S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

*Том 19, стр. 248–258 (2022)* УДК 510.5

DOI 10.33048/semi.2022.19.019 MSC 03F60

# ON THE MAXIMALITY OF DEGREES OF METRICS UNDER COMPUTABLE REDUCIBILITY

R. KORNEV

Abstract. We study the semilattice $\mathcal{CM}_c(\mathbf{X})$ of degrees of computable metrics on a Polish space $\mathbf{X}$ under computable reducibility. It is proved that this semilattice does not have maximal elements if $\mathbf{X}$ is a noncompact space. It is also shown that the degree of the standard metric on the unit interval is maximal in the respective semilattice.

**Keywords:** computable metric space, Cauchy representation, reducibility of representations, computable analysis

The present paper is a sequel to the article [1] in which the author started the investigation of the first-order properties of the degree structure of metrics on a Polish space under computable reducibility $\leq_c$. All necessary definitions, notations and preliminary results can be found there, below we briefly recall some of them.

We regard Polish spaces as structures of the kind $\mathbf{X} = (X, \tau, W, \nu)$, where $(X, \tau)$ is a Polish space in the classical sense (i. e., a separable completely metrizable space), $W$ is some countable dense subset of $X$, and $\nu$ is a numbering of $W$. Under this approach, it is natural to study a reducibility of complete metrics on $\mathbf{X}$ induced by computable reducibility of their respective Cauchy representations.

Recall that the *Cauchy representation* of a Polish metric space $(X, \rho, W, \nu)$ with a distinguished dense subset $W$ and its numbering $\nu$ is a partial mapping $\delta_\rho$ from the Baire space $\omega^\omega$ onto $X$ that puts points $x \in X$ into correspondence with sequences of elements of $W$ quickly converging to $x$, in other words, for $f \in \omega^\omega$ and $x \in X$,

$$\delta_\rho(f) = x \text{ if } w_{f(n)} \to x \text{ and } \rho(w_{f(n)}, x) < 2^{-n} \text{ for all } n,$$

Kornev, R., On the maximality of degrees of metrics under computable reducibility.



The reported study was funded by RFBR and JSPS, project number 20-51-50001.

*Received April, 7, 2021, published April, 19, 2022.*

where $w_k = \nu k$ is the $k$th point in $W$. Every $f \in \omega^\omega$ with the above property is called a *Cauchy name* for $x$.

More generally, a *representation* of a set $X$ is a partial surjection $\delta\colon \omega^\omega \to X$. This notion generalizes the notion of a numbering of a countable set. Theory of representations and computability theory on represented spaces were studied by Kreitz and Weihrauch (see [2, 3]). Computable reducibility of representations was introduced in [2], it is an analogue of the notion of reducibility of numberings. Representation $\delta_1$ of a set $X$ is said to be *computably reducible* to a representation $\delta_2$ of $X$, $\delta_1 \leq_c \delta_2$, if there is a Turing functional $\Phi_e$ such that

$$\delta(f) = \delta' \circ \Phi_e(f) \text{ for } f \in \mathrm{dom}(\delta).$$

If $\delta_X$ and $\delta_Y$ are representations of sets $X$ and $Y$, then we say that a partial function $F\colon X \to Y$ is $(\delta_X, \delta_Y)$-*computable* if there is a Turing functional $\Phi_e$ such that

$$F \circ \delta_X(f) = \delta_Y \circ \Phi_e(f) \text{ for } f \in \mathrm{dom}(F \circ \delta_X).$$

It is clear that for representations $\delta_1, \delta_2$ of $X$ it holds $\delta_1 \leq_c \delta_2$ iff the identity function $\mathrm{id}_X$ is $(\delta_1, \delta_2)$-computable.

Let $\mathbf{X} = (X, \tau, W, \nu)$ be a Polish space. The set of all complete metrics on $X$ compatible with topology $\tau$ is denoted by $M(\mathbf{X})$. Having fixed $W$ and $\nu$, we can define the notion of computable reducibility of metrics on $\mathbf{X}$: for $\rho_1, \rho_2 \in M(\mathbf{X})$, we say that $\rho_1 \leq_c \rho_2$ if $\delta_{\rho_1} \leq_c \delta_{\rho_2}$; in other words, reducibility of metrics is induced by reducibility of their Cauchy representations. Computable reducibility of metrics in $M(\mathbf{X})$ leads to the degree structure $\mathcal{M}_c(\mathbf{X})$. The subordering of $\mathcal{M}_c(\mathbf{X})$ consisting of $c$-degrees of computable metrics on $\mathbf{X}$ is denoted by $\mathcal{CM}_c(\mathbf{X})$.

In [1] it was shown that $\mathcal{CM}_c(\mathbf{X})$ is a lower semilattice for any space $\mathbf{X}$. It was also proved that the degree of a metric $\rho$ such that the computable space $(X, \rho, W, \nu)$ contains a computable limit point is not minimal in $\mathcal{CM}_c(\mathbf{X})$. Main result of the present article is dual to the second mentioned result: we show that the degree of any metric on a noncompact space $\mathbf{X}$ is not maximal in $\mathcal{CM}_c(\mathbf{X})$. In other words, our main result can be stated as follows.

**Theorem 1.** *Let* $\mathbf{X}$ *be a noncompact Polish space. Then* $\mathcal{CM}_c(\mathbf{X})$ *has no maximal elements.*

It is worth noting that this result is a direct generalization of a result from [4] that shows that the degree of the standard metric $\rho_R(x, y) = |x - y|$ on the space $\mathbf{R}$ of real numbers is not maximal in the ordering of $c$-degrees of computable metrics on $\mathbf{R}$. Theorem 1 can be fully relativized to the case of metrics of arbitrary complexity. In other words, the following holds.

**Theorem 2.** *Let* $\mathbf{X}$ *be a noncompact Polish space. Then* $\mathcal{M}_c(\mathbf{X})$ *has no maximal elements.*

Although this result is obtained by a straightforward relativization, the very possibility of this relativization is a nontrivial fact which is based on a few important observations made throughout the proof of Theorem 1.

The following theorem is related to lattice embeddings into $\mathcal{CM}_c(\mathbf{X})$ and is again a direct generalization of a result from [4].

**Theorem 3.** *Let* $\rho$ *be a computable metric on a noncompact space* $\mathbf{X}$*. Then the countable atomless Boolean algebra* $\mathrm{Int}(1 + \eta)$ *embeds into* $\mathcal{CM}_c(\mathbf{X})$ *above* $\deg_c(\rho)$ *with preservation of joins and meets.*

Theorem 1 does not necessarily hold in compact spaces, as the following counterexample shows: it turns out that $c$-degree of the standard metric on the unit interval $\mathbf{I}$ is maximal in $\mathcal{CM}_c(\mathbf{I})$. More formally, consider the Polish space $\mathbf{I} = (I, \tau_I, \mathbb{Q}, \nu_{\mathbb{Q}})$, where $\tau_I$ is the usual topology on the unit interval $I = [0,1]$, $\mathbb{Q}$ is the set of rational numbers in $I$ and $\nu_{\mathbb{Q}}$ is a Gödel numbering of $\mathbb{Q}$. Let $\rho_R$ also denote the restriction of the usual real metric to the unit interval. Then the following result holds.

**Proposition 1.** $\deg_c(\rho_R)$ *is a maximal element of* $\mathcal{CM}_c(\mathbf{I})$.

However, the following proposition shows that $\deg_c(\rho_R)$ is not a maximal element of $\mathcal{M}_c(\mathbf{I})$, that is, there exists a noncomputable metric strictly above $\rho_R$.

**Proposition 2.** *There exists a* $\mathbf{0}'$*-computable metric* $\rho^*$ *on* $\mathbf{I}$ *such that* $\rho_R <_c \rho^*$.

The paper is organized as follows. Section 1 contains the proofs of Theorems 1 and 2. Section 2 consists of the proof of Theorem 3. In the last section we prove the two results regarding the unit interval.

## 1. Proof of Theorem 1

1.1. **Proof idea.** Suppose $\mathbf{X} = (X, \tau, W, \nu)$ is a noncompact space and let $\rho \in M(\mathbf{X})$ be a computable metric on $\mathbf{X}$. We need to construct a computable metric $\rho' \in M(\mathbf{X})$ such that $\rho' >_c \rho$. More specifically, metric $\rho'$ should satisfy the following requirements for $e \in \omega$:

$\mathcal{R}_e$: $\rho' \not\leq_c \rho$ via $\Phi_e$,
$\mathcal{S}$: $\rho \leq_c \rho'$.

Our proof is based on a generalization of the construction of Theorem 1 from [4]. In that theorem the analogues of the above requirements were satisfied to obtain a computable metric (indeed, countably many pairwise nonequivalent computable metrics) strictly above the standard metric $\rho_R$ on the space of real numbers. Later, in Theorem 4.1 of [1], it was shown that the strategy for negative requirements $\mathcal{R}_e$ can be generalized to the case of an arbitrary computable metric space with a computable limit point.

Let us briefly discuss the strategy for an isolated requirement $\mathcal{R}_e$. Recall that a metric space is compact if and only if it is complete and totally bounded. Since we only work with complete metric spaces, it is clear that the noncompact space $(X, \rho)$ must not be totally bounded, that is, there exists a rational number $r \in (0, \frac{1}{3})$ such that there is no finite $3r$-net in $(X, \rho)$. Since we do not know this $r$ in advance, our proof will be non-uniform. For the rest of the proof, fix such a number $r$. Fix also a $k \in \omega$ such that $2^{-k} < r$. For natural numbers $a, b, n$, denote

$$\bar{a} = (a, \dots, a, \dots) \in \omega^\omega, \;\; a^n\bar{b} = (a, \dots, a, b, b, \dots) \in \omega^\omega \text{ ($a$ repeated $n$ times).}$$

Now back to the strategy for $\mathcal{R}_e$. We pick a special point $y = w_a \in W$ and wait until $\Phi_e(\bar{a})(k+1) \downarrow = u$. When it happens, we pick another special point $x = w_b$ such that $\rho(x, w_u) > 3r$ (it is possible by the choice of $r$). Consider the sequence $a^p\bar{b} \in \omega^\omega$, where $p = \varphi_e(\bar{a})(k+1)$. By the Use Principle we have $\Phi_e(a^p\bar{b})(k+1) = \Phi_e(\bar{a})(k+1) = u$, thus, if we arrange our construction so that $a^p\bar{b}$ is a $\rho'$-name for $w_b$, then $\rho(w_u, w_b) > 3r > r$ implies that $\Phi_e(a^p\bar{b})$ is not a $\rho$-name for $w_b$, which means that $\mathcal{R}_e$ is satisfied. In order to make $a^p\bar{b}$ a $\rho'$-name for $w_b$, it is sufficient to ensure that $\rho'(x, y) < 2^{-p+1}$. This can be done by means of a continuous deformation of a small neighbourhood $B_\rho(x,r)$ of $x$ that makes $x$ close to $y$. A construction of such a deformation is described in [1]: we embed the metric space $(X,\rho,W)$ into the Banach space $\ell^\infty$ via the Fréchet embedding $F$ defined by

$$(F(x))_i = \rho(x,w_i) - \rho(w_i,w_0),$$

then the deformation in question is given by

$$\Gamma(z) = \left(\tfrac{r-\rho(x,z)}{r}\,h\right) ^\frown \left(F(y) - \tfrac{\rho(x,z)}{r}(F(y)-F(z))\right)$$

for $z \in B_\rho(x,r)$, where $h < 2^{-p+1}$ and $^\frown$ stands for the concatenation of a real and an infinite sequence of reals. Mapping $\Gamma$ can be extended to a map $\gamma\colon X \to \ell^\infty$ by letting $\gamma(z) = 0 ^\frown F(z)$ for $z \notin B_\rho(x,r)$. Letting $\rho_1(z,v) = \|\gamma(z)-\gamma(v)\|$, we obtain a deformed metric $\rho_1$ carrying the topology $\tau$ (see Proposition 5.1 of [1]) and such that $\rho_1(x,y) = h$. Thus, $\rho_1$ satisfies $\mathcal{R}_e$. The following formulas express $\rho_1$ in terms of $\rho$; these are exactly Formulas (5.1)–(5.3) of [1].

Let $z,v$ be points of $X$; assume that $\rho(x,z) \leqslant \rho(x,v)$. Then, if $\rho(x,z) < r$ and $\rho(x,v) \geqslant r$, we have

$$\rho_1(z,v) = \max\left(\tfrac{r-\rho(x,z)}{r}\,h,\ \tfrac{\rho(x,z)}{r}\rho(z,v) + \tfrac{r-\rho(x,z)}{r}\rho(v,y)\right); \tag{1.1}$$

if $\rho(x,v) < r$,

$$\rho_1(z,v) = \max\left(\tfrac{\rho(x,v)-\rho(x,z)}{r}\,h,\ \tfrac{\rho(x,z)}{r}\rho(z,v) + \tfrac{\rho(x,v)-\rho(x,z)}{r}\rho(v,y)\right); \tag{1.2}$$

and if $\rho(x,z) \geqslant r$, then

$$\rho_1(z,v) = \rho(z,v). \tag{1.3}$$

Metric $\rho'$ satisfying all requirements $\mathcal{R}_e$ will be a pointwise limit of metrics $\rho_s$ defined at stages of the construction. At each stage $s$ we also output a finite set $A_s \subseteq W$ such that:

(1) $\rho_s(z,v) = \rho_{s+1}(z,v) = \ldots = \rho'(z,v)$ for all $z,v \in A_s$,
(2) $A_s \subseteq A_t$ for $s \leqslant t$,
(3) $\bigcup_{i\in\omega} A_s = W$.

This will imply that $\rho'$ is computable: informally speaking, we can compute the distance $\rho'(z,v)$ between two given special points $z$ and $v$ as soon as they both are contained in a set $A_s$.

Strategy for the global positive requirement $\mathcal{S}$ is also the same as in [4]. In a nutshell, strategy from [4] is based on the fact that the space $(\mathbf{R},\rho_R)$ is unbounded, thus we can choose balls of the same radius to be deformed in each $\mathcal{R}_n$-strategy, which gives us an algorithm of $c$-reduction of $\rho_R$ to the metric under construction. The same idea works here: by the choice of $r$, there exists an infinite sequence of nonintersecting balls $B_n$ of the same radius $r$. These balls can be used in the respective $\mathcal{R}_n$-strategies. The fact that the balls $B_n$ possess the same radius will permit us to prove Lemma 4 that asserts that $\rho'(z,v) \leqslant C\cdot\rho(z,v)$ for all $z,v \in X$ and some real number $C$. From the following basic lemma it then follows that $\rho \leq_c \rho'$.

**Lemma 1** ([5])**.** *Let $d,d' \in M(\mathbf{X})$. If there is a $C > 0$ such that $d'(x,y) \leqslant C\cdot d(x,y)$ for all $x,y \in X$, then $d \leq_c d'$.*

1.2. **Analytical reasoning.** In this subsection we explain the analytical reasoning behind the construction of $\rho'$ that is very similar to the one in Section 5 of [1]. Throughout the subsection, we often refer to Section 5 of [1] and use various results from there.

First of all, without loss of generality we will assume that the metric $\rho$ is bounded by 1: if this is not the case, then letting $\hat{\rho}(x,y) = \min(\rho(x,y),1)$ produces a computable metric $\hat{\rho}$ that is $c$-equivalent to $\rho$. This is an important detail. Later on we will see why it is convenient to bound the metric from above.

By the choice of the number $r$, there exists an infinite sequence $(\tilde{x}_n)_{n\in\omega}$ such that $\rho(\tilde{x}_n,\tilde{x}_m) > 3r$ for $n \neq m$. Moreover, such a sequence can be found effectively, given $r$ as a parameter. We have $B_\rho(\tilde{x}_n,r) \cap B_\rho(\tilde{x}_m,r) = \varnothing$ for $m \neq n$. Denote $x_n = \tilde{x}_{2n}$, $y_n = \tilde{x}_{2n+1}$. Points $x_n$ and $y_n$ are used in the strategy for requirement $\mathcal{R}_n$. Let $h_n$ be a suitable small positive number defined in the strategy. Similarly to the proof of Theorem 4.1 of [1], we build a sequence of metrics $\rho_n$ as follows. Let $\rho_0 = \rho$. Suppose that the metric $\rho_n$ has already been defined. Define a map $\Gamma_{n+1} \colon B_{\rho_n}(x_n,r) \to \ell^\infty$ by

$$\Gamma_{n+1}(z) = \left(\tfrac{r-\rho_n(x_n,z)}{r}\, h_n\right) ^\frown \left(F_n(y_n) - \tfrac{\rho_n(x_n,z)}{r}(F_n(y_n) - F_n(z))\right),$$

where $F_n$ is the Fréchet embedding of $(X,\rho_n,W)$ into $\ell^\infty$. Extend $\Gamma_{n+1}$ to a mapping $\gamma_{n+1} \colon X \to \ell^\infty$ by putting $\gamma_{n+1}(z) = 0 ^\frown F_n(z)$ for $z \notin B_{\rho_n}(x_n,r)$. Let $\rho_{n+1}(z,v) = \|\gamma_{n+1}(z) - \gamma_{n+1}(v)\|$.

Metric $\rho_{n+1}$ can be expressed in terms of $\rho_n$ by Formulas (1.1)–(1.3). The following lemma is an analogue of Proposition 5.4 of [1] and is proved in the same way.

**Lemma 2.** *For all $n \in \omega$, the following hold:*

1. *$\rho_n$ is a well-defined complete metric inducing topology $\tau$ on $X$;*
2. *$\rho_n(z,v) = \rho_{n-1}(z,v)$ for all $z,v \notin B_{\rho_{n-1}}(x_{n-1},r)$;*
3. *The identity map $\mathrm{id}_X$ induces a series of surjective isometries $B_{\rho_0}[x_n,r] \to B_{\rho_1}[x_n,r] \to \ldots \to B_{\rho_n}[x_n,r]$.*

*Proof.* Proceed as in the proof of Proposition 5.4 of [1]. The last clause follows from Propositions 5.2 and 5.3 of [1] and the fact that $\rho_0(x_n,\tilde{x}_m) > 3r$ for all $\tilde{x}_m \neq x_n$, thus $\min(\rho_0(x_n,x_i) - r, \rho_0(x_n,y_i)) > 2r > r$ for $i < n$. □

**Corollary 1.** *For all $z,v \in X$, there are at most two $n \in \omega$ such that $\rho_{n+1}(z,v) \neq \rho_n(z,v)$.*

*Proof.* If $z,v \notin \bigcup_{n\in\omega} B_{\rho_0}(x_n,r)$, then $\rho_n(z,v) = \rho_0(z,v)$ for all $n$. If $z \in B_{\rho_0}(x_n,r)$ for some $n$, then we may have $\rho_{n+1}(z,v) \neq \rho_n(z,v)$. A similar change may happen if $v \in B_{\rho_0}(x_m,r)$ for some $m$. Other than that, no changes will occur. □

Corollary 1 implies that there exists a pointwise limit $\rho'$ of metrics $\rho_n$ that also is a metric on $X$.

**Lemma 3.** *$\rho'$ is a complete metric inducing topology $\tau$ on $X$.*

*Proof.* Repeat the proof of Proposition 5.5 of [1]. □

**Lemma 4.** *There exists a constant $C > 0$ such that $\rho'(z,v) \leqslant C \cdot \rho_0(z,v)$ for all $z,v \in X$.*

*Proof.* First of all we show that $\rho_1(z,v) \leqslant \frac{2}{r} \cdot \rho_0(z,v)$ for all $z, v \in X$. Fix arbitrary points $z, v \in X$. By symmetry we will assume that $\rho_0(x_0, z) \leqslant \rho_0(x_0, v)$. The following three possibilities can occur.

*Case 1.* $\rho_0(x_0, z) < r$ and $\rho_0(x_0, v) \geqslant r$. Then

$$\begin{aligned}
\rho_1(z,v) &= \max\left(\tfrac{r-\rho_0(x_0,z)}{r}\, h_0,\ \tfrac{\rho_0(x_0,z)}{r}\rho_0(z,v) + \tfrac{r-\rho_0(x_0,z)}{r}\rho_0(y_0,v)\right) \\
&\leqslant \max\left(\tfrac{\rho_0(x_0,v)-\rho_0(x_0,z)}{r}\, h_0,\ \tfrac{\rho_0(x_0,z)}{r}\rho_0(z,v) + \tfrac{\rho_0(x_0,v)-\rho_0(x_0,z)}{r}\rho_0(y_0,v)\right) \\
&\leqslant \max\left(\tfrac{\rho_0(z,v)}{r}\, h_0,\ \tfrac{\rho_0(x_0,z)}{r}\rho_0(z,v) + \tfrac{\rho_0(z,v)}{r}\rho_0(y_0,v)\right) \\
&= \max\left(\tfrac{h_0}{r},\ \tfrac{\rho_0(x_0,z)+\rho_0(y_0,v)}{r}\right) \cdot \rho_0(z,v).
\end{aligned}$$

*Case 2.* $\rho_0(x_0, v) < r$. One can obtain the same boundary in the same manner.

*Case 3.* $\rho_0(x_0, z) \geqslant r$. In this case, $\rho_1(z,v) = \rho_0(z,v)$.

By construction, $h_0 \leqslant 1$. We also have $\rho_0(x_0,z) + \rho_0(y_0,v) \leqslant 2$; this is the moment when we see why it was important to bound the metric $\rho_0$ from above. Since $r < 1$, the claim follows.

To prove the statement of the lemma, fix arbitrary $z, v \in X$ and consider the following possibilities.

*Case 1.* $z, v \notin \bigcup_{n\in\omega} B_{\rho_0}(x_n, r)$. Then $\rho'(z,v) = \rho_0(z,v)$.

*Case 2.* $z \in B_{\rho_0}(x_n, r)$ and $v \in B_{\rho_0}(x_m, r)$, $n \neq m$ (say $n < m$). We have

$$\begin{aligned}
\rho_0(z,v) &= \rho_1(z,v) = \ldots = \rho_n(z,v), \\
\rho_0(z,x_n) &= \rho_1(z,x_n) = \ldots = \rho_n(z,x_n), \\
\rho_0(y_n,v) &= \rho_1(y_n,v) = \ldots = \rho_n(y_n,v).
\end{aligned}$$

Then, arguing as above, we obtain $\rho_{n+1}(z,v) \leqslant \frac{2}{r} \cdot \rho_n(z,v)$. Similarly, we have $\rho_{n+1}(v,x_m) \leqslant \frac{2}{r} \cdot \rho_n(v,x_m)$ and $\rho_{n+1}(y_m,z) \leqslant \frac{2}{r} \cdot \rho_n(y_m,z)$. Then

$$\begin{aligned}
\rho_{n+1}(z,v) &= \rho_{n+2}(z,v) = \ldots = \rho_m(z,v), \\
\rho_{n+1}(v,x_m) &= \rho_{n+2}(v,x_m) = \ldots = \rho_m(v,x_m), \\
\rho_{n+1}(y_m,z) &= \rho_{n+2}(y_m,z) = \ldots = \rho_m(y_m,z),
\end{aligned}$$

and after several careful substitutions we have

$$\begin{aligned}
\rho_{m+1}(z,v) &\leqslant \max\left(\tfrac{h_m}{r},\ \tfrac{\rho_m(v,x_m)+\rho_m(y_m,z)}{r}\right) \cdot \rho_m(z,v) \\
&\leqslant \max\left(\tfrac{h_m}{r},\ \tfrac{1}{r}\left(\tfrac{2}{r}\rho_0(v,x_m) + \tfrac{2}{r}\rho_0(y_m,z)\right)\right) \cdot \rho_m(z,v) \\
&\leqslant \max\left(\tfrac{1}{r},\ \tfrac{4}{r^2}\right) \cdot \tfrac{2}{r} \cdot \rho_0(z,v) \\
&= \tfrac{8}{r^3} \cdot \rho_0(z,v).
\end{aligned}$$

After that, the distance between these points will never change, and $\rho'(z,v) = \rho_{m+1}(z,v)$.

*Case 3.* There is only one $n$ such that $z$ or $v$ (or both) is contained in $B_{\rho_0}(x_n, r)$. A similar boundary can be obtained in this case as well.

Now, $C = \frac{8}{r^3}$ can serve as the constant from the statement of the lemma. □

We will also need the following observation.

**Lemma 5.** *The following hold:*

(1) *If* $z \notin \left(\bigcup_{n\in\omega} B_{\rho_0}[x_n, r]\right) \cup \{y_n \mid n \in \omega\}$, *then for all* $n \in \omega$

$$\rho'(z, B_{\rho_0}[x_n, r]) = \inf_{v \in B_{\rho_0}[x_n,r]} \rho'(z,v) \geqslant \min(\rho_0(x_n,z) - r, \rho_0(y_n,z)).$$

(2) *Let* $B_1 = B_{\rho_0}[x_n, r]$, $B_2 = B_{\rho_0}[x_m, r]$ *for* $m \neq n$. *Then*

$$\rho'(B_1, B_2) = \inf_{z \in B_1, v \in B_2} \rho'(z, v) \geqslant r.$$

*Proof.* (1). For all $v \in B_{\rho_0}[x_n, r]$ we have $\rho_0(z, v) = \rho_1(z, v) = \ldots = \rho_n(z, v)$, and after that $\rho_{n+1}(z, v) \geqslant \min(\rho_n(x_n, z) - r, \rho_n(y_n, z)) = \min(\rho_0(x_n, z) - r, \rho_0(y_n, z))$ by the consideration before Proposition 5.2 of [1]. After that, distance between these points never changes, and $\rho'(z, v) = \rho_{n+1}(z, v)$.

(2). Fix $m < n$ and fix two points $z \in B_{\rho_0}[x_n, r], v \in B_{\rho_0}[x_m, r]$. As above, we have $\rho_0(x_n, v) = \rho_1(x_n, v) = \ldots = \rho_m(x_n, v)$, and after that $\rho_{m+1}(x_n, v) \geqslant \min(\rho_m(x_m, x_n) - r, \rho_m(y_m, x_n)) = \min(\rho_0(x_m, x_n) - r, \rho_0(y_m, x_n)) > 2r$. Similarly, $\rho_{m+1}(y_n, v) \geqslant \min(\rho_0(x_m, y_n) - r, \rho_0(y_m, y_n)) > 2r$. Afterwards, we have $\rho_{m+1}(z, v) = \ldots = \rho_n(z, v)$ and $\rho_{n+1}(z, v) \geqslant \min(\rho_n(x_n, v) - r, \rho_n(y_n, v)) > r$. □

1.3. **Construction.** At the start of the construction, we are given a computable metric $\rho \in M(\mathbf{X})$. We have fixed a rational $r \in (0, \frac{1}{3})$ such that the space $(X, \rho)$ does not have a finite $3r$-net and have chosen a $k \in \omega$ such that $2^{-k} < r$.

*Stage* 0. Let $A_0 = \varnothing$, $\rho_0 = \rho$.

*Stage* $s+1$. Appoint a follower to requirement $\mathcal{R}_s$ as follows. Since $A_s$ is not a $3r$-net in $(X, \rho_0)$, we can find a special point $y_s$ such that $\rho_0(y_s, w) > 3r$ for all $w \in A_s$. Let $A = A_s \cup \{w_s, y_s\}$. Let $e \leqslant s$ be the least number such that the requirement $\mathcal{R}_e$ has not been met yet and $\Phi_e(\bar{a})(k+1) \downarrow= u$ in $s+1$ steps, where $w_a = y_e$ is the follower of $\mathcal{R}_e$. If no such $e$ exists, end the stage. Otherwise, choose an element $x_e = w_b$ such that $\rho_0(x_e, w) > 3r$ for all $w \in A \cup \{w_u\}$. Let $\rho_{s'}$ be the metric most recently defined in the construction. Proceed as described in Subsection 1.1 with $\rho_{s'}$ in place of $\rho$ and obtain a new metric $\rho_{s+1}$. Let $A_{s+1} = A \cup \{w_u, x_e\}$.

1.4. **Verification.** Define $\rho'$ to be the pointwise limit of metrics defined in the construction. As in Subsection 6.2 of [1] it can be shown that $\rho'$ is a computable metric. By construction, $\rho_0(x_i, x_j) > 3r$ for all $i \neq j$ and $\rho_0(x_i, y_j) > 3r$ for all $i, j$. From the discussion in Subsection 1.1 it is not hard to see that all requirements $\mathcal{R}_e$ are satisfied. By Lemmas 1 and 4, $\rho \leq_c \rho'$.

1.5. **Proof of Theorem 2.** Suppose that $\rho \in M(\mathbf{X})$ is an arbitrary metric on $\mathbf{X}$. We can carry out the construction from Theorem 1 to obtain a metric $\rho'$ satisfying all negative requirements $\mathcal{R}_e$, then Lemmas 1 and 4 guarantee that $\rho \leq_c \rho'$. It remains to note that Lemma 1 holds in arbitrary metric spaces, not necessarily computable.

## 2. Proof of Theorem 3

As mentioned in the introduction, this theorem is a generalization of a result from [4], namely of Theorem 3 of that paper. The idea of the proof is the same as in [4]: we embed the Boolean algebra $\mathcal{P}_{\mathrm{Comp}}(\omega)$ of computable subsets of $\omega$ into $\mathcal{CM}_c(\mathbf{X})$ and use the fact that $\mathrm{Int}(1+\eta)$ is embeddable into $\mathcal{P}_{\mathrm{Comp}}(\omega)$. Below we explain how to embed the 4-element Boolean algebra into $\mathcal{CM}_c(\mathbf{X})$, and then we give a hint how to generalize this construction to obtain the embedding of $\mathcal{P}_{\mathrm{Comp}}(\omega)$.

To show that the 4-element Boolean algebra embeds into $\mathcal{CM}_c(\mathbf{X})$, we construct metrics $d_0, d_1$ and $d_{01}$ such that $\deg_c(d_{01}) = \sup(\deg_c(d_0), \deg_c(d_1))$ and $\deg_c(\rho) = \inf(\deg_c(d_0), \deg_c(d_1))$ in $\mathcal{CM}_c(\mathbf{X})$. Metrics $d_0$ and $d_1$ will be built in a single construction and will satisfy the following requirements:

$\mathcal{R}_{ie}$: $d_i \nleq_c d_{1-i}$ via $\Phi_e$, for $i = 0, 1$ and $e \in \omega$,
$\mathcal{S}$: $\rho \leq_c d_0, d_1$.

Fix a rational number $r \in (0, \frac{1}{3})$ such that there is no finite $3r$-net in $(X, \rho)$. Requirements $\mathcal{R}_{ie}$ and $\mathcal{S}$ are satisfied in the exact same manner as in Theorem 1.

Metric $d_{01}$ contains all deformations defined in the course of the construction of both $d_0$ and $d_1$. More formally, at stage 0 we initially define $d_{0,0}, d_{1,0}, d_{01,0} = \rho$. Suppose that at stage $s$ we finish the $\mathcal{R}_{0e}$-strategy and introduce a deformation $\Gamma\colon B_{d_{0,s'}}(x_{0e}, r) \to \ell^\infty$:

$$\Gamma(z) = \big(\tfrac{r - d_{0,s'}(x_{0e},z)}{r}\, h_{0e}\big) ^\frown \big(F_{s'}(y_{0e}) - \tfrac{d_{0,s'}(x_{0e},z)}{r}(F_{s'}(y_{0e}) - F_{s'}(z))\big),$$

where $d_{0,s'}$ is the most recently defined approximation of the metric $d_0$ and $F_{s'}$ is the Fréchet embedding of $(X, d_{0,s'}, W)$ into $\ell^\infty$. We add an analogue of $\Gamma$ to $d_{01}$ as follows. Let $\tilde{F}_{s''}$ be the Fréchet embedding of $(X, d_{01,s''}, W)$ into $\ell^\infty$, where $d_{01,s''}$ is the most recently defined approximation of the metric $d_{01}$. Let

$$\tilde{\Gamma}(z) = \big(\tfrac{r - d_{0,s''}(x_{0e},z)}{r}\, h_{0e}\big) ^\frown \big(\tilde{F}_{s''}(y_{0e}) - \tfrac{d_{0,s''}(x_{0e},z)}{r}(\tilde{F}_{s''}(y_{0e}) - \tilde{F}_{s''}(z))\big),$$

Extend $\tilde{\Gamma}$ to a mapping $\tilde{\gamma}\colon X \to \ell^\infty$ in the usual way and let $d_{01,s}(z, v) = \|\tilde{\gamma}(z) - \tilde{\gamma}(v)\|$. In a similar way we incorporate into $d_{01}$ every deformation defined for the metric $d_1$.

To see that the $\mathcal{R}_{0e}$-strategy succeeds, let $y_{0e} = w_a$ and $x_{0e} = w_b$ be the points used in the strategy. When $\mathcal{R}_{0e}$ requires attention, say at stage $s$, let $u = \Phi_e(\bar{a})(k+1)$. Points $y_{0e}$ and $x_{0e}$ uniquely correspond to the requirement $\mathcal{R}_{0e}$, thus $y_{0e}$ is not used in the construction of $d_1$. In particular, $y_{0e}$ does not belong to any ball $B_\rho(x_{1e'}, r)$, and by Lemma 5 we have $d_1(y_{0e}, w_u) > r$ regardless of where $w_u$ is located. This guarantees the success of diagonalization against $\Phi_e$.

Observe that $\deg_c(\rho)$ is the greatest lower bound of $\deg_c(d_0)$ and $\deg_c(d_1)$. To prove this, it suffices to show that, given a $d_0$-name and a $d_1$-name for $z \in X$, we can effectively construct a $\rho$-name for $z$. Fix a $d_0$-name $f_0$ and a $d_1$-name $f_1$ for $z$. Let $S = \{x_{ie}, y_{ie} \mid i = 0, 1, e \in \omega\}$ and $T = \{x_{ie} \mid i = 0, 1, e \in \omega\}$. The construction appoints followers $v \in S$ in such a way that for each $n$ it holds $\rho(v, w_n) > 3r$ for all followers $v$ appointed after stage $n + 1$. Among the remaining (finitely many) followers, we are able to find such a $v^*$ that $\rho(v, w_n) > \frac{3r}{2}$ for all $v \neq v^*$. Clearly, this $v^*$ may be defined ambiguously, but there is a uniform computable way to find this element for each $w_n$. Consider the point $w_{f_0(k+2)}$ and find the corresponding $v^*$. We split the proof into the following three cases.

*Case 1.* $v^* = x_{1e^*}$ or $v^* = y_{1e^*}$ for some $e^* \in \omega$. Then by Lemma 5(1) we have $d_0(w_{f_0(k+2)}, B_\rho[x_{0e}, r]) > \frac{r}{2}$ for all $e$. Since $d_0(w_{f_0(k+2)}, w_{f_0(n)}) < \frac{r}{2}$ for $n \geqslant k+2$, then $w_{f_0(n)} \notin \bigcup_{e\in\omega} B_\rho[x_{0e}, r]$ for $n \geqslant k+2$, which means that $d_0(w_{f_0(m)}, w_{f_0(n)}) = \rho(w_{f_0(m)}, w_{f_0(n)})$ for all $m, n \geqslant k+2$, thus

$$\big(f_0(k+2), f_0(k+3), \ldots\big)$$

is a $\rho$-name for $z$.

*Case 2.* $v^* = y_{0e^*}$ for some $e^* \in \omega$. Then $d_0(w_{f_0(k+2)}, B_\rho[x_{0e}, r]) > \frac{r}{2}$, thus $d_0(z, B_\rho[x_{0e}, r]) > \frac{r}{4}$ for all $e \neq e^*$. The following possibilities can then hold:

(1) $z \in B_\rho[x_{0e^*}, r]$. Then clearly $\rho(z, v) > 2r$ for all $v \in S - \{x_{0e^*}\}$.

(2) $z \notin \bigcup_{e\in\omega} B_\rho[x_{0e}, r]$. In this case, $\rho(z, w_{f_0(k+2)}) = d_0(z, w_{f_0(k+2)}) < \frac{r}{4}$ by (1.3) and $\rho(z, v) > \frac{5r}{4}$ for all $v \in S - \{x_{0e^*}\}$.

By Lemma 5(1), in both of these cases for all $e \in \omega$ we have $d_1(z, B_\rho[x_{1e}, r]) > \frac{r}{4}$, i. e., $z$ is far from all balls used in the construction of $d_1$. Then, similarly to the above, $d_1(w_{f_1(m)}, w_{f_2(n)}) = \rho(w_{f_1(m)}, w_{f_1(n)})$ for all $m, n \geqslant k+2$, and

$$\big(f_1(k+2), f_1(k+3), \dots\big)$$

is a $\rho$-name for $z$.

*Case 3.* $v^* = x_{0e^*}$, $e^* \in \omega$. We break this case into the following subcases:

(1) $w_{f_0(k+2)} \notin B_\rho[x_{0e^*}, r]$. As above, we have $d_0(z, B_\rho[x_{0e}, r]) > \frac{r}{4}$ for all $e \neq e^*$, thus $z \in B_\rho[x_{0e^*}, r]$ or $z \notin \bigcup_{e\in\omega} B_\rho[x_{0e}, r]$.

(2) $w_{f_0(k+2)} \in B_\rho[x_{0e^*}, r]$. Since $d_0(w_{f_0(k+2)}, z) < \frac{r}{4}$, by Lemma 5 we must have $\rho(z, x_{0e^*}) < \frac{5r}{4}$ or $\rho(z, y_{0e^*}) < \frac{r}{4}$.

In any of these subcases, arguing as in Case 2, we obtain that $d_1(z, B_\rho[x_{1e}, r]) > \frac{r}{4}$ for all $e \in \omega$. As a result, $\big(f_1(k+2), f_1(k+3), \dots\big)$ is a $\rho$-name for $z$.

Note that we can effectively decide which of the cases 1–3 occurs. It implies that some $\rho$-name for $z$ can be computed, using $f_0$ and $f_1$.

It is not hard to see that $d_0, d_1 \leq_c d_{01}$. To prove that $\deg_c(d_{01})$ is the least upper bound of $\deg_c(d_0)$ and $\deg_c(d_1)$, it suffices to show that every $d_{01}$-name $f$ can be translated into a $d_0$-name or a $d_1$-name for the same element $z$. Fix $z \in X$ and a $d_{01}$-name $f$ for $z$. As earlier, we can compute the indices $i, e$ of a $v^* \in S$ such that $\rho(w_{f(k+2)}, v) > \frac{3r}{2}$ for all $v \in S - \{v^*\}$. Suppose that $v^* = x_{ie}$. Then Lemma 5 readily implies that

$$B_{d_0}[w_{f(k+2)}, \tfrac{r}{2}] \cap \bigcup_{x \in T - \{v^*\}} B_\rho[x, r] = \varnothing.$$

Similarly, if $v^*$ is a follower $y_{ie}$, then $B_{d_0}[w_{f(k+2)}, \frac{r}{2}] \cap \bigcup_{x\in T-\{x_{ie}\}} B_\rho[x, r] = \varnothing$. Now, suppose that $i = 0$, i. e., $v^* = x_{0e}$ or $v^* = y_{0e}$. By the above, we have $w_{f(n)} \notin \bigcup_{v\in T-\{x_{0e}\}} B_\rho[v, r]$ for all $n \geqslant k+2$. Since we add the same deformation $\Gamma$ of the ball $B_\rho(x_{0e}, r)$ to both metrics $d_0$ and $d_{01}$, then by induction on $s$ it is not hard to show that $d_{01,s}(w_{f(n)}, w_{f(m)}) = d_{0,s}(w_{f(n)}, w_{f(m)})$ for all $n, m \geqslant k+2$ and all $s$, thus $d_{01}(w_{f(n)}, w_{f(m)}) = d_0(w_{f(n)}, w_{f(m)})$ for these $n, m$, and $\big(f(k{+}2), f(k{+}3), \dots\big)$ is a $d_0$-name for $x$. We argue in the same manner in case $i = 1$.

To show that $\mathcal{P}_{\mathrm{Comp}}(\omega)$ embeds into $\mathcal{CM}_c(\mathbf{X})$, construct an infinite sequence of metrics $d_{\{i\}}$, $i \in \omega$, similarly to the above, and for any computable set $A$ construct the metric $d_A$ that contains all deformations defined for the metrics $d_i$ for all $i \in A$. Then $\deg_c(d_{A\cap B})$ will be the greatest lower bound of $\deg_c(d_A)$ and $\deg_c(d_B)$, and $\deg_c(d_{A\cup B})$ will be the least upper bound of $\deg_c(d_A)$ and $\deg_c(d_B)$, for arbitrary computable sets $A$ and $B$.

## 3. The unit interval

3.1. **Proof of Proposition 1.** We will need the following facts.

**Lemma 6** ([3], Corollary 6.2.5)**.** *Let $f$ be a continuous real function defined on a compact set $K \subseteq \mathbf{R}$. Then $\max_{x\in K} f(x)$ and $\min_{x\in K} f(x)$ are computable uniformly in $f$ and $K$.*

**Lemma 7.** *Let $\mathbf{X}$ be a Polish space and let $\rho, \rho' \in M(\mathbf{X})$ be computable metrics on $\mathbf{X}$ such that $\rho \leq_c \rho'$. Then $\rho'$ is a $(\rho, \rho, \rho_R)$-computable function.*

*Proof.* Immediate. □

Suppose that $\rho \in M(\mathbf{I})$ is a computable metric on $\mathbf{I}$ such that $\rho_R \leq_c \rho$. We need to prove that $\rho \leq_c \rho_R$. By Lemma 7, $\rho$ is $(\rho_R, \rho_R, \rho_R)$-computable. Then the projection functions $\rho_q(x) = \rho(q, x)$ are $(\rho_R, \rho_R)$-computable uniformly in $q \in \mathbb{Q}$. Denote $q_n = \nu_{\mathbb{Q}} n$.

Let $f$ be a $\rho$-name for $x \in I$. We want to compute a $\rho_R$-name for $x$. This can be done as follows: for $n \in \omega$, we will progressively compute rational $a_n, b_n$ such that $B_\rho[q_{f(k)}, 2^{-k}] \subseteq [a_n, b_n]$ for some $k$ and the length of intervals $[a_n, b_n]$ tends to zero. Since $\bigcap_k B_\rho[q_{f(k)}, 2^{-k}] = x$, in the end we obtain a $\rho_R$-name for $x$.

At stage $n$, for each $k > 0$ proceed as follows. Let $0 = c_0, c_1, \ldots, c_{kn} = q_{f(k)}$ be rational points forming a partition of interval $[0, q_{f(k)}]$ into $kn$ even subintervals. For $i \leqslant kn$, compute $\min_{y \in [0, c_i]} \rho_{q_{f(k)}}(y)$ with precision $2^{-kn}$. If for some $i$ we are able to see that $\min_{y \in [0, c_i]} \rho_{q_{f(k)}}(y) > 2^{-k}$ with said precision, pick the rightmost $c_i$ with this property and denote it by $a_n^k$, otherwise let $a_n^k = 0$. By construction, $a_n^k$ bounds $B_\rho[q_{f(k)}, 2^{-k}]$ from below. Similarly, obtain a $b_n^k$ that bounds $B_\rho[q_{f(k)}, 2^{-k}]$ from above. Continue this process until we find such a $k$ that $b_n^k - a_n^k \leqslant 2^{-n}$; this will eventually happen since the balls $B_\rho[q_{f(k)}, 2^{-k}]$ converge to $x$. This gives us the $a_n, b_n$ satisfying the properties described above.

3.2. **Proof of Proposition 2.** $\mathbf{0}'$-computable metric $\rho^* >_c \rho_R$ will have the form

$$\rho^*(x, y) = \|\gamma(x) - \gamma(y)\|,$$

where $\gamma\colon I \to I^2$ is a polygonal curve defined as follows. Consider the sequence $\mathbf{a}_0 = 0, \mathbf{a}_1 = 2^{-1}, \ldots, \mathbf{a}_n = 1 - 2^{-n}, \ldots$ We diagonalize against the $e$th functional $\Phi_e$ on the interval $[\mathbf{a}_e, \mathbf{a}_{e+1}]$ of length $2^{-e-1}$. Suppose that $\mathbf{a}_e = q_a$. Use $\mathbf{0}'$ to figure out whether the computation $\Phi_e(\bar{a})(e+3)$ ever halts. If it does, let $u = \Phi_e(\bar{a})(e+3)$, then it must hold $\rho_R(q_u, q_a) \leqslant 2^{-e-3}$, otherwise we immediately win, keeping the interval $[\mathbf{a}_e, \mathbf{a}_{e+1}]$ undeformed in the metric $\rho^*$. Let $x_e$ be the midpoint of $[\mathbf{a}_e, \mathbf{a}_{e+1}]$ and let $y_e = x_e - 2^{-e-3}, z_e = x_e + 2^{-e-3}$. In the spirit of the proof of Theorem 1, add to $\gamma$ a deformation $\Gamma_e$ of the subinterval $[y_e, z_e] \subseteq [\mathbf{a}_e, \mathbf{a}_{e+1}]$ that makes $x_e$ close to $\mathbf{a}_e$, so we also win against $\Phi_e$. More precisely, define a piecewise linear map $\Gamma_e\colon [y_e, z_e] \to I^2$ by its action on the following points:

$$\Gamma_e(y_e) = (y_e, 0),\ \Gamma_e(x_e) = (\mathbf{a}_e, h_e),\ \Gamma_e(z_e) = (z_e, 0),$$

where $h_e < \min(2^{-e}, 2^{-\varphi_e(\bar{a})(e+3)})$ (see Fig. 1).

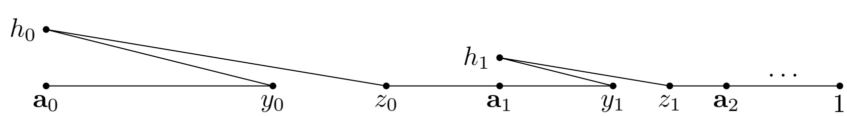


FIG. 1. First few steps of the construction.

To see why $\rho_R \leq_c \rho^*$, notice that, since the height $h_e$ of the deformation $\Gamma_e$ is bounded by $2^{-e}$ and at each step $e$ we define $\Gamma_e$ in a uniform way, only scaling the length of the interval $[y_e, z_e]$ and distance to the point $\mathbf{a}_e$ down by 2, then all the deformations $\Gamma_e$ actually share the same computable modulus of continuity $m$. Thus, $m$ is also a modulus of continiuty of the whole curve $\gamma$. Then one can effectively translate any $\rho_R$-name $f$ into a $\rho^*$-name $g$ for the same point by letting $g = f \circ m$, which means that $\rho_R \leq_c \rho^*$.

## References

[1] R. Kornev, *A semilattice of degrees of computable metrics*, Sib. Math. J., **62**:5 (2021), 822–841. Zbl 7401177

[2] Ch. Kreitz, K. Weihrauch, *Theory of representations*, Theoret. Comput. Sci., **38** (1985), 35–53. Zbl 0588.03031

[3] K. Weihrauch, *Computable analysis. An introduction*, Texts in Theoret. Comput. Sci. EATCS Ser., Springer, Berlin, 2000. Zbl 0956.68056

[4] R. Kornev, *Computable metrics above the standard real metric*, Sib. Èlectron. Mat. Izv., **18**:1 (2021), 377–392. Zbl 7360156

[5] R. Dillhage, *Computable functional analysis. Compact operators on computable Banach spaces and computable best approximation*, PhD Thesis, Fak. Math. Inform., FernUniversität Hagen, 2012

Ruslan Aleksandrovich Kornev
Sobolev Institute of Mathematics,
4, Acad. Koptyug ave.,
Novosibirsk, 630090, Russia
*Email address*: kornevrus@gmail.com