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STRUCTURE OF 4-STRAND SINGULAR PURE BRAID GROUP

T.A. KOZLOVSKAYA

Abstract. We construct a finite presentation for the singular pure braid group SP_4 on 4 strands. As consequence it was proved that the center $Z(SP_4)$, which is the infinite cyclic group, is a direct factor in SP_4. On the other side, we establish that $Z(SP_4)$ is not a direct factor in the singular braid group SG_4.

Keywords: braid group, pure braid group, singular braid group, singular pure braid group, center of group, finite presentation.

1. Introduction

The braid group B_n, $n > 1$, was defined by E. Artin [1]. Braid groups play an important role in many fields of mathematics. These groups have been of really interest in the study of classical knots and links (see [8, 18]). E. Artin gave a presentation and showed how to solve the word problem for this group. A. A. Markov [20] constructed a normal form in B_n. W. Chow [11] proved that the center of B_n is infinite cyclic group. The center of the braid group $Z(B_n)$ coincides with the center of the pure braid group $Z(P_n)$ [20].

The Baez–Birman monoid SB_n or singular braid monoid, was introduced independently by J. Baez [2] and J. Birman [9] for studying the finite type knot invariants (Vassiliev-Goussarov invariants). For monoid SB_3 the word problem was solved by A. Jarai [17] and O. Dashbach, B. Gemein [13]. In general case it was done by R. Corran [10]. L. Paris [23] proved Birman's conjecture, which says that the desingularization map η: $SB_n \to \mathbb{Z}[B_n]$ is injective. This also gives a solution for the word problem in the singular braid monoid.

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By results of R. Fenn, E. Keyman and C. Rourke [15] the singular braid monoid SB_n embeds in a group, which is said to be the group of singular braids and is denoted by SG_n. In [4] were introduced monoid and group of pseudo braids and it was proved that they are isomorphic to monoid and group of singular braids, respectively.

The kernel of the homomorphism $B_n \to S_n$ of the braid group B_n to the symmetric group S_n is the subgroup of B_n which is called the pure braid group on n strands and is denoted P_n. In [12], O. Dasbach and B. Gemein defined the singular pure braid group SP_n that is a generalization of the pure braid group P_n and is the kernel of the epimorphism $SG_n \to S_n$. O. Dasbach and B. Gemein gave a set of generators and defining relations for the group SP_n and established that this group can be constructed by HNN-extensions.

Well known that properties of of B_3 are different from properties of B_n for $n > 3$. For example, B_3 is a free product of two cyclic group with amalgamation, but B_n for $n > 3$ does not have this decomposition. The similar situation for other generalisations of the braid group (see [5] for the virtual braid groups). The singular pure braid group SP_3 is studied by V. Bardakov and T. Kozlovskaya in [6], where was found a decomposition of this group in some group constructions. Also, it was shown that the center $Z(SP_3)$ is a direct factor in SP_3. Another approach, which uses some ideas from [3], to the studying of the singular braid group SG_3 was suggested in [16].

Groups SG_n for $n > 3$ contain so called far commutativity relation which does not contain SG_3. Hence, these groups have more complicated structure. In the present article we are studying the case $n = 4$.

This article is organized as follows. In Section 2 we review some of the basic theory of braid group B_n, pure braid group P_n and singular braid monoid SB_n. In Section 3 we find a presentation of the singular pure braid group SP_4, using the idea from [7]. We find a finite presentation of SP_4, which is simpler than the presentation of O. Dasbach and B. Gemein [12]. In Section 4 we prove that the center of the singular braid group $Z(SG_4)$ is a direct factor in SP_4, but $Z(SG_4)$ is not a direct factor in SG_4. As consequence was found some other presentation of SP_4.

2. Prelimanary

We start with the definition of the braid group. The braid group B_n, $n \geq 2$, on n strands can be defined as a group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the defining relations

$$\sigma_i \, \sigma_j = \sigma_j \, \sigma_i, \quad |i - j| \geq 2,$$
$$\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}, \quad i = 1, 2, \ldots, n-2.$$

The generator σ_i is said to be the *elementary braid*. It corresponds to geometric braid in which the i-th strand passes once above the $(i+1)$-th strand, whereas the other strands are straight lines (see Fig. 2). The first relation is called far commutativity, the second relation is called the braid relation. The far commutativity relation is shown in the top of Fig. 1. A geometrical interpretation of the braid relation is given in the bottom of Fig. 1. The presentation of the braid group B_n with generators σ_i and two types of relations is the algebraic expression of the fact that any isotopy of braids can be broken down into "elementary moves" of two types that correspond to two types of relations.

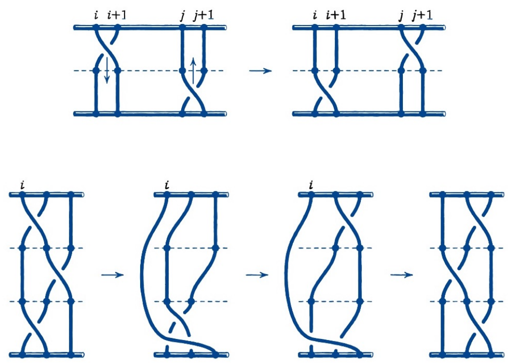

FIG. 1. The braid relations.

We recall the presentation for the monoid of singular braids on n strands.

The Baez–Birman monoid [2, 9] or *the singular braid monoid* SB_n is generated by the elements $\sigma_i^{\pm 1}$, τ_i, $i = 1, 2, \ldots, n-1$ (see Fig. 2) satisfying the following relations:

$$\sigma_i\, \sigma_i^{-1} = 1 \text{ for all } i,$$

$$\sigma_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \sigma_{i+1} \text{ and } \sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } j > i+1$$

$$\sigma_i\, \sigma_{i+1}\, \tau_i = \tau_{i+1}\, \sigma_i\, \sigma_{i+1},$$

$$\tau_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \tau_{i+1},$$

$$\tau_i\, \sigma_i = \sigma_i\, \tau_i \text{ for all } i,$$

$$\sigma_i\, \tau_j = \tau_j\, \sigma_i, \text{ for } j > i+1,$$

$$\sigma_j\, \tau_i = \tau_i\, \sigma_j, \text{ for } j > i+1,$$

$$\tau_i\, \tau_j = \tau_j\, \tau_i, \text{ for } j > i+1.$$

Geometrically the generators σ_i and τ_i are depicted in Figure 2. In pictures σ_i corresponds to canonical generator of the braid group and τ_i represents an intersection of the i-th and $(i+1)$th strand as in Figure 2. More detailed geometric interpretation of the Baez–Birman monoid can be found in the article of J. Birman [9]. In [15] it was proved that the singular braid monoid SB_n is embedded into the group SG_n which is called the *singular braid group* and has the same defining relations as SB_n.

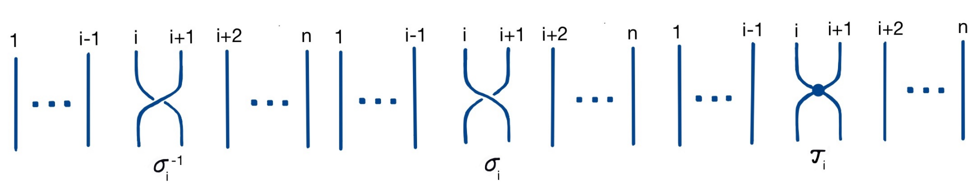

FIG. 2. The elementary braids σ_i^{-1} σ_i and τ_i .

The pure braid group P_n is the kernel of the homomorphism of B_n onto the symmetric group S_n on n symbols. This homomorphism maps σ_i to the transposition $(i, i+1)$, $i = 1, 2, \ldots, n-1$.

The group P_n is generated by a_{ij}, $1 \le i < j \le n$. These generators can be expressed by the generators of B_n as follows

$$a_{i,i+1} = \sigma_i^2,$$
$$a_{ij} = \sigma_{j-1}\,\sigma_{j-2}\ldots\sigma_{i+1}\,\sigma_i^2\,\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\,\sigma_{j-1}^{-1}, \quad i+1<j\le n.$$

In these generators P_n is defined by relations

$$a_{ik}a_{ij}a_{kj} = a_{kj}a_{ik}a_{ij},$$
$$a_{mj}a_{km}a_{kj} = a_{kj}a_{mj}a_{km}, \text{ for } m<j,$$
$$(a_{km}a_{kj}a_{km}^{-1})a_{im} = a_{im}(a_{km}a_{kj}a_{km}^{-1}), \text{ for } i<k<m<j,$$
$$a_{kj}a_{im} = a_{im}a_{kj}, \text{ for } k<i<m<j \text{ or } m<k.$$

The subgroup P_n is normal in B_n, and the quotient B_n/P_n is the symmetric group S_n. The generators of B_n act on the generator $a_{ij} \in P_n$ by the rules:

$$\sigma_k^{-1}a_{ij}\sigma_k = a_{ij}, \text{ for } k \ne i-1, i, j-1, j,$$
$$\sigma_i^{-1}a_{i,i+1}\sigma_i = a_{i,i+1},$$
$$\sigma_{i-1}^{-1}a_{ij}\sigma_{i-1} = a_{i-1,j},$$
$$\sigma_i^{-1}a_{ij}\sigma_i = a_{i+1,j}[a_{i,i+1}^{-1}, a_{ij}^{-1}], \text{ for } j \ne i+1$$
$$\sigma_{j-1}^{-1}a_{ij}\sigma_{j-1} = a_{i,j-1},$$
$$\sigma_j^{-1}a_{ij}\sigma_j = a_{ij}a_{i,j+1}a_{ij}^{-1},$$

where $[a,b] = a^{-1}b^{-1}ab = a^{-1}a^b$.

Denote by

$$U_i = \langle a_{1i}, a_{2i}, \ldots, a_{i-1,i}\rangle, \quad i = 2, \ldots, n,$$

a subgroup of P_n. It is known that U_i is a free group of rank $i-1$. One can rewrite the relations of P_n as the following conjugation rules (for $\varepsilon = \pm 1$):

$$a_{ik}^{-\varepsilon}a_{kj}a_{ik}^{\varepsilon} = (a_{ij}a_{kj})^{\varepsilon}a_{kj}(a_{ij}a_{kj})^{-\varepsilon},$$
$$a_{km}^{-\varepsilon}a_{kj}a_{km}^{\varepsilon} = (a_{kj}a_{mj})^{\varepsilon}a_{kj}(a_{kj}a_{mj})^{-\varepsilon}, \text{ for } m<j,$$
$$a_{im}^{-\varepsilon}a_{kj}a_{im}^{\varepsilon} = [a_{ij}^{-\varepsilon}, a_{mj}^{-\varepsilon}]^{\varepsilon}a_{kj}[a_{ij}^{-\varepsilon}, a_{mj}^{-\varepsilon}]^{-\varepsilon}, \text{ for } i<k<m,$$
$$a_{im}^{-\varepsilon}a_{kj}a_{im}^{\varepsilon} = a_{kj}, \text{ for } k<i<m<j \text{ or } m<k.$$

The group P_n is the semi–direct product of the normal subgroup U_n and the group P_{n-1}. Similarly, P_{n-1} is the semi–direct product of the free group U_{n-1} and the group P_{n-2}, and so on. Therefore, P_n is decomposable (see [20]) into the following semi–direct product

$$P_n = U_n \rtimes (U_{n-1} \rtimes (\ldots \rtimes (U_3 \rtimes U_2))\ldots), \quad U_i \simeq F_{i-1}, \quad i = 2, 3, \ldots, n.$$

Define the map

$$\pi : SG_n \longrightarrow S_n$$

of SG_n onto the symmetric group S_n on n symbols by actions on the generators

$$\pi(\sigma_i) = \pi(\tau_i) = (i, i+1), \quad i = 1, 2, \ldots, n-1.$$

The kernel $\ker(\pi)$ of this map is called the *singular pure braid group* and denoted by SP_n. It is clear that SP_n is a normal subgroup of index $n!$ of SG_n and we have a short exact sequence

$$1 \to SP_n \to SG_n \to S_n \to 1.$$

To find a presentation of SP_4 its possible to use the Reidemeister–Schreier method (see, for example, [19, Ch. 2.2]).

Let $m_{kl} = \rho_{k-1}\,\rho_{k-2}\ldots\rho_l$ for $l < k$ and $m_{kl} = 1$ in other cases. Then the set

$$\Lambda_n = \left\{ \prod_{k=2}^{n} m_{k,j_k} | 1 \leq j_k \leq k \right\}$$

is a Schreier set of coset representatives of SP_n in SG_n.

Define the map $^{-} : SG_n \longrightarrow \Lambda_n$ which takes an element $w \in SG_n$ into the representative $\overline{w}$ from Λ_n. In this case the element $w\overline{w}^{-1}$ belongs to SP_n. By Theorem 2.7 from [19] the group SP_n is generated by

$$S_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1},$$

where λ runs over the set Λ_n and a runs over the set of generators of SG_n.

To find defining relations of SP_n we define a rewriting process τ. It allows us to rewrite a word which is written in the generators of SG_n and presents an element in SP_n as a word in the generators of SP_n. Let us associate to the reduced word

$$u = a_1^{\varepsilon_1}\, a_2^{\varepsilon_2} \ldots a_\nu^{\varepsilon_\nu}, \quad \varepsilon_l = \pm 1, \quad a_l \in \{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \tau_1, \tau_2, \ldots, \tau_{n-1}\},$$

the word

$$\tau(u) = S_{k_1,a_1}^{\varepsilon_1}\, S_{k_2,a_2}^{\varepsilon_2} \ldots S_{k_\nu,a_\nu}^{\varepsilon_\nu}$$

in the generators of SP_n, where k_j is a representative of the $(j-1)$th initial segment of the word u if $\varepsilon_j = 1$ and k_j is a representative of the jth initial segment of the word u if $\varepsilon_j = -1$.

By [19, Theorem 2.9], the group SP_n is defined by relations

$$r_{\mu,\lambda} = \tau(\lambda\, r_\mu\, \lambda^{-1}), \quad \lambda \in \Lambda_n,$$

where r_μ is the defining relation of SG_n.

The center for the braid group was given by W. Chow [11]. It was proved that $Z(P_n)$ is an infinite cyclic group that is generated by

$$\Delta_n = (\sigma_1\sigma_2\ldots\sigma_{n-1})^n = a_{12}(a_{13}a_{23})\ldots(a_{1n}a_{2n}\ldots a_{n-1,n}).$$

It was shown that $Z(B_n) \cong Z(SG_n)$ (see [14, 24]). M. Neshchadim [21, 22] proved that $Z(P_n)$ is a direct factor in P_n but it is not a direct factor in B_n. By results of V. Bardakov and T. Kozlovskaya [6] the center $Z(SG_3)$ of SP_3 is a direct factor in SP_3.

As was shown in [6] the group SG_3 is generated by elements

$$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, b_{23},$$

and is defined by relations ($\varepsilon = \pm 1$):

$$a_{12}^{-\varepsilon} a_{23} a_{12}^{\varepsilon} = (a_{13}a_{23})^{\varepsilon} a_{23} (a_{13}a_{23})^{-\varepsilon},$$
$$a_{12}^{-\varepsilon} b_{23} a_{12}^{\varepsilon} = (a_{13}a_{23})^{\varepsilon} b_{23} (a_{13}a_{23})^{-\varepsilon},$$
$$a_{12}^{-\varepsilon} a_{13} a_{12}^{\varepsilon} = (a_{13}a_{23})^{\varepsilon} a_{13} (a_{13}a_{23})^{-\varepsilon},$$
$$a_{12}^{-\varepsilon} b_{13} a_{12}^{\varepsilon} = (a_{13}a_{23})^{\varepsilon} b_{13} (a_{13}a_{23})^{-\varepsilon},$$

$$[a_{12},\ b_{12}] = [a_{13},\ b_{13}] = [a_{23},\ b_{23}] = 1.$$

$$b_{12}^{-\varepsilon}(a_{13}a_{23}) b_{12}^{\varepsilon} = a_{13}a_{23}.$$

In fact SP_3 is generated by P_3 and the subgroup $TP_3 = \langle b_{12}, b_{13}, b_{23} \rangle$.

3. Presentation of SP_4

To find a set of generators and defining relations of SP_4 on can use the Reidemeister-Shraier method (see [19] Paragraph 2.3). For SP_3 it was done in [6]. To simplify the calculations we will use the same idea as in [7], where was found a presentation for the virtual pure braid group. Using this idea we prove the main result of the present article.

Theorem 1. *The singular pure braid group SP_4, on 4 strands is generated by elements $a_{12}, a_{13}, a_{23}, a_{14}, a_{24}, a_{34}, b_{12}, b_{13}, b_{23}, b_{14}, b_{24}, b_{34}$ is defined by relations:*

– *commutativity relations:*

(1) $$a_{12}b_{12} = b_{12}a_{12},$$

(2) $$a_{13}b_{13} = b_{13}a_{13},$$

(3) $$a_{23}b_{23} = b_{23}a_{23},$$

(4) $$a_{14}b_{14} = b_{14}a_{14},$$

(5) $$a_{24}b_{24} = b_{24}a_{24},$$

(6) $$a_{34}b_{34} = b_{34}a_{34},$$

– *conjugation by a_{12}:*

(7) $$a_{12}^{-1}a_{13}a_{12} = a_{13}a_{23}a_{13}a_{23}^{-1}a_{13}^{-1},$$

(8) $$a_{12}^{-1}b_{13}a_{12} = a_{13}a_{23}b_{13}a_{23}^{-1}a_{13}^{-1},$$

(9) $$a_{12}^{-1}a_{23}a_{12} = a_{13}a_{23}a_{13}^{-1},$$

(10) $$a_{12}^{-1}b_{23}a_{12} = a_{13}b_{23}a_{13}^{-1},$$

(11) $$a_{12}^{-1}a_{14}a_{12} = a_{14}a_{24}a_{14}a_{24}^{-1}a_{14}^{-1},$$

(12) $$a_{12}^{-1}b_{14}a_{12} = a_{14}a_{24}b_{14}a_{24}^{-1}a_{14}^{-1},$$

(13) $$a_{12}^{-1}a_{24}a_{12} = a_{14}a_{24}a_{14}^{-1},$$

(14) $$a_{12}^{-1}b_{24}a_{12} = a_{14}b_{24}a_{14}^{-1},$$

(15) $$a_{12}^{-1}a_{34}a_{12} = a_{34},$$

(16) $$a_{12}^{-1}b_{34}a_{12} = b_{34},$$

– *conjugation by a_{13}:*

(17) $$a_{13}^{-1}a_{14}a_{13} = a_{14}a_{34}a_{14}a_{34}^{-1}a_{14}^{-1},$$

$$a_{13}^{-1} b_{14} a_{13} = a_{14} a_{34} b_{14} a_{34}^{-1} a_{14}^{-1}, \tag{18}$$

$$a_{13}^{-1} a_{24} a_{13} = [a_{14}^{-1}, a_{34}^{-1}] a_{24} [a_{34}^{-1}, a_{14}^{-1}], \tag{19}$$

$$a_{13}^{-1} b_{24} a_{13} = [a_{14}^{-1}, a_{34}^{-1}] b_{24} [a_{34}^{-1}, a_{14}^{-1}], \tag{20}$$

$$a_{13}^{-1} a_{34} a_{13} = a_{14} a_{34} a_{14}^{-1}, \tag{21}$$

$$a_{13}^{-1} b_{34} a_{13} = a_{14} b_{34} a_{14}^{-1}, \tag{22}$$

– *conjugation by a_{23}:*

$$a_{23}^{-1} a_{14} a_{23} = a_{14}, \tag{23}$$

$$a_{23}^{-1} b_{14} a_{23} = b_{14}, \tag{24}$$

$$a_{23}^{-1} a_{24} a_{23} = a_{24} a_{34} a_{24} a_{34}^{-1} a_{24}^{-1}, \tag{25}$$

$$a_{23}^{-1} b_{24} a_{23} = a_{24} a_{34} b_{24} a_{34}^{-1} a_{24}^{-1}, \tag{26}$$

$$a_{23}^{-1} a_{34} a_{23} = a_{24} a_{34} a_{24}^{-1}, \tag{27}$$

$$a_{23}^{-1} b_{34} a_{23} = a_{24} b_{34} a_{24}^{-1}, \tag{28}$$

– *conjugation by b_{12}:*

$$b_{12}^{-1} (a_{13} a_{23}) b_{12} = a_{13} a_{23}, \tag{29}$$

$$b_{12}^{-1} (a_{14} a_{24}) b_{12} = a_{14} a_{24}, \tag{30}$$

$$b_{12}^{-1} a_{34} b_{12} = a_{34}, \tag{31}$$

$$b_{12}^{-1} b_{34} b_{12} = b_{34}, \tag{32}$$

– *conjugation by b_{13}:*

$$b_{13}^{-1} (a_{14} a_{34}) b_{13} = a_{14} a_{34}, \tag{33}$$

$$b_{13}^{-1} \left(a_{34}^{-1} a_{24} a_{34} \right) b_{13} = a_{34}^{-1} a_{24} a_{34}, \tag{34}$$

$$b_{13}^{-1} \left(a_{34}^{-1} b_{24} a_{34} \right) b_{13} = a_{34}^{-1} b_{24} a_{34}. \tag{35}$$

– *conjugation by b_{23}:*

$$b_{23}^{-1} a_{14} b_{23} = a_{14}, \tag{36}$$

$$b_{23}^{-1} b_{14} b_{23} = b_{14}, \tag{37}$$

$$b_{23}^{-1} (a_{24} a_{34}) b_{23} = a_{24} a_{34}, \tag{38}$$

Proof. One can see that the relations (1)–(3), (7)–(10), and (29) of the theorem are relations of SP_3. We have to show that all other relations follows from the relations of SG_4. Let us denote SP_3 by V_0. Then

$$SP_3 = V_0 = \langle a_{12},\ a_{13},\ a_{23},\ b_{12},\ b_{13},\ b_{23}\rangle\,.$$

Conjugating SP_3 by σ_3^{-1}, $\sigma_3^{-1}\sigma_2^{-1}$, $\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}$, we get three subgroups of the singular pure braid group SP_4:

$$(SP_3)^{\sigma_3^{-1}} = V_1 = \langle a_{12},\ a_{14},\ a_{24},\ b_{12},\ b_{14},\ b_{24}\rangle\,,$$

$$(SP_3)^{\sigma_3^{-1}\sigma_2^{-1}} = V_2 = \langle a_{13},\ a_{14},\ a_{34},\ b_{13},\ b_{14},\ b_{34}\rangle\,,$$

$$(SP_3)^{\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}} = V_3 = \langle a_{23},\ a_{24},\ a_{34},\ b_{23},\ b_{24},\ b_{34}\rangle\,.$$

It is easy to see that

$$SP_4 = \langle V_0,\ V_1,\ V_2,\ V_3\rangle\,.$$

The group V_1 is defined by relations:

$$\begin{gathered}
a_{12}^{-\varepsilon}a_{14}a_{12}^{\varepsilon} = (a_{14}a_{24})^{\varepsilon}a_{14}(a_{14}a_{24})^{-\varepsilon},\\
a_{12}^{-\varepsilon}b_{14}a_{12}^{\varepsilon} = (a_{14}a_{24})^{\varepsilon}b_{14}(a_{14}a_{24})^{-\varepsilon},\\
a_{12}^{-\varepsilon}a_{24}a_{12}^{\varepsilon} = (a_{14}a_{24})^{\varepsilon}a_{24}(a_{14}a_{24})^{-\varepsilon},\\
a_{12}^{-\varepsilon}b_{24}a_{12}^{\varepsilon} = (a_{14}a_{24})^{\varepsilon}b_{24}(a_{14}a_{24})^{-\varepsilon},\\
[a_{12},\ b_{12}] = [a_{14},\ b_{14}] = [a_{24},\ b_{24}] = 1,\\
b_{12}^{-\varepsilon}(a_{14}a_{24})b_{12}^{\varepsilon} = a_{14}a_{24}.
\end{gathered}$$

It is easy to see that these relations coincide with the relations (11)-(14), (1), (4), (5) and (30) from the theorem, respectively.

The group V_2 is defined by relations:

$$\begin{gathered}
a_{13}^{-\varepsilon}a_{14}a_{13}^{\varepsilon} = (a_{14}a_{34})^{\varepsilon}a_{14}(a_{14}a_{34})^{-\varepsilon},\\
a_{13}^{-\varepsilon}b_{14}a_{13}^{\varepsilon} = (a_{14}a_{34})^{\varepsilon}b_{14}(a_{14}a_{34})^{-\varepsilon},\\
a_{13}^{-\varepsilon}a_{34}a_{13}^{\varepsilon} = (a_{14}a_{34})^{\varepsilon}a_{34}(a_{14}a_{34})^{-\varepsilon},\\
a_{13}^{-\varepsilon}b_{34}a_{13}^{\varepsilon} = (a_{14}a_{34})^{\varepsilon}b_{34}(a_{14}a_{34})^{-\varepsilon},\\
[a_{13},\ b_{13}] = [a_{14},\ b_{14}] = [a_{34},\ b_{34}] = 1,\\
b_{13}^{-\varepsilon}(a_{14}a_{34})b_{13}^{\varepsilon} = a_{14}a_{34}.
\end{gathered}$$

We note that it is the relations (17), (18), (21), (22), (2), (4), (6) and (33) (see the theorem.

The group V_3 is defined by relations:

$$\begin{gathered}
a_{23}^{-\varepsilon}a_{24}a_{23}^{\varepsilon} = (a_{24}a_{34})^{\varepsilon}a_{24}(a_{24}a_{34})^{-\varepsilon},\\
a_{23}^{-\varepsilon}b_{24}a_{23}^{\varepsilon} = (a_{24}a_{34})^{\varepsilon}b_{24}(a_{24}a_{34})^{-\varepsilon},\\
a_{23}^{-\varepsilon}a_{34}a_{23}^{\varepsilon} = (a_{24}a_{34})^{\varepsilon}a_{34}(a_{24}a_{34})^{-\varepsilon},\\
a_{23}^{-\varepsilon}b_{34}a_{23}^{\varepsilon} = (a_{24}a_{34})^{\varepsilon}b_{34}(a_{24}a_{34})^{-\varepsilon},\\
[a_{23},\ b_{23}] = [a_{24},\ b_{24}] = [a_{34},\ b_{34}] = 1,\\
b_{23}^{-\varepsilon}(a_{24}a_{34})b_{23}^{\varepsilon} = a_{24}a_{34}.
\end{gathered}$$

These relations are the same as the relations (25)-(28), (3), (5), (6) and (38) from the theorem.

We see that the subgroups V_0, V_1, V_2, V_3 contains relations which follow from long relations of SG_4 and relations of the form $\sigma_i \tau_i = \tau_i \sigma_i$, $i = 1, 2, 3$. But this subgroups do not contain relations which follow from the far commutativity relations of SG_4 since SG_3 dose not contain these relations. Further we will analyze far commutativity relations of SG_4 and using Reidemeister-Shraier method find the corresponding relations of SP_4.

Lemma 1. *From the relation*

$$\sigma_3^{-1} \sigma_1^{-1} \sigma_3 \sigma_1 = 1$$

follow the relations

$$a_{12} a_{34} = a_{34} a_{12}, \quad a_{13}^{-1} a_{24} a_{13} = \left[a_{14}^{-1},\ a_{34}^{-1}\right] a_{24} \left[a_{34}^{-1},\ a_{14}^{-1}\right], \quad a_{23}^{-1} a_{14} a_{23} = a_{14},$$

that are relations (15), (19), and (23) of the theorem.

Proof. The relation

$$\sigma_3^{-1} \sigma_1^{-1} \sigma_3 \sigma_1 = 1$$

can be presented in the form

$$S_{\sigma_3, \sigma_3}^{-1} S_{\sigma_1 \sigma_3, \sigma_1}^{-1} S_{\sigma_1 \sigma_3, \sigma_3} S_{\sigma_1, \sigma_1} = 1.$$

Since

$$S_{\sigma_1, \sigma_1} = S_{\sigma_1 \sigma_3, \sigma_1} = a_{12}, \quad S_{\sigma_3, \sigma_3} = S_{\sigma_1 \sigma_3, \sigma_3} = a_{34},$$

we get the relation $a_{12} a_{34} = a_{34} a_{12}$.

Write the relation in the form

$$a_{12}^{-1} a_{34} a_{12} = a_{34}.$$

Conjugating this relation by σ_2^{-1}, we get

$$a_{13}^{-1} a_{34}^{-1} a_{24} a_{34} a_{13} = a_{34}^{-1} a_{24} a_{34}$$

Take the relation (21)

$$a_{13}^{-1} a_{34} a_{13} = a_{14} a_{34} a_{14}^{-1}.$$

Then

$$a_{14} a_{34}^{-1} a_{14}^{-1} a_{24}^{a_{13}} a_{14} a_{34} a_{14}^{-1} = a_{34}^{-1} a_{24} a_{34}.$$

We have

$$a_{13}^{-1} a_{24} a_{13} = a_{14} a_{34} a_{14}^{-1} a_{34}^{-1} a_{24} a_{34} a_{14} a_{34}^{-1} a_{14}^{-1}.$$

From the previous relation

$$a_{13}^{-1} a_{24} a_{13} = \left[a_{14}^{-1},\ a_{34}^{-1}\right] a_{24} \left[a_{34}^{-1},\ a_{14}^{-1}\right].$$

Conjugating this relation by σ_1^{-1}, we get

$$a_{23}^{-1} a_{24}^{-1} a_{14} a_{24} a_{23} = \left[a_{24}^{-1},\ a_{34}^{-1}\right] a_{24}^{-1} a_{14} a_{24} \left[a_{34}^{-1},\ a_{24}^{-1}\right].$$

Using the relation $a_{23}^{-1} a_{24} a_{23} = a_{24} a_{34} a_{24} a_{34}^{-1} a_{24}^{-1}$ which holds in V_3, we obtain relation

$$a_{23}^{-1} a_{14} a_{23} = a_{14}.$$

Conjugating $[a_{12}, a_{34}] = 1$ by other representatives of Λ_4 one can check that we did not find new relations. ☐

Lemma 2. *From the relation*

$$\sigma_1^{-1}\tau_3^{-1}\sigma_1\tau_3 = 1$$

follow the relations

$$a_{12}b_{34} = b_{34}a_{12}, \quad b_{24}^{a_{13}} = [a_{14}^{-1}, a_{34}^{-1}]b_{24}[a_{14}^{-1}, a_{34}^{-1}]^{-1}, \; a_{23}^{-1}b_{14}a_{23} = b_{14}$$

that are relations (16), (20) and (24) of the theorem.

Proof. The relation

$$\sigma_1^{-1}\tau_3^{-1}\sigma_1\tau_3 = 1$$

can be presented in the form

$$S_{\sigma_1,\sigma_1}^{-1}S_{\sigma_1\sigma_3,\tau_3}^{-1}S_{\sigma_1\sigma_3,\sigma_1}S_{\sigma_3,\tau_3} = 1.$$

Since

$$S_{\sigma_3,\tau_3} = S_{\sigma_1\sigma_3,\tau_3} = b_{34},$$

we get the relation $a_{12}b_{34} = b_{34}a_{12}$.

Conjugating this relation by σ_2^{-1} we get

$$\left(\sigma_1^{-1}\tau_3^{-1}\sigma_1\tau_3\right)^{\sigma_2^{-1}} = 1.$$

From this relation follows the relation

$$\left(a_{34}^{-1}b_{24}a_{34}\right)^{a_{13}} = a_{34}^{-1}b_{24}a_{34},$$

which gives

$$b_{24}^{a_{13}} = a_{34}^{a_{13}}a_{34}^{-1}b_{24}a_{34}a_{34}^{-a_{13}}.$$

Since in P_4 we have the relation (21)

$$a_{34}^{a_{13}} = a_{34}^{a_{14}^{-1}},$$

then we get the relation

$$b_{24}^{a_{13}} = [a_{14}^{-1}, a_{34}^{-1}]b_{24}[a_{14}^{-1}, a_{34}^{-1}]^{-1}.$$

Conjugating this relation by σ_1^{-1}, we have

$$a_{23}^{-1}a_{24}^{-1}b_{14}a_{24}a_{23} = \left[a_{24}^{-1},\ a_{34}^{-1}\right]a_{24}^{-1}b_{14}a_{24}\left[a_{34}^{-1},\ a_{24}^{-1}\right]$$

Using the relation $a_{23}^{-1}a_{24}a_{23} = a_{24}a_{34}a_{24}a_{34}^{-1}a_{24}^{-1}$ for the group V_3, we get the relation

$$a_{23}^{-1}b_{14}a_{23} = b_{14}.$$

Conjugating $[a_{12}, b_{34}] = 1$ by other representatives of Λ_4 one can check that we did not find new relations. ☐

Lemma 3. *From the relation*

$$\sigma_3^{-1}\tau_1^{-1}\sigma_3\tau_1 = 1$$

follows the relations

$$a_{34}b_{12} = b_{12}a_{34},\ b_{13}^{-1}\left(a_{34}^{-1}a_{24}a_{34}\right)b_{13} = a_{34}^{-1}a_{24}a_{34},\ b_{23}^{-1}a_{14}b_{23} = a_{14}.$$

that is the relation (31), (34) and (36) of the theorem.

Proof. We can rewrite the relation $\sigma_3^{-1}\tau_1^{-1}\sigma_3\tau_1 = 1$ in the form $a_{34}b_{12} = b_{12}a_{34}$.

Conjugating the relation $\sigma_3^{-1}\tau_1^{-1}\sigma_3\tau_1 = 1$ by σ_2^{-1}, we get

$$\left(a_{34}^{-1}a_{24}a_{34}\right)^{b_{13}} = a_{34}^{-1}a_{24}a_{34},$$

Conjugating the previous relation by σ_1^{-1}, we have

$$b_{23}^{-1}\left(a_{34}^{-1}a_{24}^{-1}a_{14}a_{24}a_{34}\right)b_{23} = a_{34}^{-1}a_{24}^{-1}a_{14}a_{24}a_{34}.$$

Take the relation

$$b_{23}^{-1}\left(a_{24}a_{34}\right)b_{23} = a_{24}a_{34}.$$

Then we obtain

$$b_{23}^{-1}a_{14}b_{23} = a_{14}.$$

Conjugating $[a_{34}, b_{12}] = 1$ by other representatives of Λ_4 one can check that we did not find new relations.

□

Lemma 4. *From the commutativity relation:*

$$\tau_1\tau_3 = \tau_3\tau_1$$

follow relations

$$b_{12}b_{34} = b_{34}b_{12},\ \left(a_{34}^{-1}b_{24}a_{34}\right)^{b_{13}} = a_{34}^{-1}b_{24}a_{34},\ b_{23}^{-1}b_{14}b_{23} = b_{14}.$$

that are relations (32), (35) and (37) of the theorem.

Proof. We can rewrite $\tau_1\tau_3 = \tau_3\tau_1$ in the form $b_{12}b_{34} = b_{34}b_{12}$.

Conjugating it by σ_2^{-1}, one can get the relation

$$\left(a_{34}^{-1}b_{24}a_{34}\right)^{b_{13}} = a_{34}^{-1}b_{24}a_{34}.$$

Conjugating the previous relation by σ_1^{-1}, we have

$$b_{23}^{-1}\left(a_{34}^{-1}a_{24}^{-1}b_{14}a_{24}a_{34}\right)b_{23} = a_{34}^{-1}a_{24}^{-1}b_{14}a_{24}a_{34}.$$

Take the relation

$$b_{23}^{-1}\left(a_{24}a_{34}\right)b_{23} = a_{24}a_{34}.$$

Then we obtain

$$b_{23}^{-1}b_{14}b_{23} = b_{14}.$$

Conjugating $[b_{12}, b_{34}] = 1$ by other representatives of Λ_4 one can check that we did not find new relations. □

Hence, all relations in theorem follows from the relations of singular braid group on 4 strands.

□

4. Center of SG_4

It is well-known [20] that the center $Z(B_n) = Z(P_n)$ is infinite cyclic group that is generated by

$$\Delta_n = (\sigma_1\sigma_2 \dots \sigma_{n-1})^n = a_{12}(a_{13}a_{23}) \dots (a_{1n}a_{2n} \dots a_{n-1,n}).$$

It was shown that $Z(B_n) \cong Z(SG_n) \cong Z(SP_n)$ (see [14, 24]). M. V. Neshchadim [21, 22] proved that $Z(P_n)$ is a direct factor in P_n, but its not a direct factor of B_n. In [6] was proved that $Z(SP_3)$ is a direct factor in SP_3. In this section we prove the same result for $n = 4$. We will use the following notations

$$\delta_k = a_{1k}a_{2k} \dots a_{k-1,k}, \quad k = 2, 3, 4.$$

Then

$$\Delta_4 = \delta_2\delta_3\delta_4.$$

Using defining relations of SP_4 can be proved

Lemma 5. *The following formulas hold in SP_4*

$$a_{14}^{a_{24}^{-1}a_{14}^{-1}\delta_3\delta_4} = a_{14}, \quad b_{14}^{a_{24}^{-1}a_{14}^{-1}\delta_3\delta_4} = b_{14}, \tag{39}$$

$$a_{24}^{a_{14}^{-1}\delta_3\delta_4} = a_{24}, \quad b_{24}^{a_{14}^{-1}\delta_3\delta_4} = b_{24}. \tag{40}$$

Lemma 6. *For any $i \in \{1, 2\}$ the following formulas hold in SP_4*

$$(a_{14}a_{24}a_{34})^{a_{i3}} = a_{14}a_{24}a_{34},$$
$$(a_{14}a_{24}a_{34})^{b_{i3}} = a_{14}a_{24}a_{34}.$$

Now we are ready to prove

Theorem 2. *The center $Z(SG_4)$ is a direct factor in SP_4. But $Z(SG_4)$ is not a direct factor in SG_4.*

Proof. As we know SP_4 is generated by

$$a_{ij}, \quad b_{ij}, \quad 1 \le i < j \le 4,$$

and is defined by the set of relations R, which we have found in Theorem 1, i.e.

$$SP_4 = \langle a_{ij}, \quad b_{ij}, \quad 1 \le i < j \le 4, \mid R\rangle. \tag{41}$$

The set of relations R is disjoint union of two subsets, $R = R_1 \sqcup R_2$, where R_1 is the set of relations which contain a_{12} and R_2 is the set of relations which do not contain a_{12}. Denote by A the set of generators SP_4 without generator a_{12} and denote by $H = \langle A\rangle \le SP_4$.

Let us prove that SP_4 also has the following presentation

$$SP_4 = \langle A, \Delta_4 \mid R_2, [\Delta_4, a] = 1, a \in A\rangle. \tag{42}$$

It is enough to prove that any relation from R_1 follows from relations R_2 and relations $[\Delta_4, a] = 1$, $a \in A$. Any relation from R_1 has one of the forms:

1) $a_{13}^{a_{12}} = a_{13}^{a_{23}^{-1} a_{13}^{-1}} \qquad b_{13}^{a_{12}} = b_{13}^{a_{23}^{-1} a_{13}^{-1}};$

2) $a_{23}^{a_{12}} = a_{23}^{a_{13}^{-1}} \qquad b_{23}^{a_{12}} = b_{23}^{a_{13}^{-1}};$

3) $a_{14}^{a_{12}} = a_{14}^{a_{24}^{-1} a_{14}^{-1}} \qquad b_{14}^{a_{12}} = b_{14}^{a_{24}^{-1} a_{14}^{-1}};$

4) $a_{24}^{a_{12}} = a_{24}^{a_{14}^{-1}} \qquad b_{24}^{a_{12}} = b_{24}^{a_{14}^{-1}};$

5) $a_{34}^{a_{12}} = a_{34} \qquad b_{34}^{a_{12}} = b_{34},$

6) $b_{12}^{a_{12}} = b_{12}.$

Since $\Delta_4 = \delta_2\delta_3\delta_4$ and $\delta_2 = a_{12}$, then $a_{12} = \Delta_4 \delta_4^{-1} \delta_3^{-1}$. Using this formula, we can remove a_{12} from the generating set of SP_4. Hence, SP_4 is generated by A, Δ_n.

Let us show that we can remove the set of relations R_1 and insert the relations $[\Delta_4, a] = 1, a \in A$. The first relation of the type 1) can be write in the form

$$a_{13}^{\Delta_4 \delta_4^{-1} \delta_3^{-1}} = a_{13}^{a_{23}^{-1} a_{13}^{-1}}.$$

The element Δ_4 lies in the center of SP_4, hence

$$a_{13}^{\delta_4^{-1}} = a_{13}^{a_{23}^{-1} a_{13}^{-1} \delta_3}.$$

Since $\delta_3 = a_{13} a_{23}$, this relation is equivalent

$$a_{13}^{\delta_4^{-1}} = a_{13} \Leftrightarrow \delta_4^{a_{13}} = \delta_4.$$

The last relation is the first relation of of Lemma 6. So, we can remove the first relation of the form 1).

By the same way, we can show that we can remove the second relation of the form 1).

The relation of the type 2) can be rewrite as

$$a_{23}^{\Delta_4 \delta_4^{-1} \delta_3^{-1}} = a_{23}^{a_{13}^{-1}}.$$

The element Δ_4 lies in the center of SP_4, consequently

$$a_{23}^{\delta_4^{-1}} = a_{23}^{a_{13}^{-1} \delta_3}.$$

This relation is equivalent

$$a_{23}^{\delta_4^{-1}} = a_{23}^{a_{23}} \Leftrightarrow \delta_4^{a_{23}} = \delta_4.$$

The last relation is the first relation of of Lemma 6. Hence, we can remove the first relation of the form 2).

Analogously, one can consider the second relation of the form 2).

3) We can write this relation in the form

$$a_{14}^{\Delta_4 \delta_4^{-1} \delta_3^{-1}} = a_{14}^{a_{24}^{-1} a_{14}^{-1}}.$$

The element Δ_4 lies in the center of SP_4, hence

$$a_{14}^{\delta_4^{-1}} = a_{14}^{a_{24}^{-1} a_{14}^{-1} \delta_3}.$$

From the first relation of Lemma 5 follows that the left side of this relation is equal to a_{14}. From the first relation of Lemma 6 follows that the right side of this relation is equal to a_{14}. Hence, we can remove the first relation of the form 3). By the same way, using the second relation of Lemma 5 and the second relation of Lemma 6 we can show that we can remove the second relation of the form 3).

The relations of the form 4) is considered by analogy.

5) We can write this relation in the form

$$a_{34}^{\Delta_4 \delta_4^{-1} \delta_3^{-1}} = a_{34}.$$

The element Δ_4 lies in the center of SP_4, then

$$a_{34}^{\delta_4^{-1}} = a_{34}^{\delta_3}.$$

The last relation is equivalent

$$a_{34}^{a_{34}^{-1} a_{24}^{-1} a_{14}^{-1}} = a_{34}^{a_{13} a_{23}}.$$

Using the following relations from Theorem 1:

$$a_{34}^{a_{13}} = a_{34}^{a_{14}^{-1}}, \ a_{34}^{a_{23}} = a_{34}^{a_{24}^{-1}}, \ a_{14}^{a_{23}} = a_{14},$$

we get

$$a_{34}^{a_{13} a_{23}} = \left(a_{34}^{a_{14}^{-1}} \right)^{a_{23}} = a_{34}^{a_{24}^{-1} a_{14}^{-1}}.$$

Therefore, 5) follows from the relation $[\Delta_4, a_{34}] = 1$ and the relations which do not contain a_{12}.

6) We can write this relation in the form

$$b_{12}^{\Delta_4 \delta_4^{-1} \delta_3^{-1}} = b_{12}.$$

The element Δ_4 lies in the center of SP_4, then

$$b_{12}^{\delta_4^{-1} \delta_3^{-1}} = b_{12}.$$

Using the conjugation rules by b_{12} one can check

$$\delta_4^{b_{12}} = \delta_4, \quad \delta_3^{b_{12}} = \delta_3,$$

i. e. 6) follows from the relation $[\Delta_4, b_{12}] = 1$ and the relations which do not contain a_{12}.

Hence, SP_4 has the presentation (42).

From this presentation follows that there are two epimorphisms

$$\pi_1 : SP_4 \to Z(SG_4), \quad \pi_1(\Delta_4) = \Delta_4, \quad \pi_1(a) = 1 \text{ for all } a \in A;$$

$$\pi_2 : SP_4 \to H, \quad \pi_2(\Delta_4) = 1, \quad \pi_2(a) = a \text{ for all } a \in A.$$

Hence, $SP_4 = \langle Z(SG_4), H \rangle$, the subgroup H has a presentation

$$H = \langle A \mid R_2 \rangle$$

and $Z(SG_4) \cap H = 1$. We proved the first part of the theorem.

The second part of the theorem follows from the fact that there exists an epimorphism $SG_4 \to B_4$ and from the fact that $Z(B_4)$ is not a direct factor of B_4. $\square$

From this theorem we get other presentation for SP_4.

Corollary 1. *The singular pure braid group SP_4, on 4 strands is generated by elements $\Delta_4, a_{13}, a_{23}, a_{14}, a_{24}, a_{34}, b_{12}, b_{13}, b_{23}, b_{14}, b_{24}, b_{34}$ is defined by relations:*

$$\Delta_4 c = c\Delta_4, \quad c \in \{a_{13}, a_{23}, a_{14}, a_{24}, a_{34}, b_{12}, b_{13}, b_{23}, b_{14}, b_{24}, b_{34}\},$$

– *commutativity relations:*

$$\begin{aligned} a_{13}b_{13} &= b_{13}a_{13}, \\ a_{23}b_{23} &= b_{23}a_{23}, \\ a_{14}b_{14} &= b_{14}a_{14}, \\ a_{24}b_{24} &= b_{24}a_{24}, \\ a_{34}b_{34} &= b_{34}a_{34}, \end{aligned}$$

– *conjugation by a_{13}:*

$$\begin{aligned} a_{13}^{-1}a_{14}a_{13} &= a_{14}a_{34}a_{14}a_{34}^{-1}a_{14}^{-1}, \\ a_{13}^{-1}b_{14}a_{13} &= a_{14}a_{34}b_{14}a_{34}^{-1}a_{14}^{-1}, \\ a_{13}^{-1}a_{24}a_{13} &= [a_{14}^{-1}, a_{34}^{-1}]a_{24}[a_{34}^{-1}, a_{14}^{-1}], \\ a_{13}^{-1}b_{24}a_{13} &= [a_{14}^{-1}, a_{34}^{-1}]b_{24}[a_{34}^{-1}, a_{14}^{-1}], \\ a_{13}^{-1}a_{34}a_{13} &= a_{14}a_{34}a_{14}^{-1}, \\ a_{13}^{-1}b_{34}a_{13} &= a_{14}b_{34}a_{14}^{-1}, \end{aligned}$$

– *conjugation by a_{23}:*

$$\begin{aligned} a_{23}^{-1}a_{14}a_{23} &= a_{14}, \\ a_{23}^{-1}b_{14}a_{23} &= b_{14}, \\ a_{23}^{-1}a_{24}a_{23} &= a_{24}a_{34}a_{24}a_{34}^{-1}a_{24}^{-1}, \\ a_{23}^{-1}b_{24}a_{23} &= a_{24}a_{34}b_{24}a_{34}^{-1}a_{24}^{-1}, \\ a_{23}^{-1}a_{34}a_{23} &= a_{24}a_{34}a_{24}^{-1}, \\ a_{23}^{-1}b_{34}a_{23} &= a_{24}b_{34}a_{24}^{-1}, \end{aligned}$$

– *conjugation by b_{12}:*

$$\begin{aligned} b_{12}^{-1}(a_{13}a_{23})b_{12} &= a_{13}a_{23}, \\ b_{12}^{-1}(a_{14}a_{24})b_{12} &= a_{14}a_{24}, \\ b_{12}^{-1}a_{34}b_{12} &= a_{34}, \\ b_{12}^{-1}b_{34}b_{12} &= b_{34}, \end{aligned}$$

– *conjugation by b_{13}:*

$$\begin{aligned} b_{13}^{-1}(a_{14}a_{34})b_{13} &= a_{14}a_{34}, \\ b_{13}^{-1}\left(a_{34}^{-1}a_{24}a_{34}\right)b_{13} &= a_{34}^{-1}a_{24}a_{34}, \\ b_{13}^{-1}\left(a_{34}^{-1}b_{24}a_{34}\right)b_{13} &= a_{34}^{-1}b_{24}a_{34}. \end{aligned}$$

– *conjugation by b_{23}:*

$$\begin{aligned} b_{23}^{-1}a_{14}b_{23} &= a_{14}, \\ b_{23}^{-1}b_{14}b_{23} &= b_{14}, \\ b_{23}^{-1}(a_{24}a_{34})b_{23} &= a_{24}a_{34}, \end{aligned}$$

References

[1] E. Artin, *Theory of braids*, Ann. Math. (2), **48** 1947, 101–126. Zbl 0030.17703

[2] J.C. Baez, *Link invariants of finite type and perturbation theory*, Lett. Math. Phys., **26**:1 (1992), 43–51. Zbl 0792.57002

[3] V.G. Bardakov, P. Bellingeri, *Combinatorial properties of virtual braids*, Topology Appl., **156**:6 (2009), 1071–1082. Zbl 1196.20044

[4] V.G. Bardakov, S. Jablan, H. Wang, *Monoid and group of pseudo braids*, J. Knot Theory Ramifications, **25**:9 (2016), Article ID 1641002. Zbl 1396.20033

[5] V.G. Bardakov, R. Mikhailov, V.V. Vershinin, J Wu, *On the pure virtual braid group* PV_3. Commun. Algebra, **44**:3 (2016), 1350–1378. Zbl 1344.20048

[6] V.G. Bardakov, T.A. Kozlovskaya, *On 3-strand singular pure braid group*, J. Knot Theory Ramifications, **29**:10 (2020), Article ID 2042001. Zbl 7291634

[7] V.G. Bardakov, J. Wu, *Lifting theorem for the virtual pure braid groups*, arXiv:2002.08686, (2020).

[8] J.S. Birman, *Braids, links and mapping class group*, Annals of Mathematics Studies, **82**, Princeton University Press and University of Tokyo Press, Princeton, 1975. Zbl 0305.57013

[9] J.S. Birman, *New points of view in knot theory*, Bull. Am. Math. Soc., New Ser., **28**:2 (1993), 253–287. Zbl 0785.57001

[10] R. Corran, *A normal form for a class of monoids including the singular braid monoid*, J. Algebra, **223**:1 (2000), 256–282. Zbl 0971.20035

[11] W.-L. Chow, *On the algebraic braid group*, Ann. Math. (2), **49**:3 (1948), 654-658. Zbl 0033.01002

[12] O.T. Dasbach, B. Gemein, *The word problem for the singular braid monoid*, arXiv:math/9809070, (1998).

[13] O.T. Dasbach, B. Gemein, *A faithful representation of the singular braid monoid on three strands*, arXiv:math/9806050, (1998).

[14] R. Fenn, D. Rolfsen, J. Zhu, *Centralizers in the braid group and singular braid monoid.* Enseign. Math., II. Sér., **42**:1-2 (1996), 75–96. Zbl 0869.20024

[15] R. Fenn, E. Keyman, C. Rourke, *The singular braid monoid embeds in a group*, J. Knot Theory Ramifications, **7**:7 (1998), 881–892. Zbl 0971.57011

[16] K. Gongopadhyay, T. Kozlovskaya, O. Mamonov, *On some decompositions of the 3-strand singular braid group*, Topology Appl., **283** (2020), Article ID 107398. Zbl 7285222

[17] A. Járai, *On the monoid of singular braids*, Topology Appl., **96**:2 (1999) 109–119. Zbl 0939.57010

[18] C. Kassel, V. Turaev, *Braid groups*, Graduate Texts in Mathematics, **247**, Springer, New York, 2008. Zbl 1208.20041

[19] W. Magnus, A. Karrass, D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers, New York etc., 1966. Zbl 0138.25604

[20] A.A. Markoff, *Foundations of the algebraic theory of braids*, Tr. Mat. Inst. Steklova, **16**, 1945. Zbl 0061.02507

[21] M.V. Neshchadim, *Inner automorphisms and some their generalizations.* Sib. Èlektron. Mat. Izv., **13** (2016), 1383–1400. Zbl 1385.20013

[22] M.V. Neshchadim, *Normal automorphisms of braid groups.* Preprint, **4**, Institute of mathimatics SB RAN, Novosibirsk, 1993.

[23] L. Paris, *Braid groups and Artin groups*, In Papadopoulos, Athanase (ed.), *Handbook of Teichmuller theory. Vol. II*, IRMA Lectures in Mathematics and Theoretical Physics, **13**, 389–451, Eur. Math. Soc., Zurich, 2009. Zbl 1230.20040

[24] V.V. Vershinin, *On the singular braid monoid*, St. Petersbg. Math. J., **21**:5 (2010), 693–704. Zbl 1201.20056

Tatyana Anatolevna Kozlovskaya
Regional Scientific and Educational Mathematical Center
of Tomsk State University,
36, Lenin ave.,
Tomsk, 634050, Russia
Email address: t.kozlovskaya@math.tsu.ru