{(14)}}.$$

The case when $i = 3$. Formula (52) from Appendix 2:

$$U_3 = [x_{2a+b}(-56p)Y_{10}, x_{-a-b}(2p)Y_5] \equiv$$
$$\equiv [x_{2a+b}(-56p), x_{-a-b}(2p)] \equiv x_a(-224p^2) \pmod{S^{(14)}}.$$

The case when $i = 4$. Formulas (45) and (47) from Appendix 2:

$$U_4 = [x_{a+b}(28p)Y_9, x_{-a}(-2p)x_{-b}(6p)Y_6] \equiv$$
$$\equiv [x_{a+b}(28p), x_{-a}(-2p)][x_{a+b}(28p), x_{-b}(6p)] \equiv$$
$$\equiv x_b(168p^2)x_a(168p^2) \pmod{S^{(14)}}.$$

The case when $i = 5$. Relations between root and diagonal elements and Table 2 from Appendix 1:

$$U_5 = [x_a(10p)\, x_b(-18p)Y_8, h_{-a}(1+2p)h_{-b}(1-6p)Y_7] \equiv$$
$$\equiv [x_a(10p), h_{-a}(1+2p)h_{-b}(1-6p)][x_b(-18p), h_{-a}(1+2p)h_{-b}(1-6p)] \equiv$$
$$\equiv x_a(-100p^2)x_b(-324p^2) \pmod{S^{(14)}}.$$

□

We will now prove Theorem 3. According to [11], the nilpotency class of a Sylow p-subgroup $PG_2(\mathbb{Z}_{p^m})$ equals $mh-1 = 6m-1$, therefore, it is regular when $6m-1 < p$. On the other hand, in [6] it is shown that the group $PG_2(\mathbb{Z}_{p^m})$ is regular if $p > |G_2| + |\Pi(G_2)| = 12 + 2 = 14$, that is, when $p \geqslant 17$. Hence, it suffices to study the cases when $p \in \{2, 3, 5, 7, 11\}$.

First, we will show that the group $PG_2(\mathbb{Z}_p)$ is not regular when $p = 2, 3, 5$. Note that in these cases $PG_2(\mathbb{Z}_p)$ coincides with the unipotent subgroup $UG_2(\mathbb{Z}_p)$ of the Chevalley group $G_2(\mathbb{Z}_p)$. The derived subgroup $UG_2(\mathbb{Z}_p)$, according to [11], belongs to the subgroup

$$H = \langle x_{a+b}(1),\ x_{2a+b}(1),\ x_{3a+b}(1),\ x_{3a+2b}(1)\rangle,$$

where every element g can be uniquely represented in the form

$$g = x_{a+b}(\alpha)\, x_{2a+b}(\beta)\, x_{3a+b}(\gamma)\, x_{3a+2b}(\delta), \qquad \alpha, \beta, \gamma, \delta \in \mathbb{Z}_p.$$

We put $A = x_a(1)$ and $B = x_b(1)$. Obviously, $[\langle A, B\rangle, \langle A, B\rangle]^p \subseteq H^p$.

The case when $p = 2$. By Lemma 13, in $PG_2(\mathbb{Z}_2)$ the following equality holds:

$$C = B^{-2}A^{-2}\,(AB)^2 = x_{a+b}(1)\,x_{2a+b}(1)\,x_{3a+b}(1).$$

Due to Lemma 12, $H^2 \subseteq x_{3a+2b}(\mathbb{Z}_p)$, therefore, $C \notin H^2$. Hence, the group $PG_2(\mathbb{Z}_2)$ is not regular. Since it is a homomorphic image of $PG_2(\mathbb{Z}_{2^m})$ for every $m \geqslant 2$, we imply that $PG_2(\mathbb{Z}_{2^m})$ is irregular for $m \geqslant 1$.

The case when $p = 3, 5$. By Lemma 12, in both cases $H^p = 1$. At the same time, by Lemma 13 $(AB)^3 = A^3B^3 \cdot x_{2a+b}(2)\,x_{3a+2b}(2)$ in $PG_2(\mathbb{Z}_3)$ and $(AB)^5 = A^5B^5 \cdot x_{3a+2b}(2)$ in $PG_2(\mathbb{Z}_5)$. Hence, the groups $PG_2(\mathbb{Z}_{3^m}), PG_2(\mathbb{Z}_{5^m})$ are irregular for $m \geqslant 1$.

Cases when $p = 7, 11$. We will prove that the group $PG_2(\mathbb{Z}_{p^2})$ is not regular for the given p. We put $A = x_a(1)x_b(1)$ and $B = x_{-3a-2b}(p)$. The element B belongs to the congruence subgroup $G_2(J)$, which is normal in $PG_2(\mathbb{Z}_{p^2})$ and is an elementary Abelian p-group. Therefore, every commutator of the elements A and B belongs to $G_2(J)$ and equals one to the power which is a multiple of p. Hence, $[\langle A, B\rangle']^p = 1$. On the other hand, every commutator of A and B, that has more than two occurrences of B, equals one, therefore,

$$B^{-p}A^{-p}\,(AB)^p = [B, A]^{\binom{p}{2}} \dots [B,\, {}_{p-1}A]^{\binom{p}{p}} = [B,\, {}_{p-1}A].$$

By Lemma 14, the commutator $[B,\, {}_{p-1}A]$ equals $x_a(10p)\,x_b(-18p)Y_8 \neq 1$ in $PG_2(\mathbb{Z}_{7^2})$ and equals $x_{3a+2b}(-168p) \neq 1$ in $PG_2(\mathbb{Z}_{11^2})$. Hence, the groups $PG_2(\mathbb{Z}_{7^2}), PG_2(\mathbb{Z}_{11^2})$ are not regular. This yields irregularity of the groups $PG_2(\mathbb{Z}_{7^m})$ and $PG_2(\mathbb{Z}_{11^m})$ for every $m \geqslant 2$.

The case when $p = 13$. We will prove that the group $PG_2(\mathbb{Z}_{13^3})$ is not regular. We put $A = x_a(1)x_b(1)$ and $B = x_{-3a-2b}(p)$. Elements A and B satisfy conditions 1) and 2) of Corollary 1. Indeed, every commutator from A and B with weight more than 12 or having two occurrences of B, belongs to the elementary Abelian p-group $G_2(J^2)$, which is centralized by the element B.

Assume that the group $PG_2(\mathbb{Z}_{p^3})$ is regular. Then by Corollary 1, there exists an element $d \in \langle A, B\rangle'$ such that

$$d^p \equiv [B, {}_{12}A]\,[B, {}_{11}A, B]^{-1} \prod_{i=1}^{5} [[B, {}_{11-i}A], [B, {}_iA]]^{(-1)^{i+1}} \pmod{S^{(14)}}. \tag{25}$$

The group $\langle A, B\rangle'$ is generated by the commutators $y_i = [B, {}_iA]$, $i = 1, 2, \dots$, and $y_{ij} = [y_i, y_j]$, $i, j = 1, 2, \dots$, that satisfy Lemma 1 and Corollary 2. Moreover, $y_{ij}^p = 1$ for every i, j, and $y_i^p = 1$, when $i \geqslant 12$. Therefore, due to Corollary 1 we can assume that

$$d = [B, A]^{\alpha_1} \dots [B, {}_{11}A]^{\alpha_{11}}$$

for some $\alpha_1, \dots, \alpha_{11}$. The element $[B, {}_iA]^p$ belongs to $S^{(i+7)}$, but does not belong to $S^{(i+8)}$ for $i = 1, \dots, 10$. Indeed, using Lemmas 14 and 1, we obtain

$$[B, {}_iA]^p = W_{i+1}^p Y_{i+2}^p [W_{i+1}, Y_{i+2}]^{\binom{p}{2}}.$$

By Lemma 4, $y_{i+2}^p \in S^{(i+9)}$, $[W_{i+1}, Y_{i+2}]^{\binom{p}{2}} \in S^{(2i+9)}$, $W_{i+1}^p \in S^{(i+7)}$, but, obviously, $W_{i+1}^p \notin S^{(i+8)}$. Since the right-hand side of (25) lies in $S^{(13)}$, and

$$d^p = [B, A]^{p\alpha_1} \dots [B, {}_{11}A]^{p\alpha_{11}},$$

then the numbers $\alpha_1, \dots, \alpha_6$ must be multiples of p. Therefore, we can assume that

$$d = [B,_7 A]^{\alpha_7} \dots [B,_{11} A]^{\alpha_{11}}.$$

Finally, $[B,_i A]^p \in S^{(14)}$ when $i \geqslant 8$, therefore,

$$d^p \equiv [B,_7 A]^{\alpha_7} \equiv x_a(10\alpha_7 p^2) x_b(-18\alpha_7 p^2) \pmod{S^{(14)}}.$$

On the other hand, using Lemmas 14 and 15, we obtain

$$[B,{}_{12}A]\,[B,{}_{11}A,B]^{-1} \prod_{i=1}^{5} [[B,{}_{11-i}A],[B,{}_iA]]^{(-1)^{i+1}} \equiv$$
$$\equiv 1 \cdot 1 \cdot x_b(-168p^2)\,(x_a(168p^2))^{-1}\, x_a(-224p^2)\,(x_b(168p^2) x_a(168p^2))^{-1} \times$$
$$\times\, x_a(-100p^2)\, x_b(-324p^2) \equiv x_a(-660p^2)\, x_b(-212p^2) \pmod{S^{(14)}}.$$

It is easy to see that both equalities $10\alpha_7 p^2 = 9p^2$ and $8\alpha_7 p^2 = 3p^2$ simultaneously are not fulfilled in the ring $\mathbb{Z}_{13^3}$ given any α_7. Therefore, the group $PG_2(\mathbb{Z}_{13^3})$, along with the groups $PG_2(\mathbb{Z}_{13^m})$, $m \geqslant 3$, is not regular. Theorem 3 is proved.

6. Appendix 1

According to [13, p.319], in a three-dimentional Eucledian space with the orthonormal basic ε_1, ε_2, ε_3, we choose two vectors

$$a = \varepsilon_1 - \varepsilon_2, \qquad b = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

Then the set of vectors

$$-3a-2b,\ -3a-b,\ -2a-b,\ -a-b,\ -a,\ -b,\ a,\ b,\ a+b, 2a+b,\ 3a+b,\ 3a+2b$$

forms a root system of type G_2, and the roots

$$a,\ b,\ a+b,\ 2a+b,\ 3a+b,\ 3a+2b$$

form its subsystem of positive roots. The roots a and b form a fundamental system of roots.

Structural constants of the Lie algebra of type G_2 are listed in the following lemma.

Lemma 16. *We put*

$$N_{a,b} = \epsilon_1,\ N_{a,a+b} = 2\epsilon_2,\ N_{a,2a+b} = 3\epsilon_3,\ N_{b,3a+b} = \epsilon_4$$

and suppose that $\epsilon_5 = \frac{\epsilon_1\epsilon_3}{\epsilon_4}$. *Then nonzero constants* $N_{r,s}$ *have the forms provided in the following table:*

Table 1.

$N_{r,s}$	$-3a-2b$	$-3a-b$	$-2a-b$	$-a-b$	$-a$	$-b$	a	b	$a+b$	$2a+b$	$3a+b$	$3a+2b$
$-3a-2b$								ϵ_4	$-\epsilon_5$	ϵ_5	$-\epsilon_4$	
$-3a-b$						ϵ_4	ϵ_3			$-\epsilon_3$		$-\epsilon_4$
$-2a-b$				$-3\epsilon_5$	$3\epsilon_3$		$2\epsilon_2$		$-2\epsilon_2$		$-\epsilon_3$	ϵ_5
$-a-b$			$3\epsilon_5$		$2\epsilon_2$		$3\epsilon_1$	$-\epsilon_1$		$-2\epsilon_2$		$-\epsilon_5$
$-b$		$-\epsilon_4$			ϵ_1				$-\epsilon_1$			ϵ_4
$-a$			$-3\epsilon_3$	$-2\epsilon_2$		$-\epsilon_1$			$3\epsilon_1$	$2\epsilon_2$	ϵ_3	
a		$-\epsilon_3$	$-2\epsilon_2$	$-3\epsilon_1$				ϵ_1	$2\epsilon_2$	$3\epsilon_3$		
b	$-\epsilon_4$			ϵ_1			$-\epsilon_1$				ϵ_4	
$a+b$	ϵ_5		$2\epsilon_2$		$-3\epsilon_1$	ϵ_1	$-2\epsilon_2$			$-3\epsilon_5$		
$2a+b$	$-\epsilon_5$	ϵ_3		$2\epsilon_2$	$-2\epsilon_2$		$-3\epsilon_3$		$3\epsilon_5$			
$3a+b$	ϵ_4		ϵ_3		$-\epsilon_3$			$-\epsilon_4$				
$3a+2b$		ϵ_4	$-\epsilon_5$	ϵ_5		$-\epsilon_4$						

Proof. According to [12, Theorem 4.2.2.], structural constants of an simple Lie algebra of type Φ over $\mathbb{C}$ satisfy the following relations:

(i) $N_{s,r} = -N_{r,s},\ r, s \in \Phi$;

(ii) $\dfrac{N_{r_1,r_2}}{(r_3,r_3)} = \dfrac{N_{r_2,r_3}}{(r_1,r_2)} = \dfrac{N_{r_3,r_1}}{(r_2,r_2)}$, if $r_1, r_2, r_3 \in \Phi$ and $r_1 + r_2 + r_3 = 0$;

(iii) $N_{r,s}N_{-r,-s} = -(p+1)^2,\ r, s, r+s \in \Phi$;

(iv) $\dfrac{N_{r_1,r_2}N_{r_3,r_4}}{(r_1+r_2, r_1+r_3)} + \dfrac{N_{r_2,r_3}N_{r_1,r_4}}{(r_2+r_3, r_2+r_3)} + \dfrac{N_{r_3,r_1}N_{r_2,r_4}}{(r_3+r_1, r_3+r_1)} = 0,$

if $r_1, r_2, r_3, r_4 \in \Phi$, $r_1 + r_2 + r_3 + r_4 = 0$ and there are no opposite pairs among the roots r_1, r_2, r_3, r_4.

From the equalities

$$b + (-a-b) + a = 0,\ (a+b) + (-2a-b) + a = 0,$$
$$(2a+b) + (-3a-b) + a = 0,\ (3a+b) + (-3a-2b) + b = 0$$

and item (ii) it follows that

$$\frac{N_{b,-a-b}}{2} = \frac{N_{-a-b,a}}{6} = \frac{N_{a,b}}{2} = \frac{\epsilon_1}{2}, \quad \frac{N_{a+b,-2a-b}}{2} = \frac{N_{-2a-b,a}}{2} = \frac{N_{a,a+b}}{2} = \epsilon_2,$$
$$\frac{N_{2a+b,-3a-b}}{2} = \frac{N_{-3a-b,a}}{2} = \frac{N_{a,2a+b}}{6} = \frac{\epsilon_3}{2},$$
$$\frac{N_{3a+b,-3a-2b}}{6} = \frac{N_{-3a-2b,b}}{6} = \frac{N_{b,3a+b}}{6} = \frac{\epsilon_4}{6}.$$

Next, the equality

$$(a+b) + (2a+b) + (-b) + (-3a-b) = 0$$

and item (iv) yield that

$$\frac{N_{a+b,2a+b}N_{-b,-3a-b}}{6} + \frac{N_{-b,a+b}N_{2a+b,-3a-b}}{2} = 0,$$

hence,

$$N_{a+b,2a+b} = -3\frac{N_{-b,a+b}N_{2a+b,-3a-b}}{N_{-b,-3a-b}} = -3\frac{(-\epsilon_1)\epsilon_3}{-\epsilon_4} = -3\frac{\epsilon_1\epsilon_3}{\epsilon_4} = -3\epsilon_5.$$

Finally, from the equality

$$(2a+b) + (-3a-2b) + (a+b) = 0$$

it follows that

$$\frac{N_{2a+b,-3a-2b}}{2} = \frac{N_{-3a-2b,a+b}}{2} = \frac{N_{a+b,2a+b}}{6} = -\frac{\epsilon_5}{2}.$$

The remaining structural constants $N_{r,s}$ are defined from the relations:

$$N_{r,s} = N_{-s,-r} = -N_{s,r} = -N_{-r,-s}.$$

□

The following table lists the values of the dot product (h_s, r).

Table 2.

$h_r \setminus s$	a	b	$a+b$	$2a+b$	$3a+b$	$3a+2b$
h_a	2	-3	-1	1	3	0
h_b	-1	2	1	0	-1	1
h_{a+b}	-1	3	2	1	0	3
h_{2a+b}	1	0	1	2	3	3
h_{3a+b}	1	-1	0	1	2	1
h_{3a+2b}	0	1	1	1	1	2

7. Appendix 2

The Chevalley's commutator formulas for the type G_2

Let Φ be a reduced indecomposable root system, K is a field or associative-commutative ring with a unit. According to [12, Theorem 5.2.2], the commutator

$$[x_s(u), x_r(t)] = x_s(u)^{-1} x_r(t)^{-1} x_s(u) x_r(t),$$

where $r, s \in \Phi$ and $u, t \in K$, equals identity, if $r + s \notin \Phi$ and $r \neq -s$, and can be decomposed into a product of root elements by the formula

$$[x_s(u), x_r(t)] = \prod_{\substack{ir+js \in \Phi,\\ i,j>0}} x_{ir+js}(C_{ij,rs}(-t)^i u^j),$$

if $r + s \in \Phi$. Co-factors of the product are located with respect to the increase of the sum $i + j$, and the constants $C_{ij,rs}$ are integers and are defined by the formulas [12, Theorem 5.2.2]:

$$C_{i1,rs} = M_{r,s,i},\ C_{1j,rs} = (-1)^j M_{s,r,j},\ C_{32,rs} = \frac{1}{3} M_{r+s,r,2},\ C_{23,rs} = -\frac{2}{3} M_{s+r,s,2}.$$

In turn, the numbers $M_{r,s,i}$ are expressed with respect to the structural constants $N_{r,s}$ of the corresponding Lie algebra by the formula [12, p. 61]

$$M_{r,s,i} = \frac{1}{i!} N_{r,s} N_{r,r+s} \dots N_{r,(i-1)r+s},$$

The list of formulas.

Positive roots

(26) $[x_b(u), x_a(t)] =$

$$= x_{a+b}(-\epsilon_1 tu)\, x_{2a+b}(\epsilon_1\epsilon_2 t^2 u)\, x_{3a+b}(-\epsilon_1\epsilon_2\epsilon_3 t^3 u)\, x_{3a+2b}(-\epsilon_2\epsilon_5 t^3 u^2),$$

$$(27)\quad [x_a(u), x_b(t)] = \\ = x_{a+b}(\epsilon_1 tu)\, x_{2a+b}(-\epsilon_1\epsilon_2 tu^2)\, x_{3a+b}(\epsilon_1\epsilon_2\epsilon_3 tu^3)\, x_{3a+2b}(-2\epsilon_2\epsilon_5 t^2u^3),$$

$$(28)\quad [x_{a+b}(u), x_a(t)] = x_{2a+b}(-2\epsilon_2 tu)\, x_{3a+b}(3\epsilon_2\epsilon_3 t^2u)\, x_{3a+2b}(-3\epsilon_2\epsilon_5 tu^2),$$

$$(29)\quad [x_{2a+b}(u), x_a(t)] = x_{3a+b}(-3\epsilon_3 tu),$$

$$(30)\quad [x_{3a+b}(u), x_b(t)] = x_{3a+2b}(-\epsilon_4 tu),$$

$$(31)\quad [x_{2a+b}(u), x_{a+b}(t)] = x_{3a+2b}(3\epsilon_5 tu),$$

Negative roots

$$(32)\quad [x_{-b}(u), x_{-a}(t)] = \\ = x_{-a-b}(\epsilon_1 tu)\, x_{-2a-b}(\epsilon_1\epsilon_2 t^2u)\, x_{-3a-b}(\epsilon_1\epsilon_2\epsilon_3 t^3u)\, x_{-3a-2b}(-\epsilon_2\epsilon_5 t^3u^2),$$

$$(33)\quad [x_{-a}(u), x_{-b}(t)] = \\ = x_{-a-b}(-\epsilon_1 tu)\, x_{-2a-b}(-\epsilon_1\epsilon_2 tu^2)\, x_{-3a-b}(-\epsilon_1\epsilon_2\epsilon_3 tu^3),$$

$$(34)\quad [x_{-a-b}(u), x_{-a}(t)] = x_{-2a-b}(2\epsilon_2 tu)\, x_{-3a-b}(3\epsilon_2\epsilon_3 t^2u)\, x_{-3a-2b}(-3\epsilon_2\epsilon_5 tu^2),$$

$$(35)\quad [x_{-2a-b}(u), x_{-a}(t)] = x_{-3a-b}(3\epsilon_3 tu),$$

$$(36)\quad [x_{-3a-b}(u), x_{-b}(t)] = x_{-3a-2b}(\epsilon_4 tu),$$

$$(37)\quad [x_{-2a-b}(u), x_{-a-b}(t)] = x_{-3a-2b}(-3\epsilon_5 tu),$$

Mixed roots

$$(38)\quad [x_{-a-b}(u), x_a(t)] = x_{-b}(3\epsilon_1 tu),$$

$$(39)\quad [x_{-2a-b}(u), x_a(t)] = x_{-a-b}(2\epsilon_2 tu)\, x_{-b}(3\epsilon_1\epsilon_2 t^2u)\, x_{-3a-2b}(3\epsilon_2\epsilon_5 tu^2),$$

$$(40)\quad [x_{-3a-b}(u), x_a(t)] = \\ = x_{-2a-b}(\epsilon_3 tu)\, x_{-a-b}(\epsilon_2\epsilon_3 t^2u)\, x_{-b}(\epsilon_1\epsilon_2\epsilon_3 t^3u)\, x_{-3a-2b}(\epsilon_2\epsilon_5 t^3u^2),$$

$$(41)\quad [x_a(u), x_{-3a-b}(t)] = \\ = x_{-2a-b}(-\epsilon_3 tu)\, x_{-a-b}(-\epsilon_2\epsilon_3 tu^2)\, x_{-b}(-\epsilon_1\epsilon_2\epsilon_3 tu^3)\, x_{-3a-2b}(2\epsilon_2\epsilon_5 t^2u^3),$$

$$(42)\quad [x_b(u), x_{-a-b}(t)] = \\ = x_{-a}(\epsilon_1 tu)\, x_{-2a-b}(-\epsilon_1\epsilon_2 t^2u)\, x_{-3a-2b}(\epsilon_1\epsilon_2\epsilon_5 t^3u)\, x_{-3a-b}(-\epsilon_2\epsilon_3 t^3u^2),$$

$$(43)\quad [x_{-a-b}(u), x_b(t)] = \\ = x_{-a}(-\epsilon_1 tu)\, x_{-2a-b}(\epsilon_1\epsilon_2 tu^2)\, x_{-3a-2b}(-\epsilon_1\epsilon_2\epsilon_5 tu^3)\, x_{-3a-b}(-2\epsilon_2\epsilon_3 t^2u^3),$$

$$(44)\quad [x_{-3a-2b}(u), x_b(t)] = x_{-3a-b}(\epsilon_4 tu),$$

$$(45)\quad [x_{-a}(u), x_{a+b}(t)] = x_b(3\epsilon_1 tu),$$

$$\begin{aligned}(46)\quad &[x_{-b}(u), x_{a+b}(t)] = \\ &= x_a(-\epsilon_1 tu)\, x_{2a+b}(-\epsilon_1\epsilon_2 t^2 u)\, x_{3a+2b}(-\epsilon_1\epsilon_2\epsilon_5 t^3 u)\, x_{3a+b}(-\epsilon_2\epsilon_3 t^3 u^2),\end{aligned}$$

$$\begin{aligned}(47)\quad &[x_{a+b}(u), x_{-b}(t)] = \\ &= x_a(\epsilon_1 tu)\, x_{2a+b}(\epsilon_1\epsilon_2 tu^2)\, x_{3a+2b}(\epsilon_1\epsilon_2\epsilon_5 tu^3)\, x_{3a+b}(-2\epsilon_2\epsilon_3 t^2 u^3),\end{aligned}$$

$$(48)\quad [x_{-2a-b}(u), x_{a+b}(t)] = x_{-a}(-2\epsilon_2 tu)\, x_b(-3\epsilon_1\epsilon_2 t^2 u)\, x_{-3a-b}(3\epsilon_2\epsilon_3 tu^2),$$

$$\begin{aligned}(49)\quad &[x_{-3a-2b}(u), x_{a+b}(t)] = \\ &= x_{-2a-b}(-\epsilon_5 tu)\, x_{-a}(\epsilon_2\epsilon_5 t^2 u)\, x_b(\epsilon_1\epsilon_2\epsilon_5 t^3 u)\, x_{3a+b}(\epsilon_2\epsilon_3 t^3 u^2),\end{aligned}$$

$$\begin{aligned}(50)\quad &[x_{a+b}(u), x_{-3a-2b}(t)] = \\ &= x_{-2a-b}(\epsilon_5 tu)\, x_{-a}(-\epsilon_2\epsilon_5 tu^2)\, x_b(-\epsilon_1\epsilon_2\epsilon_5 tu^3)\, x_{3a+b}(2\epsilon_2\epsilon_3 t^2 u^3),\end{aligned}$$

$$(51)\quad [x_{-a}(u), x_{2a+b}(t)] = x_{a+b}(2\epsilon_2 tu)\, x_{3a+2b}(-3\epsilon_2\epsilon_5 t^2 u)\, x_b(-3\epsilon_1\epsilon_2 tu^2),$$

$$(52)\quad [x_{-a-b}(u), x_{2a+b}(t)] = x_a(-2\epsilon_2 tu)\, x_{3a+b}(-3\epsilon_2\epsilon_3 t^2 u)\, x_{-b}(3\epsilon_1\epsilon_2 tu^2),$$

$$\begin{aligned}(53)\quad &[x_{-3a-b}(u), x_{2a+b}(t)] = \\ &= x_{-a}(-\epsilon_3 tu)\, x_{a+b}(-\epsilon_2\epsilon_3 t^2 u)\, x_{3a+2b}(\epsilon_1\epsilon_2\epsilon_5 t^3 u)\, x_b(-\epsilon_1\epsilon_2 t^3 u^2),\end{aligned}$$

$$\begin{aligned}(54)\quad &[x_{2a+b}(u), x_{-3a-b}(t)] = \\ &= x_{-a}(\epsilon_3 tu)\, x_{a+b}(\epsilon_2\epsilon_3 tu^2)\, x_{3a+2b}(-\epsilon_2\epsilon_3\epsilon_5 tu^3)\, x_b(-2\epsilon_1\epsilon_2 t^2 u^3),\end{aligned}$$

$$\begin{aligned}(55)\quad &[x_{-3a-2b}(u), x_{2a+b}(t)] = \\ &= x_{-a-b}(\epsilon_5 tu)\, x_a(-\epsilon_2\epsilon_5 t^2 u)\, x_{3a+b}(-\epsilon_2\epsilon_3\epsilon_5 t^3 u)\, x_{-b}(\epsilon_1\epsilon_2 t^3 u^2),\end{aligned}$$

$$\begin{aligned}(56)\quad &[x_{2a+b}(u), x_{-3a-2b}(t)] = \\ &= x_{-a-b}(-\epsilon_5 tu)\, x_a(\epsilon_2\epsilon_5 tu^2)\, x_{3a+b}(\epsilon_2\epsilon_3\epsilon_5 tu^3)\, x_{-b}(2\epsilon_1\epsilon_2 t^2 u^3),\end{aligned}$$

$$\begin{aligned}(57)\quad &[x_{-a}(u), x_{3a+b}(t)] = \\ &= x_{2a+b}(\epsilon_3 tu)\, x_{a+b}(-\epsilon_2\epsilon_3 tu^2)\, x_b(\epsilon_1\epsilon_2\epsilon_3 tu^3)\, x_{3a+2b}(2\epsilon_2\epsilon_5 t^2 u^3),\end{aligned}$$

$$\begin{aligned}(58)\quad &[x_{3a+b}(u), x_{-a}(t)] = \\ &= x_{2a+b}(-\epsilon_3 tu)\, x_{a+b}(\epsilon_2\epsilon_3 t^2 u)\, x_b(-\epsilon_1\epsilon_2\epsilon_3 t^3 u)\, x_{3a+2b}(\epsilon_2\epsilon_5 t^3 u^2),\end{aligned}$$

$$\begin{aligned}(59)\quad &[x_{-2a-b}(u), x_{3a+b}(t)] = \\ &= x_a(-\epsilon_3 tu)\, x_{-a-b}(\epsilon_2\epsilon_3 tu^2)\, x_{-3a-2b}(\epsilon_2\epsilon_3\epsilon_5 tu^3)\, x_{-b}(-2\epsilon_1\epsilon_2 t^2 u^3),\end{aligned}$$

$$\begin{aligned}(60)\quad &[x_{3a+b}(u), x_{-2a-b}(t)] = \\ &= x_a(\epsilon_3 tu)\, x_{-a-b}(-\epsilon_2\epsilon_3 t^2 u)\, x_{-3a-2b}(-\epsilon_2\epsilon_3\epsilon_5 t^3 u)\, x_{-b}(-\epsilon_1\epsilon_2 t^3 u^2),\end{aligned}$$

$$(61)\quad [x_{-3a-2b}(u), x_{3a+b}(t)] = x_{-b}(-\epsilon_4 tu),$$

$$(62)\quad [x_{-a-b}(u), x_{3a+2b}(t)] = \\ = x_{2a+b}(-\epsilon_5 tu)\, x_a(-\epsilon_2\epsilon_5 tu^2)\, x_{-b}(\epsilon_1\epsilon_2\epsilon_5 tu^3)\, x_{3a+b}(2\epsilon_2\epsilon_3 t^2u^3),$$

$$(63)\quad [x_{3a+2b}(u), x_{-a-b}(t)] = \\ = x_{2a+b}(\epsilon_5 tu)\, x_a(\epsilon_2\epsilon_5 t^2u)\, x_{-b}(-\epsilon_1\epsilon_2\epsilon_5 t^3u)\, x_{3a+b}(\epsilon_2\epsilon_3 t^3u^2),$$

$$(64)\quad [x_{-2a-b}(u), x_{3a+2b}(t)] = \\ = x_{a+b}(\epsilon_5 tu)\, x_{-a}(\epsilon_2\epsilon_5 tu^2)\, x_{-3a-b}(-\epsilon_2\epsilon_3\epsilon_5 tu^3)\, x_b(2\epsilon_1\epsilon_2 t^2u^3),$$

$$(65)\quad [x_{3a+2b}(u), x_{-2a-b}(t)] = \\ = x_{a+b}(-\epsilon_5 tu)\, x_{-a}(-\epsilon_2\epsilon_5 t^2u)\, x_{-3a-b}(\epsilon_2\epsilon_3\epsilon_5 t^3u)\, x_b(\epsilon_1\epsilon_2 t^3u^2),$$

$$(66)\quad [x_{-3a-b}(u), x_{3a+2b}(t)] = x_{a+b}(-\epsilon_4 tu).$$

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