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GRÖBNER–SHIRSHOV BASIS AND HOCHSCHILD COHOMOLOGY OF THE GROUP Γ_5^4

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Abstract. In this paper, we construct a Gröbner—Shirshov basis for the group Γ_5^4 with respect to the tower order on the words. By using this result, we apply the discrete algebraic Morse theory to find explicitly the first two differentials of the Anick resolution for Γ_5^4, and calculate the first and second Hochschild cohomology groups of the group algebra of Γ_5^4 with coefficients in the trivial 1-dimensional bimodule over a field $\Bbbk$ of characteristic zero.

Keywords: Gröbner—Shirshov basis, Anick resolution, Hochschild cohomology.

1. Introduction

V. O. Manturov et al. in [7]–[12] introduced a family of groups G_n^k depending on two positive integers $n > k$, and formulated the following principle: If a dynamical system describing a motion of n particles possesses a nice codimension one property governed by k particles, then it has topological invariants valued in the group G_n^k. Another series of groups Γ_n^4 corresponding to the motion of points in $\mathbb{R}^3$ was proposed [7, 13] from the point of view of Delaunay triangulations.

In terms of generators and defining relations the group Γ_n^4 is defined as follows. It is generated by the set

$$\{d_{(i,j,k,l)} \mid \{i,j,k,l\} \subseteq \{1,2,\ldots,n\},\ |\{i,j,k,l\}| = 4\}.$$

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subject to the following relations:

$$\begin{aligned}
&d^2_{(i,j,k,l)} = 1,\\
&d_{(i,j,k,l)}d_{(s,t,u,v)} = d_{(s,t,u,v)}d_{(i,j,k,l)}, \quad |\{i,j,k,l\} \cap \{s,t,u,v\}| < 3,\\
&d_{(i,j,k,l)}d_{(i,j,k,m)}d_{(i,j,l,m)}d_{(i,k,l,m)}d_{(j,k,l,m)} = 1,\\
&d_{(i,j,k,l)} = d_{(l,i,j,k)} = d_{(k,l,i,j)} = d_{(j,k,l,i)} =\\
&d_{(i,l,k,j)} = d_{(j,i,l,k)} = d_{(k,j,i,l)} = d_{(l,k,j,i)} \text{ for distinct } i,j,k,l.
\end{aligned}$$

for distinct i, j, k, l. In the papers [7, 13], the authors studied homomorphisms from the group of pure braids on n strands to the product of copies of Γ^4_n, and also studied the group of pure braids in $\mathbb{R}^3$, which is described by a fundamental group of the restricted configuration space of $\mathbb{R}^3$. They defined a group homomorphism from the group of pure braids in $\mathbb{R}^3$ to Γ^4_n, and gave some comments about relations between the restricted configuration space of $\mathbb{R}^3$ and triangulations of the 3-dimensional ball and Pachner moves. Therefore, it is useful to study these groups from combinatorial point if view.

This paper is devoted to finding Gröbner—Shirshov basis and cohomology of Γ^4_n for $n = 5$ (the first infinite example in the series). In [22], it was done for the first infinite example of the series G^k_n.

Gröbner basis and Gröbner—Shirshov basis were invented independently by A.I. Shirshov [19] for ideals of Lie algebras and, implicitly, associative algebras, by H. Hironaka [20] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [21] for ideals of the polynomial algebras. Gröbner basis and Gröbner—Shirshov basis theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra. It is a powerful tool to solve the following classical problems: normal form, word problem, conjugacy problem, rewriting system, embedding theorem, PBW theorem, growth function, (co)homology, etc., (see, e.g., [16] and references therein). It is well known that the first cohomology group of an associative algebra is related to its derivations, and the elements of the second one are interpreted as its null extensions. Homological algebra also introduces important numerical invariants in the group theory, e.g., (co)homological dimension and Euler characteristic. In some way, the (co)homology groups carry in-depth information about the properties of a group or an algebra, e.g., the property of a finite group to be nilpotent or solvable can be characterized in homological terms. So we can say that homological methods allow us to get important information about the structure of an algebra.

The most important cohomology theory for associative algebras is the Hochschild cohomology. Given an associative algebra A over a field $\Bbbk$ and an A-bimodule W, the set of Hochschild cochains $C^m(A, W)$ consists of all linear maps $A^{\otimes m} \to W$, $m \geq 0$, and the mth Hochschild cohomology group is defined as $H^m(A, W) = \operatorname{Ker} \Delta_m / \operatorname{Im} \Delta_{m-1}$, where $\Delta_m : C^m(A, W) \to C^{m+1}(A, W)$ is the Hochschild differential

$$(1) \quad \Delta_m f(r_1, \ldots, r_{m+1}) = r_1 f(r_2, \ldots, r_{m+1}) + \sum_{i=1}^{m} (-1)^i f(r_1, \ldots, r_i r_{i+1}, \ldots, r_{m+1}) + (-1)^{m+1} f(r_1, \ldots, r_m) r_{m+1}.$$

One of the most important features of the Hochschild cohomology groups is that the first one, $H^1(A, W)$, describes outer derivations from A to W and the second one, $H^2(A, W)$, is in one-to-one correspondence with the equivalence classes of null extensions [15]

$$0 \to W \to E \to A \to 0, \quad W^2 = 0.$$

It is often a difficult problem to find the groups $H^m(A, W)$ for a given algebra A and A-bimodule W. The first important step of solution is to find a long exact sequence of A-bimodules starting from A, a resolution of A. Next, one needs to calculate the derived functor to $\mathrm{Hom}(\cdot, W)$ to find the cohomology groups. The most natural resolution known as the bar-resolution is easy to define but it is not convenient for the calculation of cohomologies. A smaller resolution was proposed by D.J. Anick [1]: he found a way to construct a free resolution for an associative algebra which is homotopy equivalent to the bar-resolution. The Anick resolution found numerous applications in combinatorial algebra, see [2]. This resolution is more convenient for the calculation of cohomology groups, but the computation of differentials according to the original Anick algorithm described in [1] is extremely hard. In order to visualize the computation of differentials it is possible to use the discrete algebraic Morse theory [3, 4, 5, 6, 22] based on the concept of a Morse matching. In this paper, we choose a tower order on the free monoid generated by a well–ordered set X. By using this order, we find a Gröbner—Shirshov basis for the group Γ_5^4. Then we apply the discrete algebraic Morse theory to find explicitly the differentials δ_1 and δ_2 of the Anick resolution for Γ_5^4. As an application, we calculate the second Hochschild cohomology group of the group algebra of Γ_5^4 with coefficients in the trivial 1-dmensional $\Bbbk\Gamma_5^4$-bimodule M for char $\Bbbk = 0$. It turns out that there exist five inequivalent 1-dimensional null extensions of $\Bbbk\Gamma_5^4$ by means of the trivial module.

2. Composition–Diamond Lemma for Associative Algebras

Let $\Bbbk$ be a field, X be a nonempty set, $\Bbbk\langle X\rangle$ be the free associative algebra generated by X. Denote by X^* the free monoid generated by X, where the empty word is denoted by 1. Let $\deg(u)$ stand for the length (number of letters) of a word $u = x_1x_2\dots x_n \in X^*$, $x_i \in X$. In particular, $\deg(u) = 0$ means $u = 1$.

A well order $<$ on X^* is said to be monomial if $u > v$ implies $wu > wv$ and $uw > vw$ for all $u, v, w \in X^*$. Let us fix a monomial order $<$ on X^*. An arbitrary $f \in \Bbbk\langle X\rangle$ may be written as

$$f = \sum_{i=1}^{n} \alpha_i u_i,$$

where $\alpha_i \in \Bbbk \setminus \{0\}$, $u_i \in X^*$, $n \in \mathbb{N}$, and $u_1 > u_2 > \dots > u_n$. Define the leading monomial of f, denoted by $\overline{f}$, to be u_1. If $\alpha_1 = 1$, then we say f is a monic polynomial.

Let us recall the main statements of the Gröbner—Shirshov basis theory for associative algebras according to [16, 18, 19].

Definition 1. Let $f, g \in \Bbbk\langle X\rangle$ be monic polynomials. Define the following two kinds of *compositions*:

- If $w = \overline{f}b = a\overline{g} \in X^*$, where $a, b \in X^*$, $\deg(\overline{f}) + \deg(\overline{g}) > \deg(w)$, then $(f, g)_w = fb - ag$ is called an intersection composition of f and g with respect to w;
- If $w = \overline{f} = a\overline{g}b \in X^*$, $a, b \in X^*$, then $(f, g)_w = f - agb$ is called an inclusion composition of f and g with respect to w.

Let $S \subseteq \Bbbk\langle X\rangle$ be a nonempty set of polynomials. Given $f \in \Bbbk\langle X\rangle$, $w \in X^*$, if $f = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in \Bbbk$, $a_i, b_i \in X^*$, $s_i \in S$, and $a_i \overline{s_i} b_i < w$ with respect to the fixed monomial order, then f is said to be trivial modulo (S, w), it is denoted by

$$f \equiv 0 \pmod{(S, w)}.$$

Definition 2. A nonempty set $S \subset \Bbbk\langle X\rangle$ of monic polynomials is called a *Gröbner—Shirshov basis* in $\Bbbk\langle X\rangle$ if for every $f, g \in S$ having a composition we have $(f, g)_w \equiv 0$ modulo (S, w).

Theorem 1 (Composition–Diamond Lemma)**.** *Let $<$ be a monomial order on X^*, $S \subseteq \Bbbk\langle X\rangle$ be a nonempty set of monic polynomials, and let $\mathrm{Id}\,(S)$ be the ideal of $\Bbbk\langle X\rangle$ generated by S. Then the following statements are equivalent.*

(i) *S is a Gröbner—Shirshov basis in $\Bbbk\langle X\rangle$.*
(ii) *If $0 \neq f \in \mathrm{Id}\,(S)$ then $\overline{f} = a\overline{s}b$ for some $s \in S$, $a, b \in X^*$.*
(iii) *The set $\mathrm{Irr}\,(S) = \{z \in X^* \mid z \neq a\overline{s}b,\ a, b \in X^*,\ s \in S\}$ is a linear basis of the algebra $\Bbbk\langle X \mid S\rangle := \Bbbk\langle X\rangle/\mathrm{Id}\,(S)$.*

If a subset S of $\Bbbk\langle X\rangle$ is not a Gröbner—Shirshov basis then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner—Shirshov basis S^{comp}. Such a process is known as the Shirshov algorithm since [19]. If S is a set of semigroup (or binomial) relations (that is, polynomails of the form $u - v$, where $u, v \in X^*$), then every their composition has the same form. As a result, the set S^{comp} also consists of semigroup relations.

Let $M = \mathrm{sgp}\,\langle X \mid S\rangle := X^*/\rho(S)$ be a semigroup presentation, where $S \subset X^* \times X^*$ and $\rho(S)$ is the congruence on X^* generated by S. Then S may be considered is a set of semigroup relations in $\Bbbk\langle X\rangle$, and hence one can find the corresponding Gröbner—Shirshov basis S^{comp}. The latter does not depend on $\Bbbk$ and, as mentioned above, it consists of semigroup relations. We will call S^{comp} a Gröbner—Shirshov basis for M. This is the same as a Gröbner—Shirshov basis for the semigroup algebra $\Bbbk M = \Bbbk\langle X \mid S\rangle$.

The Gröbner—Shirshov basis of a given algebra highly depends on the set X of generators and on the chosen monomial order on X^*. In what follows, we will use the following one known as the *tower order*. Let X be a well-ordered set. Given a word $u \in X^*$, $u \neq 1$, denote by $\max(u)$ the largest letter occurring in u, and let n_u be the number of occurrences of $a = \max(u)$ in u. Then u may be uniquely presented as

$$u = u_0 a u_1 a u_2 \dots u_{n_u - 1} a u_{n_u}, \quad u_i \in X^*,\ 0 \leq i \leq n_u.$$

Here $\max(u_i) < \max(u)$ for every nonempty u_i.

For $u \in X^*$, $u \neq 1$, define the sequence

$$\mathrm{wt}\,(u) = (\max(u), n_u, u_0, \dots, u_{n_u}).$$

If $u, v \in X^*$ are two nonempty words then set $u > v$ if only and only if $\mathrm{wt}\,(u) > \mathrm{wt}\,(v)$ lexicographically, where $u_0, u_1, \dots, u_{n_u}$ are compared by induction. Such an order is called a *tower order* on X^* which is a monomial order.

3. Gröbner–Shirshov basis for Γ_5^4

Definition 3 ([13]). The group Γ_5^4 is generated (as a semigroup) by the set of elements

$$\begin{aligned}A_1 = \{&a = d_{(1,2,3,4)}, b = d_{(1,2,4,3)}, c = d_{(1,3,2,4)}, d = d_{(1,2,3,5)}, e = d_{(1,2,5,3)},\\ &f = d_{(1,3,2,5)}, h = d_{(1,2,4,5)}, g = d_{(1,2,5,4)}, k = d_{(1,4,2,5)}, l = d_{(1,3,4,5)},\\ &m = d_{(1,3,5,4)}, n = d_{(1,4,3,5)}, q = d_{(2,3,4,5)}, r = d_{(2,3,5,4)}, s = d_{(2,4,3,5)}\}.\end{aligned}$$

subject to the following relations:

$$\begin{aligned}S_1 = \{&x^2 = 1,\ x \in A_1,\\ &adhlq = qadhl = lqadh = hlqad = dhlqa = 1,\\ &bhdns = sbhdn = nsbhd = dnsbh = hdnsb = 1,\\ &cflhr = rcflh = hrcfl = lhrcf = flhrc = 1,\\ &dagmr = rdagm = mrdag = gmrda = agmrd = 1,\\ &egans = segan = nsega = anseg = ganse = 1,\\ &fcmgq = qfcmg = gqfcm = mgqfc = cmgqf = 1,\\ &hbemr = rhbem = mrhbe = emrhb = bemrh = 1,\\ &geblq = qgebl = lqgeb = blqge = eblqg = 1,\\ &kcmes = skcme = eskcm = meskc = cmesk = 1\}.\end{aligned}$$

Lemma 1. *Let*

$$A_2 = \{e, g, k, l, m, n, q, r, s\},$$

and

$$\begin{aligned}S_2 = \{&x^2 = 1,\ x \in A_2,\ ge = nsegns,\\ &gnse = esng, le = rnsemrnkmgqksrmgqlrm, lrnse = egqlskqgmknrm,\\ &kse = rnsemrnkm, krnse = semknrm, me = rnsemrns, mrnse = esnrm,\\ &qrnse = nsegnkqgmknrm, qnse = rnsemrnkmgqkng, lqg = rmgqlrm,\\ &lrmg = qgmrlq, gqlnse = mregqlsnrm, gmre = qlnsemrnsrmgqlrm\}.\end{aligned}$$

Then $\Gamma_5^4 = \mathrm{sgp}\,\langle A_2 \mid S_2\rangle$.

Proof. The group Γ_5^4 is indeed generated by A_2 since a, b, c, d, f, h may be expressed via A_2:

$$a = nseg,\ b = emrh,\ c = mesk,\ d = nsemr,\ f = cmgq,\ h = rmgql.$$

It remains to show that S_2 hold in Γ_5^4 and, conversely, that S_1 hold in $\mathrm{sgp}\,\langle A_2 \mid S_2\rangle$. For example,

$$1 = adhlq = anseg = agmrd.$$

Since $a^2 = d^2 = h^2 = 1$, then

$$\begin{aligned}&a = dhlq = nseg = gmrd, \quad nseg = gesn, \quad d = nsemr, \quad nsemr = rmesn,\\ &h = rmgql, \quad rmgql = lqgmr.\end{aligned}$$

So $ge = nsegns$, $gnse = esng$, $me = rnsemrns$, $lqg = rmgqlrm$, $lrmg = gmrl$. Conversely, let

$$a = nseg,\ d = nsemr,\ h = rmgql.$$

Then $a^2 = d^2 = h^2 = 1$ and

$$drm = nse, \quad hlq = rmg, \quad drmg = nseg, \quad dhlq = drmg,$$

$$a = dhlq = nseg = drmg, \quad 1 = adhlq = anseg = agmrd.$$

Similarly, we can get the rest of the relations. □

Theorem 2. *Let $<$ be the tower order on A_2^* based on $e > g > k > l > m > n > q > r > s$. Then*

$$S = S_2 \cup \{g(qlrm)^j qlnse - (mrlq)^j mregqlsnrm, g(qlrm)^j nse - (mrlq)^j esng,$$
$$g(mrlq)^j mre - (qlrm)^j qlnsemrnsrmgqlrm, g(mrlq)^j e - (qlrm)^j nsegns; j \geq 1\}$$

is a Gröbner–Shirshov basis for Γ_5^4.

Proof. Let us show that the last four series of relations in S hold in Γ_5^4. For example,

$$lrmg = qgmrlq \Rightarrow gmrlq = qlrmg \Rightarrow gmrlqmre = qlrm(gmre)$$
$$\Rightarrow gmrlqmre = qlrmqlnsemrnsrmgqlrm.$$

By induction, assume that

$$g(mrlq)^{j-1} mre = (qlrm)^{j-1} qlnsemrnsrmgqlrm,$$

then

$$\begin{aligned}
g(mrlq)^j mre &= g(mrlq)^{j-1} mrlqmre \\
&= g(mrlq)^{j-1} mreelqmre \\
&= (qlrm)^{j-1} qlnsemrnsrmgqlr(me)lqmre \\
&= (qlrm)^{j-1} qlnsemrnsrmgqlr(rnsemrns)lqmre \\
&= (qlrm)^{j-1} qlnsemrnsrm(gqlnse)mrnslqmre \\
&= (qlrm)^{j-1} qlnsemrnsrm(mregqlsnrm)mrnslqmre \\
&= (qlrm)^{j-1} qlnse(mrnse)gmre \\
&= (qlrm)^{j-1} qlnse(esnrm)gmre \\
&= (qlrm)^{j-1} qlrm(gmre) \\
&= (qlrm)^j (gmre) \\
&= (qlrm)^j qlnsemrnsrmgqlrm.
\end{aligned}$$

Similarly, we can get other relations. Let us check some compositions:

$$\begin{aligned}
(lrmg)mre - lrm(gmre) &= q(gmrlqmre) - lrmqlnsemrnsrmgqlrm \\
&= (qq)lrmqlnsemrnsrmgqlrm - lrmqlnsemrnsrmgqlrm \\
&= 0;
\end{aligned}$$

$$\begin{aligned}
(gg)mrlqmre - g(gmrlqmre) &= mrlqmre - (gqlrmqlnse)mrnsrmgqlrm \\
&= mrlqmre - mrlqmregqlsnrmmrnsrmgqlrm \\
&= mrlqmre - mrlqmregq(lrmg)qlrm \\
&= mrlqmre - mrlqmregqqgmrlqqlrm \\
&= mrlqmre - mrlqmre = 0;
\end{aligned}$$

$$
\begin{aligned}
(lqg)qlnse - lq(gqlnse) &= rm(gqlrmqlnse) - lqmregqlsnrm \\
&= rmmrlqmregqlsnrm - lqmregqlsnrm = 0;
\end{aligned}
$$

$$
\begin{aligned}
(gg)qlrmqlnse &- g(gqlrmqlnse)qlrmqlnse - (gmrlqmre)gqlsnrm \\
&= qlrmqlnse - qlrmqlnsemrnsrmgq(lrmg)qlsnrm \\
&= qlrmqlnse - qlrmqlnsemrnsrmgqqgmrlqqlsnrm \\
&= qlrmqlnse - qlrmqlnse = 0.
\end{aligned}
$$

This proves triviality of compositions for $j = 1$. Proceed by induction for $j > 1$. For example, we have

$$
\begin{aligned}
&lrm(g(mrlq)^j mre) - (lrmg)(mrlq)^j mre = \\
&= lrm(qlrm)^j qlnsemrnsrmgqlrm - qgmrlq(mrlq)^j mre \\
&= lrm(qlrm)^j qlnsemrnsrmgqlrm - q(g(mrlq)^{j+1} mre) \\
&= lrm(qlrm)^j qlnsemrnsrmgqlrm - q((qlrm)^{j+1} qlnsemrnsrmgqlrm) \\
&= lrm(qlrm)^j qlnsemrnsrmgqlrm - lrm(qlrm)^j qlnsemrnsrmgqlrm = 0;
\end{aligned}
$$

$$
\begin{aligned}
g(g(mrlq)^j mre) - (gg)(mrlq)^j mre &= (g(qlrm)^j qlnse)mrnsrmgqlrm - (mrlq)^j mre \\
&= (mrlq)^j mregq(lrmg)qlrm - (mrlq)^j mre \\
&= (mrlq)^j mregqqgmrlqqlrm - (mrlq)^j mre \\
&= (mrlq)^j mre - (mrlq)^j mre = 0;
\end{aligned}
$$

$$
\begin{aligned}
lq(g(mrlq)^j e) - (lqg)(mrlq)^j e &= lq(qlrm)^j nsegns - rmgqlrm(mrlq)^j e \\
&= lqqlrm(qlrm)^{j-1} nsegns - rm(g(mrlq)^{j-1} e) \\
&= rm(qlrm)^{j-1} nsegns - rm(qlrm)^{j-1} nsegns = 0.
\end{aligned}
$$

$$
\begin{aligned}
g(g(mrlq)^j e) - (gg)(mrlq)^j e &= (g(qlrm)^j nse)gns - (mrlq)^j e \\
&= (mrlq)^j esnggns - (mrlq)^j e \\
&= (mrlq)^j e - (mrlq)^j e = 0.
\end{aligned}
$$

Similarly, we can check that any compostion in S is trivial. □

4. Morse matching

In this section, we state main definitions and known results related to the construction of the Anick chain complex via algebraic Morse theory following [14, 17, 4]. The approach is based on the following idea. Suppose $\Bbbk$ is a field and Λ is a unital associative $\Bbbk$-algebra. Let $B = (B_n, d_n)_{n\geq 0}$ be a chain complex of free left Λ-modules, and let X_n be a basis of B_n over Λ. One may represent the complex with a weighted oriented graph $\Gamma(B)$ whose vertices are $\bigcup_n X_n$ and there is an edge from $x \in X_n$ to $y \in X_{n-1}$ if the distribution of $d_n(x) \in B_{n-1}$ contains λy for $0 \neq \lambda \in \Lambda$ (the weight of the edge is set to be this λ).

Assume M is a particular matching in the graph $\Gamma(B)$, i.e., a subset of edges such that neither of vertices belongs to more than one edge from M. In addition, assume the weights of edges from M are invertible central elements of Λ Then transform the graph $\Gamma(B)$ in the following way: invert the direction of all edges from M and replace their weights λ with $-\lambda^{-1}$. If the resulting graph $\Gamma_M(B)$ has no directed cycles then M is called a *Morse matching*. The vertices that do not belong to an edge from M are said to be *critical cells*.

For a Morse matching graph $\Gamma_M(B)$, one may construct another chain complex of free Λ-modules $(B_{(m)}, \delta_m)_{m\geq 0}$, where $B_{(m)}$ is spanned over Λ by the set $X_{(m)}$ of all critical cells from X_m and the calculation of δ_m is based on the consideration of all paths in $\Gamma_M(B)$ (see, e.g., [14]). Namely,

$$\delta_m(x) = \sum_{y\in X_{(m-1)}} \Gamma(x,y)y, \quad y \in X_{(m)},$$

where $\Gamma(w, w')$ is the sum of path weights in the Morse matching graph $\Gamma_M(B)$. The new complex $(B_{(m)}, \delta_m)_{m\geq 0}$ is homotopy equivalent to the initial one, but it is smaller since we choose only critical cells as generators of the modules $B_{(m)}$.

For the bar-resolution B, there is a natural way to choose a Morse matching M in $\Gamma(B)$ in such a way that the resulting complex constructed from $\Gamma_M(B)$ coincides with the Anick resolution. We will describe the construction in bimodule setting which is easy to get by replacing Λ with $\Lambda \otimes \Lambda^{op}$.

Suppose Λ has an augmentation $\varepsilon : \Lambda \to \Bbbk$, A is a set of generators for Λ. Then Λ is a homomorphic image of the free associative algebra $\Bbbk\langle A\rangle$ generated by A. Assume $\leq$ is a monomial order on the free monoid A^*, and GSB_Λ is a Gröbner—Shirshov basis of Λ. The latter may be considered as a confluent set of defining relations for the algebra Λ, each of relations is of the form $u - f$, where $u \in A^*$, $f \in \Bbbk\langle A\rangle$, $u \geq \bar{f}$, $\bar{f}$ is the leading monomial of f relative to $\leq$. Denote by V the set of all leading terms u of relations from GSB_Λ. Following Anick [1], V is called the set of *obstructions*.

The cokernel of $\varepsilon : \Bbbk \to \Lambda$ is denoted by $\Lambda/\Bbbk$. The set of all non-trivial words in A^* that do not contain a word from V as a subword (i.e., the set of V-reduced words [18]) forms a linear basis of $\Lambda/\Bbbk$. This is one of the equivalent conditions in the Composition-Diamond Lemma about Gröbner—Shirshov bases for associative algebras (see Theorem (1)).

The two-sided bar resolution $(B_m(\Lambda, \Lambda), d_m)_{m\geq 0}$ is an exact sequence of Λ-bimodules

$$0 \leftarrow \Bbbk \leftarrow \Lambda \leftarrow B_0(\Lambda, \Lambda) \leftarrow B_1(\Lambda, \Lambda) \leftarrow \dots,$$

where

$$B_m(\Lambda, \Lambda) := \Lambda \otimes (\Lambda/\Bbbk)^{\otimes(m+1)} \otimes \Lambda.$$

We will use the standard convention denoting $1 \otimes \lambda_1 \otimes \dots \otimes \lambda_{m+1} \otimes 1 \in B_m(\Lambda, \Lambda)$ by $[\lambda_1|\dots|\lambda_{m+1}]$ for $\lambda_i \in \Lambda/\Bbbk$. The differential $d_m : B_m \to B_{m-1}$ is defined as follows:

$$d_m([\lambda_1|\dots|\lambda_{m+1}]) = \lambda_1[\lambda_2|\dots|\lambda_{m+1}] + (-1)^{m+1}[\lambda_1|\dots|\lambda_m]\lambda_{m+1} + \\ + \sum_{i=1}^{m}(-1)^i[\lambda_1|\dots|\lambda_i\lambda_{i+1}|\dots|\lambda_{m+1}].$$

The role of critical cells $X_{(m)}$ is played by so called m-chains. By definition, 0-chains are the elements of A, 1-chains are the words from V. For $m \geq 1$, a word $v = x_{i_1}\dots x_{i_t} \in A^*$ is said to be an *m-prechain* if and only if there exist indices $a_j, b_j \in \mathbb{Z}$, $1 \leq j \leq m$, satisfying the following conditions:

(1) $1 = a_1 < a_2 \leq b_1 < \dots < a_m \leq b_{m-1} < b_m = t$;
(2) $x_{i_{a_j}} \dots x_{i_{b_j}} \in V$ for $1 \leq j \leq m$.

That is, v is obtained by m obstructions consecutively intersecting with each other. An m-prechain $x_{i_1}\dots x_{i_t}$ is an *m-chain* if the integers a_j, b_j can be chosen in such a way that for every $1 \le l \le m$ and for every $s < b_l$ the prefix $x_{i_1}\dots x_{i_s}$ is not an l-prechain. Denote the set of all m-chains by $V^{(m)}$.

According to [14, 4, 17], there is a way to choose a Morse matching in the graph $\Gamma(B)$ corresponding to the bar-resolution in such a way that $X_{(m)} = V^{(m)}$ and the new complex is exactly the Anick resolution

$$0 \leftarrow \Bbbk \overset{\varepsilon}{\leftarrow} \Lambda \overset{\delta_0}{\leftarrow} B_{(0)} \overset{\delta_1}{\leftarrow} B_{(1)} \overset{\delta_2}{\leftarrow} B_{(2)} \leftarrow \dots,$$

where

$$B_{(m)} = \Lambda \otimes \Bbbk V^{(m)} \otimes \Lambda, \quad m \ge 0.$$

We suppress the construction of graphs in the algorithm sketched above in order to replace graphical manipulations with algebraic expressions. Our aim is to evaluate $\delta_n(w) \in \Lambda \otimes \Bbbk V^{(n-1)} \otimes \Lambda$ for a given $w \in V^{(n)}$. For $w = w_1 w_2 \dots w_n w_{n+1} \in V^{(n)}$, where all w_i are V-reduced words, denote $[w] = [w_1|w_2|\dots|w_n|w_{n+1}] \in (\Lambda/\Bbbk)^{\otimes(n+1)}$. In the sequel we will often omit symbols | in the elements of the Anick resolution.

We replace the computation of path weights in the Morse matching graph for the bar-resolution with an algebraic procedure based on the consecutive application of Λ-bimodule homomorphisms. Define two Λ-bilinear operators δ'_n and δ''_n whose recurrent application leads us to the desired δ_n. This is just a computational method to get the same relations as in the "graphical way" applied in [22].

(1) Given $[w] \in V^{(n)}$, calculate the ordinary differential

$$\delta'_n[w] = w_1[w_2|\dots|w_n|w_{n+1}] + (-1)^n[w_1|w_2|\dots|w_n]w_{n+1} + \sum_{j=1}^{n}(-1)^j[w_1|w_2|\dots|s_j|\dots|w_{n+1}] \in \Lambda \otimes (\Lambda/\Bbbk)^{\otimes n} \otimes \Lambda,$$

where s_j are the V-reduced forms of the products $w_j w_{j+1}$, $j = 1, \dots, n$;

(2) If $[v] \in (\Lambda/\Bbbk)^{\otimes n}$ belongs to $V^{(n-1)}$, then

$$\delta''_n[v] = [v]$$

and the computation is finished. Otherwise, suppose $[v] = [v_1|\dots|v_n]$ does not belong to $V^{(n-1)}$ (here all v_ks are V-reduced) then there exists the largest integer $i \ge 0$ such that $v_1 \dots v_i \in V^{(i-1)}$, v_{i+1} may be presented as $v_{i+1} = v'_{i+1}v''_{i+1}$, and $[v_1|\dots|v'_{i+1}]$ belongs to $V^{(i)}$ [4]. Then set

$$\delta''_n[v] = (-1)^i \delta'_n([v_1|\dots|v'_{i+1}|v''_{i+1}|\dots|v_n]) + [v].$$

If such an index $i \ge 0$ does not exist then set $\delta''_n([v]) = 0$.

After finitely many steps, the computation of δ''_n finishes. Therefore,

$$\delta_n(w) = (\delta''_n)^k \delta'_n[w],$$

where $k = k(w) \ge 1$ is a sufficiently large integer.

For example let us to calculate $\delta_1([w])$ for $[w] = [l|rmg]$. First, compute

$$\delta'_1([l|rmg]) = l[rmg] - [qgmrlq] + [l]rmg.$$

Next, apply δ''_1 to each summand:

$$\delta''_1([rmg]) = \delta'_1([r|mg]) + [rmg] = r[mg] + [r]mg, \quad i = 0,$$

$$\delta_1''([mg]) = \delta_1'([m|g]) + [mg] = m[g] + [m]g.$$

Therefore, $(\delta_1'')^2[rmg] = rm[g] + r[m]g + [r]mg$. Similarly,

$(\delta_1'')^5([qgmrlq]) = qgmrl[q] + qgmr[l]q + qgm[r]lq + qg[m]rlq + q[g]mrlq + [q]gmrlq.$

Hence,

$$\delta_1([lrmg]) = l(rm[g] + r[m]g + [r]mg) - (qgmrl[q] + qgmr[l]q + qgm[r]lq + qg[m]rlq + q[g]mrlq + [q]gmrlq) + [l]rmg.$$

Let us also consider an example of the calculation of δ_2. In the following we will often omit the bars in the expressions for $[w]$ if $w \in V^{\otimes n}$ in order to make the formulas shorter.

Suppose $[w] = [g|g|e]$ which can be written as $[w] = [g^2e]$. Then calculate

$$\delta_2'([g^2e]) = g[ge] - 0 + [g|nsegns] - [g^2]e.$$

Since $[ge], [g^2] \in V^{(1)}$, it is enough to calculate $(\delta_2'')^k([g|nsegns])$ for $k = 1, 2, \ldots$ Here

$$\delta_2''([g|nsegns]) = -\delta_2'([g|nse|gns]) + [g|nsegns] = -g[nse|gns] + [esng|gns] + [g|nse]gns, \quad i = 1.$$

Note that $[g|nse] \in V^{(1)}$ and calculate

$$\delta_2''([esng|gns]) = \delta_2'([e|sng|gns]) + [esng|gns] = e[sng|gns] + 0 - [e|sng]gns,$$

$$\delta_2''([sng|gns]) = \delta_2'([s|ng|gns]) + [sng|gns] = s[ng|gns] + [s|s] - [s|ng]gns$$

(here $i = 0$ in both cases). For the words $ensg$ and sng there are no $i \geq 0$ satisfying the condition of Step (2) of the algorithm, so

$$\delta_2''([e|sng]) = \delta_2''([s|ng]) = 0.$$

It remains to calculate

$$\delta_2''([ng|gns]) = \delta_2'([n|g|gns]) + [ng|gns] = n[g|gns] + [n|ns] - [n|g]gns,$$

where $\delta_2''([n|g]) = 0$ and

$$\delta_2''([g|gns]) = -\delta_2'([g|g|ns]) + [g|gns] = -g[g|ns] + 0 + [g|g]ns, \quad i = 1,$$

$$\delta_2''([n|ns]) = -\delta_2'([n|n|s]) + [n|ns] = -n[n|s] + 0 + [n|n]s.$$

Here $[g|g], [n|n] \in V^{(1)}$ and $\delta_2''([g|ns]) = \delta_2''([n|s]) = 0$. Collecting similar terms leads us to the desired expression for $\delta_2[g^2e]$ (see the next section).

Let Λ be a $\Bbbk$-algebra as above, and let W be a unital Λ-bimodule. Consider the Hochschild cochain complex $(C^m(\Lambda, W), \Delta_m)_{m\geq 0}$ where Δ_m is given by (1). We may use the Anick resolution to calculate $H^m(\Lambda, W) = \operatorname{Ker}\Delta_m / \operatorname{Im}\Delta_{m-1}$ via

$$\Delta_m(f) = f\delta_m, \quad f : B_{(m-1)} \to W.$$

Indeed, $H^m(\Lambda, W) = Z^m/\beta^m$, where

$$\beta^m := \operatorname{Im}\Delta_{m-1} = \{f : B_{(m-1)} \to W \mid f = g\delta_{m-1},\, g : B_{(m-2)} \to W\},$$
$$Z^m := \operatorname{Ker}\Delta_m = \{f : B_{(m-1)} \to W \mid f\delta_m = 0\}.$$

5. The Anick complex for Γ_5^4

To find the Anick complex, we need two steps. First, we have to find the set of obstructions for Γ_5^4 relative to the given Gröbner—Shirshov basis (the set of leading terms in $S = \mathrm{GSB}_{\Gamma_5^4}$) and the set of 2-chains. Next, build a Morse graph and calculate the path weights $\Gamma(w, w')$ for every m-chain w and $(m-1)$-chain w' for $m = 1, 2$. Alternatively, we may apply the algorithm described in the previous section to find the Anick differentials δ_m, $m = 1, 2$.

For $m = 1$ we have the following set of obstructions

$$V^{(1)} = \{e^2, g^2, k^2, l^2, m^2, n^2, q^2, r^2, s^2, ge, gnse, le, lrnse, kse, krnse, me, mrnse, qrnse, qnse, lqg, lrmg, gqlnse, gmre, g(qlrm)^j qlnse, g(qlrm)^j nse, g(mrlq)^j mre, g(mrlq)^j e \mid j \geq 1\}.$$

Let us calculate δ_1 using theorem 2 as stated in the previous section:

$$\delta_1[x^2] = x[x] + [x]x, \quad x \in \{e, g, k, l, m, n, q, r, s\}.$$

$$\delta_1[ge] = g[e] + [g]e - [n]segns - n[s]egns - ns[e]gns - nse[g]ns - nseg[n]s - nsegn[s];$$

$$\delta_1[gnse] = [g]nse + g[n]se + gn[s]e + gns[e] - [e]sng - e[s]ng - es[n]g - esn[g];$$

$$\delta_1[mrnse] = [m]rnse + m[r]nse + mr[n]se + mrn[s]e + mrns[e] - [e]snrm - e[s]nrm - es[n]rm - esn[r]m - esnr[m];$$

$$\delta_1[me] = [m]e + m[e] - [r]nsemrns - r[n]semrns - rn[s]emrns - rns[e]mrns - rnse[m]rns - rnsem[r]ns - rnsemr[n]s - rnsemrn[s];$$

$$\begin{aligned}\delta_1[le] = {} & [l]e + l[e] - [r]nsemrnkmgqksrmgqlrm - r[n]semrnkmgqksrmgqlrm \\ & - rn[s]emrnkmgqksrmgqlrm - rns[e]mrnkmgqksrmgqlrm \\ & - rnse[m]rnkmgqksrmgqlrm - rnsem[r]nkmgqksrmgqlrm \\ & - rnsemr[n]kmgqksrmgqlrm - rnsemrn[k]mgqksrmgqlrm \\ & - rnsemrnk[m]gqksrmgqlrm - rnsemrnkm[g]qksrmgqlrm \\ & - rnsemrnkmg[q]ksrmgqlrm - rnsemrnkmgq[k]srmgqlrm \\ & - rnsemrnkmgqk[s]rmgqlrm - rnsemrnkmgqks[r]mgqlrm \\ & - rnsemrnkmgqksr[m]gqlrm - rnsemrnkmgqksrm[g]qlrm \\ & - rnsemrnkmgqksrmg[q]lrm - rnsemrnkmgqksrmgq[l]rm \\ & - rnsemrnkmgqksrmgql[r]m - rnsemrnkmgqksrmgqlr[m];\end{aligned}$$

$$\begin{aligned}\delta_1[lrnse] = {} & [l]rnse + l[r]nse + lr[n]se + lrn[s]e + lrns[e] - [e]gqlskqgmknrm \\ & - e[g]qlskqgmknrm - eg[q]lskqgmknrm - egq[l]skqgmknrm \\ & - egql[s]kqgmknrm - egqls[k]qgmknrm - egqlsk[q]gmknrm \\ & - egqlskq[g]mknrm - egqlskqg[m]knrm - egqlskqgm[k]nrm \\ & - egqlskqgmk[n]rm - egqlskqgmkn[r]m - egqlskqgmknr[m];\end{aligned}$$

$$\begin{aligned}\delta_1[kse] = {} & [k]se + k[s]e + ks[e] - [r]nsemrnkm - r[n]semrnkm - rn[s]emrnkm \\ & - rns[e]mrnkm - rnse[m]rnkm - rnsem[r]nkm - rnsemr[n]km \\ & - rnsemrn[k]m - rnsemrnk[m];\end{aligned}$$

$$\begin{aligned}\delta_1[krnse] &= [k]rnse + k[r]nse + kr[n]se + krn[s]e + krns[e] - [s]emknrm\\ &\quad - s[e]mknrm - se[m]knrm - sem[k]nrm - semk[n]rm - semkn[r]m\\ &\quad - semknr[m];\end{aligned}$$

$$\begin{aligned}\delta_1[qrnse] &= [q]rnse + q[r]nse + qr[n]se + qrn[s]e + qrns[e] - [n]segnkqgmknrm\\ &\quad - n[s]egnkqgmknrm - ns[e]gnkqgmknrm - nse[g]nkqgmknrm\\ &\quad - nseg[n]kqgmknrm - nsegn[k]qgmknrm - nsegnk[q]gmknrm\\ &\quad - nsegnkq[g]mknrm - nsegnkqg[m]knrm - nsegnkqgm[k]nrm\\ &\quad - nsegnkqgmk[n]rm - nsegnkqgmkn[r]m - nsegnkqgmknr[m];\end{aligned}$$

$$\begin{aligned}\delta_1[qnse] &= [q]nse + q[n]se + qn[s]e + qns[e] - [r]nsemrnkmgqkng\\ &\quad - r[n]semrnkmgqkng - rn[s]emrnkmgqkng - rns[e]mrnkmgqkng\\ &\quad - rnse[m]rnkmgqkng - rnsem[r]nkmgqkng - rnsemr[n]kmgqkng\\ &\quad - rnsemrn[k]mgqkng - rnsemrnk[m]gqkng - rnsemrnkm[g]qkng\\ &\quad - rnsemrnkmg[q]kng - rnsemrnkmgq[k]ng - rnsemrnkmgqk[n]g\\ &\quad - rnsemrnkmgqkn[g];\end{aligned}$$

$$\begin{aligned}\delta_1[gqlnse] &= [g]qlnse + g[q]lnse + gq[l]nse + gql[n]se + gqln[s]e + gqlns[e]\\ &\quad - [m]regqlsnrm - m[r]egqlsnrm - mr[e]gqlsnrm - mre[g]qlsnrm\\ &\quad - mreg[q]lsnrm - mregq[l]snrm - mregql[s]nrm - mregqls[n]rm\\ &\quad - mregqlsn[r]m - mregqlsnr[m];\end{aligned}$$

$$\begin{aligned}\delta_1[gmre] &= [g]mre + g[m]re + gm[r]e + gmr[e] - [q]lnsemrnsrmgqlrm\\ &\quad - q[l]nsemrnsrmgqlrm - ql[n]semrnsrmgqlrm - qln[s]emrnsrmgqlrm\\ &\quad - qlns[e]mrnsrmgqlrm - qlnse[m]rnsrmgqlrm - qlnsem[r]nsrmgqlrm\\ &\quad - qlnsemr[n]srmgqlrm - qlnsemrn[s]rmgqlrm - qlnsemrns[r]mgqlrm\\ &\quad - qlnsemrnsr[m]gqlrm - qlnsemrnsrm[g]qlrm - qlnsemrnsrmg[q]lrm\\ &\quad - qlnsemrnsrmgq[l]rm - qlnsemrnsrmgql[r]m - qlnsemrnsrmgqlr[m];\end{aligned}$$

$$\begin{aligned}\delta_1[lqg] &= [l]qg + l[q]g + lq[g] - [r]mgqlrm - r[m]gqlrm - rm[g]qlrm - rmg[q]lrm\\ &\quad - rmgq[l]rm - rmgql[r]m - rmgqlr[m];\end{aligned}$$

$$\begin{aligned}\delta_1[lrmg] &= [l]rmg + l[r]mg + lr[m]g + lrm[g] - [q]gmrlq - q[g]mrlq - qg[m]rlq\\ &\quad - qgm[r]lq - qgmr[l]q - qgmrl[q];\end{aligned}$$

$$\begin{aligned}\delta_1[g(qlrm)^j qlnse] &= [g](qlrm)^j qlnse + \sum_{i=0}^{j-1} g(qlrm)^i[q]lrm(qlrm)^{j-i-1}qlnse\\ &\quad + \sum_{i=0}^{j-1} g(qlrm)^i q[l]rm(qlrm)^{j-i-1}qlnse + g(qlrm)^j[q]lnse\\ &\quad + \sum_{i=0}^{j-1} g(qlrm)^i ql[r]m(qlrm)^{j-i-1}qlnse + g(qlrm)^j q[l]nse\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{j-1} g(qlrm)^i qlr[m](qlrm)^{j-i-1} qlnse + g(qlrm)^j ql[n]se \\
&-\sum_{i=0}^{j-1} (mrlq)^i [m]rlq(mrlq)^{j-i-1} mregqlsnrm + g(qlrm)^j qln[s]e \\
&-\sum_{i=0}^{j-1} (mrlq)^i m[r]lq(mrlq)^{j-i-1} mregqlsnrm + g(qlrm)^j qlns[e] \\
&-\sum_{i=0}^{j-1} (mrlq)^i mr[l]q(mrlq)^{j-i-1} mregqlsnrm \\
&-\sum_{i=0}^{j-1} (mrlq)^i mrl[q](mrlq)^{j-i-1} mregqlsnrm \\
&-(mrlq)^j mregq[l]snrm - (mrlq)^j mreg[q]lsnrm \\
&-(mrlq)^j [m]regqlsnrm - (mrlq)^j m[r]egqlsnrm \\
&-(mrlq)^j mre[g]qlsnrm - (mrlq)^j mregql[s]nrm \\
&-(mrlq)^j mregqlsn[r]m - (mrlq)^j mregqlsnr[m] \\
&-(mrlq)^j mregqls[n]rm - (mrlq)^j mr[e]gqlsnrm;
\end{aligned}
$$

$$
\begin{aligned}
\delta_1[g(mrlq)^j mre] &= [g](mrlq)^j mre + \sum_{i=0}^{j-1} g(mrlq)^i [m]rlq(mrlq)^{j-i-1} mre \\
&+\sum_{i=0}^{j-1} g(mrlq)^i m[r]lq(mrlq)^{j-i-1} mre \\
&+\sum_{i=0}^{j-1} g(mrlq)^i mr[l]q(mrlq)^{j-i-1} mre \\
&+\sum_{i=0}^{j-1} g(mrlq)^i mrl[q](mrlq)^{j-i-1} mre \\
&+g(mrlq)^j [m]re + g(mrlq)^j m[r]e + g(mrlq)^j mr[e] \\
&-\sum_{i=0}^{j-1} (qlrm)^i [q]lrm(qlrm)^{j-i-1} qlnsemrnsrmgqlrm \\
&-\sum_{i=0}^{j-1} (qlrm)^i q[l]rm(qlrm)^{j-i-1} qlnsemrnsrmgqlrm \\
&-\sum_{i=0}^{j-1} (qlrm)^i ql[r]m(qlrm)^{j-i-1} qlnsemrnsrmgqlrm \\
&-\sum_{i=0}^{j-1} (qlrm)^i qlr[m](qlrm)^{j-i-1} qlnsemrnsrmgqlrm \\
&-(qlrm)^j [q]lnsemrnsrmgqlrm - (qlrm)^j q[l]nsemrnsrmgqlrm \\
&-(qlrm)^j ql[n]semrnsrmgqlrm - (qlrm)^j qln[s]emrnsrmgqlrm
\end{aligned}
$$

$$
\begin{aligned}
&-(qlrm)^j qlns[e]mrnsrmgqlrm-(qlrm)^j qlnse[m]rnsrmgqlrm\\
&-(qlrm)^j qlnsem[r]nsrmgqlrm-(qlrm)^j qlnsemr[n]srmgqlrm\\
&-(qlrm)^j qlnsemrn[s]rmgqlrm-(qlrm)^j qlnsemrns[r]mgqlrm\\
&-(qlrm)^j qlnsemrnsr[m]gqlrm-(qlrm)^j qlnsemrnsrm[g]qlrm\\
&-(qlrm)^j qlnsemrnsrmg[q]lrm-(qlrm)^j qlnsemrnsrmgq[l]rm\\
&-(qlrm)^j qlnsemrnsrmgql[r]m-(qlrm)^j qlnsemrnsrmgqlr[m];
\end{aligned}
$$

$$
\begin{aligned}
\delta_1[g(qlrm)^j nse] = {} & [g](qlrm)^j nse+\sum_{i=0}^{j-1} g(qlrm)^i[q]lrm(qlrm)^{j-i-1}nse\\
&+\sum_{i=0}^{j-1} g(qlrm)^i q[l]rm(qlrm)^{j-i-1}nse-(mrlq)^j esn[g]\\
&+\sum_{i=0}^{j-1} g(qlrm)^i ql[r]m(qlrm)^{j-i-1}nse-(mrlq)^j es[n]g\\
&+\sum_{i=0}^{j-1} g(qlrm)^i qlr[m](qlrm)^{j-i-1}nse+g(qlrm)^j[n]se\\
&-\sum_{i=0}^{j-1}(mrlq)^i[m]rlq(mrlq)^{j-i-1}esng+g(qlrm)^j ns[e]\\
&-\sum_{i=0}^{j-1}(mrlq)^i m[r]lq(mrlq)^{j-i-1}esng+g(qlrm)^j n[s]e\\
&-\sum_{i=0}^{j-1}(mrlq)^i mr[l]q(mrlq)^{j-i-1}esng-(mrlq)^j e[s]ng\\
&-\sum_{i=0}^{j-1}(mrlq)^i mrl[q](mrlq)^{j-i-1}esng-(mrlq)^j[e]sng;
\end{aligned}
$$

$$
\begin{aligned}
\delta_1[g(mrlq)^j e] = {} & [g](mrlq)^j e+g(mrlq)^j[e]+\sum_{i=0}^{j-1} g(mrlq)^i[m]rlq(mrlq)^{j-i-1}e\\
&+\sum_{i=0}^{j-1} g(mrlq)^i m[r]lq(mrlq)^{j-i-1}e+\sum_{i=0}^{j-1} g(mrlq)^i mr[l]q(mrlq)^{j-i-1}e\\
&+\sum_{i=0}^{j-1} g(mrlq)^i mrl[q](mrlq)^{j-i-1}e-(qlrm)^j[n]segns\\
&-\sum_{i=0}^{j-1}(qlrm)^i[q]lrm(qlrm)^{j-i-1}nsegns-(qlrm)^j ns[e]gns\\
&-\sum_{i=0}^{j-1}(qlrm)^i q[l]rm(qlrm)^{j-i-1}nsegns-(qlrm)^j n[s]egns
\end{aligned}
$$

$$- \sum_{i=0}^{j-1} (qlrm)^i ql[r]m(qlrm)^{j-i-1} nsegns - (qlrm)^j nse[g]ns$$
$$- \sum_{i=0}^{j-1} (qlrm)^i qlr[m](qlrm)^{j-i-1} nsegns - (qlrm)^j nseg[n]s$$
$$- (qlrm)^j nsegn[s].$$

For $m = 2$ we have

$$V^{(2)} = \{e^3, g^3, k^3, l^3, m^3, n^3, q^3, r^3, s^3, g^2e, ge^2, g^2nse, gnse^2, m^2rnse, mrnse^2, m^2e, me^2, k^2se, kse^2, k^2rnse, krnse^2, q^2rnse, qrnse^2, q^2nse, qnse^2, l^2e, le^2, l^2rnse, lrnse^2, l^2qg, lqg^2, l^2rmg, lrmg^2, g^2qlnse, gqlnse^2, g^2mre, gmre^2, g^2(qlrm)^j qlnse, g^2(mrlq)^j mre, g^2(qlrm)^j nse, g^2(mrlq)^j e, g(mrlq)^j e^2, g(qlrm)^j nse^2, g(qlrm)^j qlnse^2, g(mrlq)^j mre^2, lqg(mrlq)^j e, lrmg(mrlq)^j e, lqg(qlrm)^j nse, lrmg(qlrm)^j nse, lrmg(mrlq)^j mre, lqg(mrlq)^j mre, lqg(qlrm)^j qlnse, lrmg(qlrm)^j qlnse, lqge, lqgnse, lqgqlnse, lqgmre, lrmge, lrmgnse, lrmgmre, lrmgqlnse \mid j \geq 1\}.$$

Let us construct a Morse matching and evaluate the differential $\delta_2 : B_{(3)} \to B_{(2)}$. In the simplest case $w = x^3$, $x \in \{e, g, k, l, m, n, q, r, s\}$, we have $\delta_2[x^3] = x[x^2] - [x^2]x$ for $x \in \{e, g, k, l, m, n, q, r, s\}$. In a similar way, one may calculate

$$\delta_2[g^2e] = g[ge] - [g^2]e + [gnse]gns + esn[g^2]ns + es[n^2]s + e[s^2];$$
$$\delta_2[ge^2] = g[e^2] - [ge]e - nse[gnse] - ns[e^2]sng - n[s^2]ng - [n^2]g;$$
$$\delta_2[g^2nse] = g[gnse] - [g^2]nse + [ge]sng + nsegn[s^2]ng + nseg[n^2]g + nse[g^2];$$
$$\delta_2[gnse^2] = gns[e^2] - [gnse]e - esn[ge] - es[n^2]segns - e[s^2]egns - [e^2]gns;$$
$$\delta_2[m^2rnse] = m[mrnse] - [m^2]rnse + [me]snrm + rnsemrn[s^2]nrm + rnsemr[n^2]rm + rnsem[r^2]m + rnse[m^2];$$
$$\delta_2[mrnse^2] = mrns[e^2] - [mrnse]e - esnr[me] - esn[r^2]nsemrns - es[n^2]semrns - e[s^2]emrns - [e^2]mrns;$$
$$\delta_2[m^2e] = m[me] - [m^2]e + [mrnse]mrns + esnr[m^2]rns + esn[r^2]ns + es[n^2]s + e[s^2];$$
$$\delta_2[me^2] = m[e^2] - [me]e - rnse[mrnse] - rns[e^2]snrm - rn[s^2]nrm - r[n^2]rm - [r^2]m;$$
$$\delta_2[k^2se] = k[kse] - [k^2]se + [krnse]mrnkm + semknr[m^2]rnkm + semkn[r^2]nkm + semk[n^2]km + sem[k^2]m + se[m^2];$$
$$\delta_2[kse^2] = ks[e^2] - [kse]e - rnsemrnk[me] - rnsemrn[krnse]mrns - rns[e^2]snrks - rn[s^2]nrks - r[n^2]rks - [r^2]ks - knr[m^2]rns - kn[r^2]ns - k[n^2]s - [m^2]ks - rnse[mrnse]mks;$$
$$\delta_2[k^2rnse] = k[krnse] - [k^2]rnse + [kse]mknrm + rnsemrnk[m^2]knrm + rnsemrn[k^2]nrm + rnsemr[n^2]rm + rnsem[r^2]m + rnse[m^2];$$

$$\begin{aligned}\delta_2[krnse^2] &= krns[e^2] - [krnse]e - semknr[me] - sem[kse]mrns - k[m^2]rns\\ &- se[mrnse]mrnkrns - nr[m^2]rnkrns - n[r^2]nkrns - [n^2]krns\\ &- [s^2]krns - semkn[r^2]nsemrns - semk[n^2]semrns - s[e^2]skrns;\end{aligned}$$

$$\begin{aligned}\delta_2[q^2rnse] &= q[qrnse] - [q^2]rnse + [qnse]gnkqgmknrm\\ &+ rnsemrnkmgqk[n^2]kqgmknrm + rnsemr[n^2]rm + rnsem[r^2]m\\ &+ rnsemrnkmgq[k^2]qgmknrm + rnsemrnkmg[q^2]gmknrm + rnse[m^2]\\ &+ rnsemrnkm[g^2]mknrm + rnsemrnk[m^2]knrm + rnsemrn[k^2]nrm\\ &+ rnsemrnkmgqkn[g^2]nkqgmknrm;\end{aligned}$$

$$\begin{aligned}\delta_2[qrnse^2] &= qrns[e^2] - [qrnse]e - nsegnkqgmknr[me] - gmk[n^2]kmgqrns\\ &- nsegnkqgmk[n^2]semrns - nsegnkqgm[kse]mrns - qkn[g^2]nkrns\\ &- nsegnkq[ge]skrns - nsegnk[qnse]gnkrns - q[m^2]rns\\ &- nse[gnse]gqrns - [g^2]qrns - ns[e^2]snqrns - n[s^2]nqrns - [n^2]qrns\\ &- knr[m^2]rnkrns - kn[r^2]nkrns - k[n^2]krns - qk[s^2]krns\\ &- qk[n^2]krns - q[k^2]rns - gmknr[m^2]rnkmgqrns - gmkn[r^2]nkmgqrns\\ &- nsegnkqg[mrnse]mrnkrns - gm[k^2]mgqrns - g[m^2]gqrns\\ &- nsegnkqgmkn[r^2]nsemrns - nsegn[krnse]mrnkmgqrns;\end{aligned}$$

$$\begin{aligned}\delta_2[q^2nse] &= q[qnse] - [q^2]nse + [qrnse]mrnkmgqkng\\ &+ nsegnkqgmkn[r^2]nkmgqkng + nsegnkqgmk[n^2]kmgqkng\\ &+ nsegnkqgm[k^2]mgqkng + nsegnkqg[m^2]gqkng + nsegnkq[g^2]qkng\\ &+ nsegnk[q^2]kng + nsegn[k^2]ng + nseg[n^2]g + nse[g^2]\\ &+ nsegnkqgmknr[m^2]rnkmgqkng;\end{aligned}$$

$$\begin{aligned}\delta_2[qnse^2] &= qns[e^2] - [qnse]e - rnsemrnkmgqkn[ge] - rnsemrnkmgqk[n^2]segns\\ &- rnsemrnkmgq[kse]gns - rnsemrnkmg[qrnse]mrnkmgns - [r^2]qns\\ &- rnsemrnkm[gnse]gnkqns - rnsemrnk[me]skqns - knr[m^2]rnkqns\\ &- rnse[mrnse]mqns - rns[e^2]snrqns - rn[s^2]nrqns - r[n^2]rqns\\ &- [m^2]qns - [k^2]qns - k[n^2]kqns - kn[r^2]nkqns - qgm[k^2]mgns\\ &- k[s^2]kqns - k[n^2]kqns - kn[g^2]nkqns - q[g^2]ns - qg[m^2]gns\\ &- qgmk[n^2]kmgns - qgmkn[r^2]nkmgns - qgmknr[m^2]rnkmgns\\ &- rnsemrn[krnse]mrnkqns;\end{aligned}$$

$$\begin{aligned}\delta_2[l^2e] &= l[le] - [l^2]e + [lrnse]mrnkmgqksrmgqlrm + emrl[q^2]lrm\\ &+ egqlskqgmkn[r^2]nkmgqksrmgqlrm + egqlskqgmk[n^2]kmgqksrmgqlrm\\ &+ egqlskqgm[k^2]mgqksrmgqlrm + egqlskqg[m^2]gqksrmgqlrm\\ &+ egqlskq[g^2]qksrmgqlrm + egqlsk[q^2]ksrmgqlrm\end{aligned}$$

$$+ egqls[k^2]srmgqlrm + egql[s^2]rmgqlrm + egq[lrmg]qlrm$$
$$+ e[g^2] + eg[q^2]g + e[m^2] + em[r^2]m + emr[l^2]rm$$
$$+ egqlskqgmknr[m^2]rnkmgqksrmgqlrm;$$

$$\delta_2[le^2] = l[e^2] - [le]e - rnsemrnkmgqksrmgqlr[me] - k[s^2]kl - lsnr[m^2]rns$$
$$- rnsemrnkmgqksrm[gqlnse]mrns - rnsemrnkmgq[kse]gql - k[n^2]kl$$
$$- rnsemrnkmg[qrnse]mrnkmgql - rnsemrnkm[gnse]gnkl - [m^2]l$$
$$- rnsemrn[krnse]mrnkl - rnse[mrnse]ml - rns[e^2]snrl - rn[s^2]nrl$$
$$- [r^2]l - [k^2]l - k[n^2]kl - kn[r^2]nkl - knr[m^2]rnkl - [q^2]l - q[g^2]ql$$
$$- qgm[k^2]mgql - qgmk[n^2]kmgql - qgmkn[r^2]nkmgql - r[n^2]rl$$
$$- rnsemrnkmgqksr[m^2]regql - rnsemrnkmgqks[r^2]egql - kn[g^2]nkl$$
$$- rnsemrnkmgqksrmgql[r^2]nsemrns - lsn[r^2]ns - ls[n^2]s - l[s^2]$$
$$- rnsemrnk[me]skl - qgmknr[m^2]rnkmgql - qg[m^2]gql;$$

$$\delta_2[l^2rnse] = l[lrnse] - [l^2]rnse + [le]gqlskqgmknrm$$
$$+ rnsemrnkmgqksrmgq[lrmg]qlskqgmknrm$$
$$+ rnsemrnkmgqksrmg[q^2]gmrskqgmknrm$$
$$+ rnsemrnkmgqksrm[g^2]mrskqgmknrm$$
$$+ rnsemrnkmgqksr[m^2]rskqgmknrm$$
$$+ rnsemrnkmgqk[s^2]kqgmknrm + rnsemrnkmgq[k^2]qgmknrm$$
$$+ rnsemrnkmg[q^2]gmknrm + rnsemrnkm[g^2]mknrm$$
$$+ rnsemrn[k^2]nrm + rnsemr[n^2]rm + rnsem[r^2]m + rnse[m^2]$$
$$+ rnsemrnkmgqksl[q^2]lskqgmknrm + rnsemrnkmgqks[l^2]skqgmknrm$$
$$+ rnsemrnkmgqks[r^2]skqgmknrm + rnsemrnk[m^2]knrm;$$

$$\delta_2[lrnse^2] = lrns[e^2] - [lrnse]e - egqlskqgmknr[me] - egqlskqgmkn[r^2]nsemrns$$
$$- egqlskqgmk[n^2]semrns - egqlskqgm[kse]mrns - lknr[m^2]rnkrns$$
$$- l[m^2]rns - egqlskq[ge]skrns - lk[n^2]krns - lkn[r^2]nkrns - l[k^2]rns$$
$$- egqlsk[qnse]gnkrns - lk[s^2]krns - egqls[krnse]mrnkmgqrns$$
$$- lk[n^2]krns - lkn[g^2]nkrns - egql[s^2]egqrns - rmgqlrm[m^2]gqrns$$
$$- rmgqlr[k^2]mgqrns - rmgqlrk[n^2]kmgqrns - rmgqlrkn[r^2]nkmgqrns$$
$$- rmgqlrknr[m^2]rnkmgqrns - egq[le]gqrns - egqlskqg[mrnse]mrnkrns$$
$$- skqgmkn[r^2]nkmgqkslrns - skqgmk[n^2]kmgqkslrns$$
$$- skqg[m^2]gqkslrns - skq[g^2]qkslrns - sk[q^2]kslrns - s[k^2]slrns$$
$$- eg[qrnse]mrnkmgqkslrns - e[gnse]gnslrns - rmgq[lrmg]qrns$$
$$- [e^2]lrns - [s^2]lrns - s[n^2]slrns - sn[g^2]nslrns - [r^2]lrns$$
$$- rm[g^2]mrlrns - rmg[q^2]gmrlrns - l[q^2]rns - r[m^2]rlrns$$
$$- skqgm[k^2]mgqkslrns - skqgmknr[m^2]rnkmgqkslrns;$$

$$\begin{aligned}
\delta_2[l^2qg] &= l[lqg] - [l^2]qg + [lrmg]qlrm + qgmrl[q^2]lrm + qgmr[l^2]rm \\
&\quad + qgm[r^2]m + qg[m^2]; \\
\delta_2[lqg^2] &= lq[g^2] - [lqg]g - rmgq[lrmg] - rmg[q^2]gmrlq \\
&\quad - rm[g^2]mrlq - r[m^2]rlq - [r^2]lq; \\
\delta_2[l^2rmg] &= l[lrmg] - [l^2]rmg + [lqg]mrlq + rmgqlr[m^2]rlq \\
&\quad + rmgql[r^2]lq + rmgq[l^2]q + rmg[q^2]; \\
\delta_2[lrmg^2] &= lrm[g^2] - [lrmg]g - qgmr[lqg] - qgm[r^2]mgqlrm \\
&\quad - qg[m^2]gqlrm - q[g^2]qlrm - [q^2]lrm;
\end{aligned}$$

$$\begin{aligned}
\delta_2[g^2qlnse] &= g[gqlnse] - [g^2]qlnse + [gmre]gqlsnrm + qlnsemrns[r^2]snrm \\
&\quad + qlnsemrns[q^2]lsnrm + qlnsemrns[l^2]snrm + qlnse[m^2] \\
&\quad + qlnsemrnsrm[g^2]mrsnrm + qlnsemrnsr[m^2]rsnrm \\
&\quad + qlnsemrn[s^2]nrm + qlnsemr[n^2]rm + qlnsem[r^2]m \\
&\quad + qlnsemrnsrmgq[lrmg]qlsnrm + qlnsemrnsrmg[q^2]gmrsnrm;
\end{aligned}$$

$$\begin{aligned}
\delta_2[gqlnse^2] &= gqlns[e^2] - [gqlnse]e - mregqlsnr[me] - mregqlsn[r^2]nsemrns \\
&\quad - mregqls[n^2]semrns - mregql[s^2]emrns - mregq[le]mrns \\
&\quad - mreg[qrnse]mrnkmgqksrmgqlns - mre[gnse]gnsrmgqlns \\
&\quad - m[r^2]mgqlns - [m^2]gqlns - mrsn[g^2]nsrmgqlns \\
&\quad - mr[s^2]rmgqlns - gql[r^2]ns - mrskqgmknr[m^2]rnkmgqksrmgqlns \\
&\quad - mrskqgmkn[r^2]nkmgqksrmgqlns - mrskqgmk[n^2]kmgqksrmgqlns \\
&\quad - mrskqgm[k^2]mgqksrmgqlns - mrskqg[m^2]gqksrmgqlns \\
&\quad - mrskq[g^2]qksrmgqlns - mrsk[q^2]ksrmgqlns - mrs[k^2]srmgqlns \\
&\quad - gqlr[m^2]rns - mr[e^2]rmgqlns - mrs[n^2]srmgqlns;
\end{aligned}$$

$$\begin{aligned}
\delta_2[g^2mre] &= g[gmre] - [g^2]mre + [gqlnse]mrnsrmgqlrm + mregq[lrmg]qlrm \\
&\quad + mre[g^2] + mremrl[q^2]lrm + mremr[l^2]rm + mrem[r^2]m + mre[m^2] \\
&\quad + mregqlsnr[m^2]rnsmgqlrm + mregqlsn[r^2]nsmgqlrm \\
&\quad + mregql[s^2]mgqlrm + mreg[q^2]g + mregqls[n^2]smgqlrm;
\end{aligned}$$

$$\begin{aligned}
\delta_2[gmre^2] &= gmr[e^2] - [gmre]e - qlnsemrnsrmgqlr[me] - qlnse[mrnse]gql \\
&\quad - qlnsemrnsrm[gqlnse]mrns - gmrsnr[m^2]rns - gmrsn[r^2]ns \\
&\quad - gmr[s^2] - qlnsemrnsr[m^2]regql - qlnsemrns[r^2]egql - [q^2]gmr \\
&\quad - qlns[e^2]snrmgql - qln[s^2]nrmgql - ql[n^2]rmgql - q[lrmg]ql \\
&\quad - gmrl[q^2]l - gmr[l^2] - qlnsemrnsrmgql[r^2]nsemrns - gmrs[n^2]s;
\end{aligned}$$

$$\begin{aligned}\delta_2[g^2(qlrm)^j qlnse] &= g[g(qlrm)^j qlnse] - [g^2](qlrm)^j qlnse\\
&+ (qlrm)^j qlnsemrnsrmgq[lrmg]qlsnrm\\
&+ (qlrm)^j qlnsemrns[l^2]snrm + (qlrm)^j qlnse[m^2]\\
&+ (qlrm)^j qlnsemrnsrm[g^2]mrsnrm\\
&+ (qlrm)^j qlnsemrns[r^2]snrm + (qlrm)^j qlnsemrn[s^2]nrm\\
&+ (qlrm)^j qlnsemr[n^2]rm + (qlrm)^j qlnsem[r^2]m\\
&+ [g(mrlq)^j mre]gqlsnrm + (qlrm)^j qlnsemrns[q^2]lsnrm\\
&+ (qlrm)^j qlnsemrnsrmg[q^2]gmrsnrm\\
&+ (qlrm)^j qlnsemrnsr[m^2]rsnrm;\end{aligned}$$

$$\begin{aligned}\delta_2[g^2(mrlq)^j mre] &= g[g(mrlq)^j mre] - [g^2](mrlq)^j mre\\
&+ (mrlq)^j mregq[lrmg]qlrm + (mrlq)^j mreg[q^2]g\\
&+ (mrlq)^j mremrl[q^2]lrm + (mrlq)^j mremr[l^2]rm\\
&+ (mrlq)^j mre[m^2] + (mrlq)^j mregqlsnr[m^2]rnsrmgqlrm\\
&+ (mrlq)^j mregqlsn[r^2]nsrmgqlrm + (mrlq)^j mrem[r^2]m\\
&+ (mrlq)^j mregql[s^2]rmgqlrm\\
&+ [g(qlrm)^j qlnse]mrnsrmgqlrm\\
&+ (mrlq)^j mregqls[n^2]srmgqlrm;\end{aligned}$$

$$\begin{aligned}\delta_2[g^2(qlrm)^j nse] &= g[g(qlrm)^j nse] - [g^2](qlrm)^j nse + [g(mrlq)^j e]sng\\
&+ (qlrm)^j nsegn[s^2]ng + (qlrm)^j nseg[n^2]g + (qlrm)^j nse[g^2];\end{aligned}$$

$$\begin{aligned}\delta_2[g^2(mrlq)^j e] &= g[g(mrlq)^j e] - [g^2](mrlq)^j e + [g(qlrm)^j nse]gns\\
&+ (mrlq)^j esn[g^2]ns + (mrlq)^j es[n^2]s + (mrlq)^j e[s^2];\end{aligned}$$

$$\begin{aligned}\delta_2[g(mrlq)^j e^2] &= g(mrlq)^j[e^2] - [g(mrlq)^j e]e - (qlrm)^j nse[gnse]\\
&- (qlrm)^j n[s^2]ng - (qlrm)^j[n^2]g\\
&- \sum_{i=1}^{j}(qlrm)^{j-i}[q^2]g(mrlq)^i - (qlrm)^j ns[e^2]sng\\
&- \sum_{i=1}^{j}(qlrm)^{j-i}q[lrmg](mrlq)^{i-1};\end{aligned}$$

$$\begin{aligned}\delta_2[g(qlrm)^j nse^2] &= g(qlrm)^j ns[e^2] - [g(qlrm)^j nse]e - (mrlq)^j esn[ge]\\
&- (mrlq)^j e[s^2]egns - (mrlq)^j[e^2]gns\\
&- \sum_{i=1}^{j}(mrlq)^{j-i}mr[lqg](qlrm)^{i-1}ns - (mrlq)^j es[n^2]segns\end{aligned}$$

$$- \sum_{i=1}^{j} (mrlq)^{j-i} m[r^2]mg(qlrm)^i ns$$

$$- \sum_{i=1}^{j} (mrlq)^{j-i} [m^2]g(qlrm)^i ns;$$

$$\begin{aligned}
\delta_2[g(qlrm)^j qlnse^2] = {} & g(qlrm)^j qlns[e^2] - [g(qlrm)^j qlnse]e \\
& - (mrlq)^j mregqlsnr[me] - (mrlq)^j mregqlsn[r^2]nsemrns \\
& - (mrlq)^j mregqls[n^2]semrns - (mrlq)^j mregql[s^2]emrns \\
& - (mrlq)^j mregq[le]mrns - (mrlq)^j gqlr[m^2]rns \\
& - (mrlq)^j mre[gnse]gnsrmgqlns - (mrlq)^j mr[e^2]rmgqlns \\
& - (mrlq)^j m[r^2]mgqlns - (mrlq)^j mrsn[g^2]nsrmgqlns \\
& - (mrlq)^j mrs[n^2]srmgqlns - (mrlq)^j mr[s^2]rmgqlns \\
& - (mrlq)^j mrskqgmknr[m^2]rnkmgqksrmgqlns \\
& - (mrlq)^j mrskqgmkn[r^2]nkmgqksrmgqlns \\
& - (mrlq)^j mrskqgmk[n^2]kmgqksrmgqlns - (mrlq)^j gql[r^2]ns \\
& - (mrlq)^j mrskqgm[k^2]mgqksrmgqlns - (mrlq)^j [m^2]gqlns \\
& - (mrlq)^j mrskqg[m^2]gqksrmgqlns - (mrlq)^j mrs[k^2]srmgqlns \\
& - (mrlq)^j mrskq[g^2]qksrmgqlns - (mrlq)^j mrsk[q^2]ksrmgqlns \\
& - \sum_{i=1}^{j} (mrlq)^{j-i} m[r^2]mg(qlrm)^i qlns \\
& - \sum_{i=1}^{j} (mrlq)^{j-i} [m^2]g(qlrm)^i qlns \\
& - \sum_{i=1}^{j} (mrlq)^{j-i} mr[lqg](qlrm)^{i-1} qlns \\
& - (mrlq)^j mreg[qrnse]mrnkmgqksrmgqlns;
\end{aligned}$$

$$\begin{aligned}
\delta_2[g(mrlq)^j mre^2] = {} & g(mrlq)^j mr[e^2] - [g(mrlq)^j mre]e \\
& - (qlrm)^j qlnsemrnsrmgql[r^2]nsemrns - g(mrlq)^j mrsn[r^2]ns \\
& - (qlrm)^j qlnsemrnsrm[gqlnse]mrns - g(mrlq)^j mrsnr[m^2]rns \\
& - g(mrlq)^j mrs[n^2]s - g(mrlq)^j mr[s^2] \\
& - (qlrm)^j qlnsemrns[r^2]egql - (qlrm)^j qlnse[mrnse]gql \\
& - (qlrm)^j qlns[e^2]snrmgql - (qlrm)^j qln[s^2]nrmgql \\
& - (qlrm)^j q[lrmg]ql - (qlrm)^j [q^2]gmr - g(mrlq)^j mrl[q^2]l \\
& - \sum_{i=1}^{j} (qlrm)^{j-i} [q^2]g(mrlq)^i mr \\
& - (qlrm)^j qlnsemrnsrmgqlr[me] - (qlrm)^j ql[n^2]rmgql
\end{aligned}$$

$$- g(mrlq)^j mr[l^2] - (qlrm)^j qlnsemrnsr[m^2]regql$$
$$- \sum_{i=1}^{j} (qlrm)^{j-i} q[lrmg](mrlq)^{i-1} mr;$$

$$\begin{aligned}\delta_2[lqg(mrlq)^j e] &= lq[g(mrlq)^j e] - [lqg](mrlq)^j e + l[q^2]lrm(qlrm)^{j-1} nsegns \\ &+ [l^2]rm(qlrm)^{j-1} nsegns - rmgqlr[m^2]rlq(mrlq)^{j-1} e \\ &- rmgql[r^2]lq(mrlq)^{j-1} e - rmgq[l^2]q(mrlq)^{j-1} e \\ &- rm[g(mrlq)^{j-1} e] - rmg[q^2](mrlq)^{j-1} e;\end{aligned}$$

$$\begin{aligned}\delta_2[lrmg(mrlq)^j e] &= lrm[g(mrlq)^j e] - [lrmg](mrlq)^j e - [q^2]lrm(qlrm)^j nsegns \\ &- q[g(mrlq)^{j+1} e];\end{aligned}$$

$$\begin{aligned}\delta_2[lqg(qlrm)^j nse] &= lq[g(qlrm)^j nse] - [lqg](qlrm)^j nse - r[m^2]rlq(mrlq)^j esng \\ &- rm[g(qlrm)^{j+1} nse] - [r^2]lq(mrlq)^j esng;\end{aligned}$$

$$\begin{aligned}\delta_2[lrmg(qlrm)^j nse] &= lrm[g(qlrm)^j nse] - [lrmg](qlrm)^j nse \\ &- qgmr[l^2]rm(qlrm)^{j-1} nse - qgm[r^2]m(qlrm)^{j-1} nse \\ &- qg[m^2](qlrm)^{j-1} nse - q[g(qlrm)^{j-1} nse] \\ &+ l[r^2]lq(mrlq)^{j-1} esng + [l^2]q(mrlq)^{j-1} esng \\ &- qgmrl[q^2]lrm(qlrm)^{j-1} nse + lr[m^2]rlq(mrlq)^{j-1} esng;\end{aligned}$$

$$\begin{aligned}\delta_2[lrmg(mrlq)^j mre] &= lrm[g(mrlq)^j mre] - [lrmg](mrlq)^j mre \\ &- q[g(mrlq)^{j+1} mre] - [q^2]lrm(qlrm)^j qlnsemrnsrmgqlrm;\end{aligned}$$

$$\begin{aligned}\delta_2[lqg(mrlq)^j mre] &= lq[g(mrlq)^j mre] - [lqg](mrlq)^j mre - rm[g(mrlq)^{j-1} mre] \\ &+ l[q^2]lrm(qlrm)^{j-1} qlnsemrnsrmgqlrm \\ &+ [l^2]rm(qlrm)^{j-1} qlnsemrnsrmgqlrm \\ &- rmgql[r^2]lq(mrlq)^{j-1} mre - rmgq[l^2]q(mrlq)^{j-1} mre \\ &- rmgqlr[m^2]rlq(mrlq)^{j-1} mre - rmg[q^2](mrlq)^{j-1} mre;\end{aligned}$$

$$\begin{aligned}\delta_2[lqg(qlrm)^j qlnse] &= lq[g(qlrm)^j qlnse] - [lqg](qlrm)^j qlnse \\ &- rm[g(qlrm)^{j+1} qlnse] - [r^2]lq(mrlq)^j mregqlsnrm \\ &- r[m^2]rlq(mrlq)^j mregqlsnrm;\end{aligned}$$

$$\begin{aligned}\delta_2[lrmg(qlrm)^j qlnse] &= lrm[g(qlrm)^j qlnse] - [lrmg](qlrm)^j qlnse \\ &- qgmrl[q^2]lrm(qlrm)^{j-1} qlnse \\ &- qgmr[l^2]rm(qlrm)^{j-1} qlnse - qgm[r^2]m(qlrm)^{j-1} qlnse \\ &- qg[m^2](qlrm)^{j-1} qlnse + lr[m^2]rlq(mrlq)^{j-1} mregqlsnrm \\ &+ [l^2]q(mrlq)^{j-1} mregqlsnrm - q[g(qlrm)^{j-1} qlnse] \\ &+ l[r^2]lq(mrlq)^{j-1} mregqlsnrm;\end{aligned}$$

$$\begin{aligned}\delta_2[lqge] &= lq[ge] - [lqg]e + l[qnse]gns - rmgqlr[me] + [lrnse]mrnkmgqks\\ &- rmgql[r^2]nsemrns - rm[gqlnse]mrns - r[m^2]regql - [r^2]egql\\ &- egqlsn[r^2]ns - egqls[n^2]s - egql[s^2] + egqlskqgmknr[m^2]rnkmgqks\\ &+ egqlskqgmkn[r^2]nkmgqks + egqlskqgmk[n^2]kmgqks\\ &+ egqlskqg[m^2]gqks + egqlskq[g^2]qks + egqlsk[q^2]ks + egqls[k^2]s + egql[s^2]\\ &+ egqls[n^2]s + egqlsn[g^2]ns - egqlsnr[m^2]rns + egqlskqgm[k^2]mgqks;\end{aligned}$$

$$\delta_2[lqgnse] = lq[gnse] - [lqg]nse - rm[gqlrmnse] - r[m^2]rlqesng - [r^2]lqesng;$$

$$\begin{aligned}\delta_2[lqgqlnse] &= lq[gqlnse] - [lqg]qlnse - rm[gqlrmqlnse] - r[m^2]rlqmregqlsnrm\\ &- [r^2]lqmregqlsnrm;\end{aligned}$$

$$\begin{aligned}\delta_2[lqgmre] &= lq[gmre] - [lqg]mre + l[q^2]lnsemrnsrmgqlrm\\ &- rmgqlr[m^2]re - rmgql[r^2]e - rmgq[le] + [l^2]nsemrnsrmgqlrm\\ &- rm[gnse]gnsrmgqlrm - r[me]rmgqlrm - [r^2]nsemrnsrmgqlrm\\ &- nsemrnkqgmkn[r^2]nkmgqksrmgqlrm\\ &- nsemrnkqgm[k^2]mgqksrmgqlrm - nsemrnkqg[m^2]gqksrmgqlrm\\ &- nsemrnkq[g^2]qksrmgqlrm - nsemrnk[q^2]ksrmgqlrm\\ &- nsemrnkqgmknr[m^2]rnkmgqksrmgqlrm\\ &- nsemrn[n^2]srmgqlrm - nsemrns[s^2]rmgqlrm\\ &- rmg[qrnse]mrnkmgqksrmgqlrm - nsemrn[k^2]srmgqlrm\\ &- nsemrnkqgmk[n^2]kmgqksrmgqlrm - nsemr[g^2]nsrmgqlrm;\end{aligned}$$

$$\delta_2[lrmge] = lrm[ge] - [lrmg]e - q[gmrlqe] - [q^2]lrmnsegns;$$

$$\begin{aligned}\delta_2[lrmgnse] &= lrm[gnse] - [lrmg]nse + lr[me]sng - qgmrl[qnse]\\ &- qgmr[lrnse]mrnkmgqkng - q[gmre]gqlsng\\ &- lnsemrnsrmgq[lrmg]qlsng - lnsemrnsl[q^2]lsng\\ &- lnsemrn[s^2]ng - lnsemr[n^2]g - lnsemrns[r^2]sng\\ &- lnsemrnsrm[g^2]mrsng - lnsemrnsrmg[q^2]gmrsng\\ &- lnsemrkqgmknr[m^2]rnkmgqkng\\ &- lnsemrkqgmk[n^2]kmgqkng - lnsemrkqgm[k^2]mgqkng\\ &- lnsemrkqg[m^2]gqkng - lnsemrk[q^2]kng - lnsemr[k^2]ng\\ &+ lnsemrn[s^2]ng + lnsemr[n^2]g + l[r^2]nsemrg\\ &- lnsemrnsr[m^2]rsng - lnsemrns[l^2]sng - [q^2]lnsemrg\\ &- lnsemrkqgmkn[r^2]nkmgqkng - lnsemrkq[g^2]qkng;\end{aligned}$$

$$\begin{aligned}\delta_2[lrmgmre] &= lrm[gmre] - [lrmg]mre - q[gmrlqmre]\\ &- [q^2]lrmqlnsemrnsrmgqlrm;\end{aligned}$$

$$
\begin{aligned}
\delta_2[lrmgqlnse] &= lrm[gqlnse] - [lrmg]qlnse + lr[m^2]regqlsnrm \\
&+ [le]gqlsnrm + rnsemrnkmgqksrmgq[lrmg]qlsnrm \\
&+ rnsemrnkmgqksrmg[q^2]gmrsnrm + l[r^2]egqlsnrm \\
&+ rnsemrnkmgqksr[m^2]rsnrm + rnsemrnkmgqks[r^2]snrm \\
&+ rnsemrnkmgqk[s^2]nrm + rnsemrnkmgqksl[q^2]lsnrm \\
&+ rnsemrnkmgqks[l^2]snrm - qgmrl[q^2]lnse - qgmr[l^2]nse \\
&- q[ge]snrm - [qnse]grm - rnsemrnkmgqkn[g^2]rm \\
&- rnsemrnkmgqk[s^2]nrm - rnsemrnkmgqkn[n^2]rm \\
&+ rnsemrnkmgqksrm[g^2]mrsnrm - qg[mrnse].
\end{aligned}
$$

6. Application of Anick resolution on Γ_5^4

Let $M = \Bbbk$ be a one-dimensional bimodule over $\Lambda = \Bbbk\Gamma_5^4$ where char $\Bbbk = 0$. Let us assume the bimodule is symmetric: $x1 = 1x = \epsilon(x)$, where $x \in A_2$ are the generators of Γ_5^4, $1 \in \Bbbk$ is the generator of M, and $\epsilon : \{e, g, k, l, m, n, q, r, s\} \to \{1, -1\}$ is an arbitrary function. By Lemma 1 an arbitrary function ϵ like that completely determines a 1-dimensional bimodule over $\Bbbk\Gamma_5^4$ denoted by M_ϵ.

Let us fix a function ϵ and denote the bimodule M obtained as above by M_ϵ. For $i \geq 0$, consider the Λ-bilinear map $\tau_i : B_{(i)} \to \Bbbk V^{(i)}$ defined by

$$\tau_i : \lambda_1 \otimes v \otimes \lambda_2 \mapsto \epsilon(\lambda_1)\epsilon(\lambda_2)v, \quad v \in V^{(i)},\ \lambda_1, \lambda_2 \in \Lambda.$$

Then the map δ_m induces a linear map $\hat{\delta}_m$ from $\Bbbk V^{(m)}$ to $\Bbbk V^{(m-1)}$ such that

$$\tau_{m-1}\delta_m = \hat{\delta}_m \tau_m.$$

The corresponding conjugate map $\hat{\delta}_n^*$ acts from the space $(\Bbbk V^{(m-1)})^*$ to $(\Bbbk V^{(m)})^*$. Since

$$\mathrm{Hom}_{\Lambda-\Lambda}(B_{(m)}, M) \simeq \mathrm{Hom}(\Bbbk V^{(m)}, M) \simeq (\Bbbk V^{(m)})^*,$$

we have

$$Z^m(\Lambda, M) \simeq \mathrm{Ker}\, \hat{\delta}_m^*, \quad \beta^m(\Lambda, M) \simeq \mathrm{Im}\, \hat{\delta}_{m-1}^*.$$

This is straightforward to check that $\mathrm{rank}\, \hat{\delta}_1 = 9 = |V^{(0)}|$. It remains to find $\dim \mathrm{Ker}\, \hat{\delta}_2^*$. According to the Fredholm principle, $\dim \mathrm{Ker}\, \hat{\delta}_2^* = \dim(\mathrm{Im}\, \hat{\delta}_2)^\perp = \mathrm{codim}_{V^{(1)}} \mathrm{Im}\, \hat{\delta}_2$. By definition,

$$H^m := H^m(\Bbbk\Gamma_5^4, M) \simeq \mathrm{Ker}\, \hat{\delta}_m^* / \mathrm{Im}\, \hat{\delta}_{m-1}^*.$$

Hence we obtain

$$\dim H^2 = \mathrm{codim}_{\Bbbk V^{(1)}} \mathrm{Im}\, \hat{\delta}_2 - 9. \tag{2}$$

The similar formula for $\dim H^1$ implies the following (quite expectable)

Corollary 1. *The first Hochschild cohomology group $H^1(\Bbbk\Gamma_5^4, M_\epsilon)$ is trivial for every ϵ.*

In order to find the dimension of the second Hochschild cohomology group $H^2(\Bbbk\Gamma_5^4, M)$ for $M = M_\epsilon$ one needs to calculate $\mathrm{codim}_{\Bbbk V^{(1)}} \mathrm{Im}\, \hat{\delta}_2$ by means of the formulae from Section 5.

Corollary 2. *The second Hochschild cohomology group of $\Bbbk\Gamma_5^4$ with trivial coefficients is 5-dimensional in case $\epsilon(x) = 1$ for every $x \in A_2$.*

Proof. Suppose $x1 = 1x = \epsilon(x) = 1$ for all $x \in A_2$. Let us calculate $\hat{\delta}_2$. For example,

$$\hat{\delta}_2(q^2nse) = \hat{\delta}_2(q^2rnse) = 2[g^2] + 2[k^2] + 2[m^2] + 2[n^2] + [r^2] + [qrnse] + [qnse],$$

Proceed in the same way with all formulas for δ_2 to get a matrix of the map $\hat{\delta}_2$ and use elementary transformations to reduce it (partially) to the row echelon form.

As a result, we obtain the following vectors from $\Bbbk V^{(1)}$ (corresponding to the rows of the echelon form) to span $\operatorname{Im}\hat{\delta}_2$:

$$\begin{aligned}
e_1 &= 2[g^2] + 2[k^2] + 2[m^2] + 2[n^2] + [r^2] + [qrnse] + [qnse],\\
e_2 &= [l^2] + 2[m^2] + [n^2] + 2[q^2] + 2[r^2] + [s^2] + [lrmg] + [gqlnse] + [gmre]\\
e_3 &= 2[m^2] + [n^2] + [r^2] + [kse] + [krnse],\\
e_4 &= -[n^2] - [r^2] - [s^2] - [me] - [mrnse],\\
e_5 &= 3[q^2] + 2[r^2] + \frac{5}{2}[s^2] + [le] + [lrnse] - \frac{1}{2}[kse] - \frac{1}{2}[krnse] + \frac{3}{2}[me]\\
&\quad + \frac{3}{2}[mrnse] - [qrnse] - [qnse] + [lrmg],\\
e_6 &= [r^2] - [ge] - [gnse] + [me] + [mrnse],\\
e_7 &= -\frac{7}{6}[s^2] - \frac{4}{3}[ge] - \frac{7}{3}[gnse] - \frac{2}{3}[le] + \frac{1}{3}[lrnse] + \frac{5}{6}[kse] + \frac{5}{6}[krsne] + \frac{5}{6}[me]\\
&\quad + \frac{11}{6}[mrnse] - \frac{1}{3}[qrnse] + \frac{2}{3}[qnse] + \frac{1}{3}[lrmg] - [gqlnse],\\
e_8 &= \frac{5}{7}[ge] + [gnse] - \frac{1}{7}[le] - \frac{3}{7}[lrnse] - \frac{4}{7}[kse] - \frac{4}{7}[krsne] - \frac{4}{7}[me] - \frac{6}{7}[mrnse]\\
&\quad + \frac{3}{7}[qrnse] + \frac{1}{7}[qnse] + [lqg] + \frac{4}{7}[lrmg] + \frac{2}{7}[gqlnse],\\
e_9 &= \frac{8}{5}[gnse] + \frac{1}{5}[le] - \frac{2}{5}[lrnse] - \frac{1}{5}[kse] - \frac{1}{5}[krnse] - \frac{1}{5}[me] - \frac{4}{5}[mrnse]\\
&\quad + \frac{2}{5}[qrnse] - \frac{1}{5}[qnse] - \frac{2}{5}[lqg] + \frac{1}{5}[lrmg] + \frac{3}{5}[gqlnse] - [gqlrmnse],\\
e_{10} &= \frac{1}{8}[le] - \frac{1}{4}[lrnse] - \frac{1}{8}[kse] - \frac{1}{8}[krnse] - \frac{1}{8}[me] - \frac{1}{2}[mrnse] + \frac{1}{4}[qrnse]\\
&\quad - \frac{1}{8}[qnse] - \frac{1}{4}[lqg] + \frac{1}{8}[lrmg] + \frac{11}{8}[gqlnse] + \frac{3}{8}[gqlrmnse] - [qdlrmqlnse],\\
e_{11} &= [lrnse] + [kse] + [krnse] + [me] + 2[mrnse] - [qrnse] - [lqg] - [lrmg]\\
&\quad - 3[gqlnse] - 2[gqlrmnse] + 2[gqlrmqlnse] - [gmrlqe],\\
e_{12} &= [gqlnse] + [gmre] + [gqlrmqlnse] + [gmrlqmre],\\
e_{13}^{(j)} &= [gmre] - [gmrlqmre] - [g(mrlq)^{j-1}e] + [g(mrlq)^j e],\\
e_{14}^{(j)} &= -[gqlrmnse] - [gmrlqe] + [g(mrlq)^j e] + [g(qlrm)^j nse],\\
e_{15}^{(j)} &= -[gqlrmqlnse] - [gmrlqmre] + [g(mrlq)^j mre] + [g(qlrm)^j qlnse],\\
e_{16}^{(j)} &= [gmrlqe] - 2[g(mrlq)^{j-1}e] + [g(mrlq)^j e],\\
e_{17}^{(j)} &= [gmrlqmre] - 2[g(mrlq)^{j-1}e] + 2[g(mrlq)^j e] - [g(mrlq)^j mre],
\end{aligned}$$

$$e_{18}^{(j)} = [g(mrlq)^{j-1}e] - [g(mrlq)^{j-1}mre] - [g(mrlq)^j e] + [g(mrlq)^j mre],$$
$$e_{19}^{(j)} = -[g(mrlq)^{j-1}mre] - [g(qlrm)^{j-1}qlnse] + [g(mrlq)^j mre] + [g(qlrm)^j qlnse],$$
$$e_{20}^{(j)} = [g(qlrm)^{j-1}qlnse] - [g(mrlq)^{j-1}nse] - [g(qlrm)^j qlnse] + [g(mrlq)^j nse],$$
$$e_{21}^{(j)} = [g(mrlq)^{j-1}nse] + [g(mrlq)^j e] - [g(qlrm)^j nse] - [g(mrlq)^{j+1}e],$$
$$e_{22}^{(j)} = -[g(mrlq)^j e] + [g(mrlq)^j mre] + [g(mrlq)^{j+1}e] - [g(mrlq)^{j+1}mre],$$
$$e_{23}^{(j)} = [g(mrlq)^j mre] + [g(qlrm)^j qlnse] - [g(mrlq)^{j+1}mre] - [g(qlrm)^{j+1}qlnse],$$
$$e_{24}^{(j)} = -[g(qlrm)^j qlnse] + [g(qlrm)^j nse] + [g(qlrm)^{j+1}qlnse] - [g(qlrm)^{j+1}nse].$$

Here $j \geq 2$. Now we may choose a linear basis of $\Bbbk V^{(1)}/\mathrm{Im}\,\hat{\delta}_2$. Note that the classes of $e^2, k^2, kse, krnse, me, mrnse, qrnse, qnse, lqg, lrmg$ must belong to the desired basis. Moreover, if we compare e_{13} and e_{16} for j and $j+1$ then we conclude that

$$g(mrlq)^j e \equiv g(mrlq)^{j-1}e \pmod{\mathrm{Im}\,\hat{\delta}_2}.$$

Then proceed in the same way with e_{14} to get

$$g(qlrm)^j nse \equiv g(qlrm)^{j-1}nse \pmod{\mathrm{Im}\,\hat{\delta}_2}.$$

Similarly, e_{18} and e_{20} provide us with the same relations for $g(mrlq)^j mre$ and $g(qlrm)^j qlnse$. One may easily see that there are no more relations for the elements of $V^{(1)}$, so the codimension of $\mathrm{Im}\,\hat{\delta}_2$ is 14. It follows from (2) that $\dim H^2 = 5$. $\square$

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