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EXPONENTIAL TIGHTNESS FOR INTEGRAL – TYPE FUNCTIONALS OF CENTERED INDEPENDENT DIFFERENTLY DISTRIBUTED RANDOM VARIABLES

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Abstract. Exponential tightness is proved for a sequence of integral – type random fields constructed by centered independent differently distributed random variables. This result is proven using sufficient conditions for the exponential tightness of a sequence of continuous random fields of arbitrary form, which are also obtained in this paper.

Keywords: random field, Cramer's moment condition, large deviations principle, moderate deviations principle, exponential tightness.

1. Introduction, main results

An important problem related to the proof of the large deviations principle (LDP) for sequences of random elements is the proof of the exponential tightness of this sequence. Moreover, if the random elements belong to some complete separable metric space, then the exponential tightness is a necessary condition for fulfillment of the LDP (see e.g. [1], [2, Remark(a), p. 8]). In the present paper we will be interested in sufficient conditions for exponential tightness of sequences of continuous random fields, in particular, for integral functionals of centered independent differently distributed random variables.

In what follows we will assume that all random elements are defined on some probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. Let's recall the definition of exponential tightness.

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Definition 1. *The sequence of random elements $\mathcal{F}_n \in \mathbb{X}$, $n \in \mathbb{N}$ is exponentially tight (ET) in a metric space $\mathbb{X}$ with a normalizing function (NF) $\psi(n)$: $\lim\limits_{n\to\infty} \psi(n) = \infty$, if for any $C > 0$ there exists a compact $K_C \subseteq \mathbb{X}$ such that*

$$\limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbf{P}(\mathcal{F}_n \notin K_C) \le -C.$$

For a fixed $d \in \mathbb{N}$ we denote by

$$\mathbb{R}^d_+ := \{\vec{\mathbf{u}} = (u_1, \dots, u_d) \in \mathbb{R}^d : \ u_1 \ge 0, \dots, u_d \ge 0\}, \quad \mathbb{Z}^d_+ := \mathbb{Z}^d \cap \mathbb{R}^d_+$$

positive quadrant of the space $\mathbb{R}^d$ and the lattice $\mathbb{Z}^d$, respectively. It is convenient for us to use the norm

$$|\vec{\mathbf{u}}| := \max_{1\le i\le d} |u_i|$$

in $\mathbb{R}^d$. We will write $\vec{\mathbf{u}} \le \vec{\mathbf{v}}$ or $\vec{\mathbf{u}} < \vec{\mathbf{v}}$ if the corresponding inequality is satisfied for each coordinate.

Let's consider an array of independent random variables

$$X_{\vec{\mathbf{i}},n}, \quad n \in \mathbb{N}, \quad \vec{\mathbf{i}} = (i_1, \dots, i_d) \in \mathbb{Z}^d_+. \tag{1.1}$$

Throughout the paper we will assume that for a fixed $\alpha \in (0,1]$ the following uniform Cramer moment condition is satisfied

$[\mathbf{C}_0(\alpha)]$. *For some $\lambda > 0$, $M < \infty$ the inequality*

$$\mathbf{E}\exp\{\lambda |X_{\vec{\mathbf{i}},n}|^\alpha\} \le M, \quad n \in \mathbb{N}, \quad \vec{\mathbf{i}} \in \mathbb{Z}^d_+$$

is true.

In this case we fix the sequence $x = x(n)$ such that

$$\lim_{n\to\infty} \frac{x(n)}{n^{\frac{d}{2}}} = \infty, \quad \lim_{n\to\infty} \frac{x^{2-\alpha}(n)}{n^d} = 0. \tag{1.2}$$

Denote by

$$\Delta(\vec{\mathbf{u}}] := (u_1 - \Delta, u_1] \times \cdots \times (u_d - \Delta, u_d],$$

for $\vec{\mathbf{u}} = (u_1, \cdots, u_d) \in \mathbb{R}^d_+$, $\Delta > 0$. Let

$$\Delta_n(\vec{\mathbf{u}}] := \Delta(\vec{\mathbf{u}}]\Big|_{\Delta = \frac{1}{n}}$$

be "semi−open" cube in $\mathbb{R}^d$ with an edge $\frac{1}{n}$ and the "upper right vertex" (the vertex farthest from the origin) at the point $\vec{\mathbf{u}} \in \mathbb{R}^d$.

A random field is a random process that is indexed by a $d-$dimensional parameter $\vec{\mathbf{u}}$. It is easy to see that since $\vec{\mathbf{u}}$ belongs to the unit cube $[0,1]^d$ in space $\mathbb{R}^d$, there is an unique $\vec{\mathbf{i}}(\vec{\mathbf{u}}) \in \mathbb{Z}^d_+$ such that $\vec{\mathbf{u}} \in \Delta_n\left(\frac{1}{n}\vec{\mathbf{i}}(\vec{\mathbf{u}})\right]$. Let's define a random field

$$\mathcal{P}_n(\vec{\mathbf{u}}) := \frac{n^d}{x} X_{\vec{\mathbf{i}}(\vec{\mathbf{u}}),n}, \quad \vec{\mathbf{u}} \in [0,1]^d. \tag{1.3}$$

Thus, the field (1.3) is defined by the array (1.1) (Cramer's condition $[\mathbf{C}_0(\alpha)]$ is satisfied for its elements) and the sequence $x = x(n)$ (see (1.2)). As a rule, we will assume that the means are equal to zero:

$$\mathbf{E}X_{\vec{\mathbf{i}},n} = 0, \quad n \in \mathbb{N}, \quad \vec{\mathbf{i}} \in \mathbb{Z}^d_+. \tag{1.4}$$

It is convenient to interpret the field $\mathcal{P}_n = \mathcal{P}_n(\vec{\mathbf{u}})$ as a density with respect to the Lebesgue measure $\mu(d\vec{\mathbf{u}})$ of a random generalized measure

$$\mathcal{M}_n = \mathcal{M}_n(B) := \int_{\vec{\mathbf{u}}\in B} \mathcal{P}_n(\vec{\mathbf{u}})\mu(d\vec{\mathbf{u}}), \quad B \subset [0,1]^d$$

defined on a Borel $\sigma-$algebra of the subsets of the unit cube $[0,1]^d$. Let us assign to the field $\mathcal{P}_n$ the "distribution function" corresponding to it

$$\mathcal{G}_n = \mathcal{G}_n(\vec{\mathbf{t}}) := \mathcal{M}_n(B_{\vec{\mathbf{t}}}), \quad \vec{\mathbf{t}} \in [0,1]^d, \tag{1.5}$$

where

$$B_{\vec{\mathbf{t}}} := \{\vec{\mathbf{u}} \in \mathbb{R}^d_+ : \ \vec{\mathbf{u}} < \vec{\mathbf{t}}\}.$$

We denote by $\mathbb{C}[0,1]^d$ the space of continuous functions $f = f(\vec{\mathbf{t}})$ which mapped the cube $[0,1]^d$ to the real line $\mathbb{R}$. Let us define the uniform norm

$$||f|| := \sup_{\vec{\mathbf{t}}\in[0,1]^d} |f(\vec{\mathbf{t}})|$$

on the space $\mathbb{C}[0,1]^d$.

It is convenient to consider functions $\mathcal{G}_n$ as elements of a normalized space

$$\mathbb{C}_0[0,1]^d := \{f = f(\vec{\mathbf{t}}) \in \mathbb{C}[0,1]^d : \quad f(\vec{\mathbf{t}}) = 0 \quad \text{if} \quad \vec{\mathbf{t}} \in \partial\mathbb{R}^d_+\},$$

equipped with the uniform metric, where $\partial\mathbb{R}^d_+$ is the quadrant $\mathbb{R}^d_+$ boundary.

We note that in the case $d = 1$ the sequence $\mathcal{G}_n$ is a continuous polygon constructed by inhomogeneous random walk; in the case $d = 2$ the sequence $\mathcal{G}_n$ can be interpreted as an integral functional of the graphon. Graphon is an important characteristic of a random graph (there is a connection between the limit properties of the graphon and the limit properties of the structure of the graph by which the graphon is constructed, see [3], [4]).

Let us now formulate the main result of the paper.

Theorem 1.1. *Let the conditions $[\mathbf{C}_0(\alpha)]$, (1.2), (1.4) are met, then the sequence of random fields $\mathcal{G}_n$ is exponentially tight in the metric space $\mathbb{C}[0,1]^d$ with the normalizing function $\psi(n) = \frac{x^2}{n^d}$.*

Theorem 1.1 will be proved in section 3. It follows from theorem 1.1

Corollary 1.1. *Let the conditions of theorem 1.1 are met except condition (1.4) (the terms have zero mean), which will be replaced by the condition: $\mathcal{E}_n := \mathbf{E}\mathcal{G}_n$ is relatively compact in $\mathbb{C}[0,1]^d$. Then the statement of theorem 1.1 holds.*

P r o o f of corollary 1.1. It follows from the condition $[\mathbf{C}_0(\alpha)]$ that

$$\mathbf{E}|X_{\vec{\mathbf{i}},n}| = \mathbf{E}|X_{\vec{\mathbf{i}},n}|\mathbf{I}(|X_{\vec{\mathbf{i}},n}| \le z(\alpha,\lambda)) + |X_{\vec{\mathbf{i}},n}|\mathbf{I}(|X_{\vec{\mathbf{i}},n}| > z(\alpha,\lambda))$$

$$\le z(\alpha,\lambda) + \mathbf{E}e^{\lambda|X_{\vec{\mathbf{i}},n}|^\alpha} \le z(\alpha,\lambda) + M,$$

where $z(\alpha,\lambda) := \min\{z > 0 : e^{\lambda z^\alpha} \ge z\}$.

Therefore, the condition $[\mathbf{C}_0(\alpha)]$ (with a suitable choice of $\tilde{\lambda} > 0$, $\tilde{M} < \infty$) will be satisfied also for

$$\widetilde{X}_{\vec{\mathbf{i}},n} := X_{\vec{\mathbf{i}},n} - \mathbf{E}X_{\vec{\mathbf{i}},n}, \ n \in \mathbb{N}, \ \vec{\mathbf{i}} = (i_1,\ldots,i_d) \in \mathbb{Z}^d_+.$$

Let us consider a sequence of integral functionals $\widetilde{\mathcal{G}}_n$ constructed by the family $\widetilde{X}_{\vec{\mathbf{i}},n}$. Since the family $\widetilde{X}_{\vec{\mathbf{i}},n}$ satisfies the condition $[\mathbf{C}_0(\alpha)]$, it follows from theorem 1.2 that the sequence $\widetilde{\mathcal{G}}_n$ is ET, i.e. for any $C > 0$ there exists a compact K_C such that

$$\limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbf{P}(\widetilde{\mathcal{G}}_n \notin K_C) \leq -C. \tag{1.6}$$

We denote by K the closure of the set $\{\mathcal{E}_n,\ n \in \mathbb{N}\}$. Let's consider a compact

$$\widetilde{K}_C := \{\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 :\ \mathcal{H}_1 \in K,\ \mathcal{H}_2 \in K_C\}.$$

By virtue of the fact that $\mathcal{G}_n = \widetilde{\mathcal{G}}_n + \mathcal{E}_n$, we have

$$\mathbf{P}(\mathcal{G}_n \notin \widetilde{K}_C) \leq 1 - \mathbf{P}(\widetilde{\mathcal{G}}_n \in K_C, \mathcal{E}_n \in K)$$

$$= 1 - \mathbf{P}(\widetilde{\mathcal{G}}_n \in K_C) = \mathbf{P}(\widetilde{\mathcal{G}}_n \notin K_C).$$

Therefore, it follows from (1.6)

$$\limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbf{P}(\mathcal{G}_n \notin \widetilde{K}_C) \leq \limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \mathbf{P}(\widetilde{\mathcal{G}}_n \notin K_C) \leq -C.$$

□

The rest of the paper consists of three sections 2–4. Section 2 contains the conditions under which a sequence of arbitrary random fields $\mathcal{F}_n \in \mathbb{C}_0[0,1]^d$ has the ET property (see theorem 2.1, corollary 2.1). Theorem 1.1 is proved in section 3. Section 4 (appendix) contains the lemma 4.1, which plays an important role in the proof of theorem 1.1.

2. Sufficient conditions for exponential tightness of a sequence of continuous random fields

The following result contains sufficient conditions for the sequence of random fields to be ET in the space $\mathbb{C}[0,1]^d$.

Theorem 2.1. *Let for any $n \in \mathbb{N}$ a random field $\mathcal{F}_n = \mathcal{F}_n(\vec{\mathbf{t}})$ belongs to the space $\mathbb{C}_0[0,1]^d$ with probability 1, $\lim\limits_{n\to\infty} \psi(n) = \infty$ and for any $\varepsilon > 0$ the equality*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{\psi(n)} \ln \sup_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in[0,1]^d:\ \vec{\mathbf{s}}<\vec{\mathbf{t}},\ |\vec{\mathbf{s}}-\vec{\mathbf{t}}|=\delta} \mathbf{P}\left(\sup_{\vec{\mathbf{s}}\leq\vec{\mathbf{r}}\leq\vec{\mathbf{t}}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{s}})| > \varepsilon \right) = -\infty \tag{2.1}$$

is satisfied. Then the sequence $\mathcal{F}_n$ is ET in the space $\mathbb{C}[0,1]^d$ with NF $\psi(n)$.

P r o o f. Let's denote

$$g_n(\delta,\varepsilon) := \frac{1}{\psi(n)} \ln \mathbf{P}\left(\sup_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in[0,1]^d:\ \vec{\mathbf{s}}<\vec{\mathbf{t}},\ |\vec{\mathbf{s}}-\vec{\mathbf{t}}|\leq\delta} \left(\sup_{\vec{\mathbf{s}}\leq\vec{\mathbf{r}}\leq\vec{\mathbf{t}}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{s}})| \right) > \varepsilon \right).$$

The following lemma plays an important role in the proof of theorem 2.1.

Lemma 2.1. *It follows from the condition (2.1) that*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} g_n(\delta,\varepsilon) = -\infty. \tag{2.2}$$

P r o o f of lemma 2.1. For simplicity, we prove that (2.1) implies (2.2) for the case $d = 2$.

We denote by $l(\delta) := [\frac{1}{\delta}]$. Consider the vectors

$$\vec{\mathbf{u}}_{k,j} := \left(\frac{k}{l(\delta)}, \frac{j}{l(\delta)}\right),\ \vec{\mathbf{v}}_{k,j} := \left(\frac{k-2}{l(\delta)}, \frac{j-2}{l(\delta)}\right),\ 2 \le k, j \le l(\delta).$$

It is easy to see that if $\vec{\mathbf{s}}, \vec{\mathbf{t}} \in [0,1]^2$, $\vec{\mathbf{s}} < \vec{\mathbf{t}}$ and $|\vec{\mathbf{s}} - \vec{\mathbf{t}}| \le \delta$ then there are k, j such that

$$\vec{\mathbf{v}}_{k,j} \le \vec{\mathbf{s}} < \vec{\mathbf{t}} \le \vec{\mathbf{u}}_{k,j},$$

and hence

$$\sup_{\vec{\mathbf{s}} \le \vec{\mathbf{r}} \le \vec{\mathbf{t}}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{s}})| \le \sup_{\vec{\mathbf{v}}_{k,j} \le \vec{\mathbf{r}} \le \vec{\mathbf{u}}_{k,j}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{v}}_{k,j})|$$

$$+|\mathcal{F}_n(\vec{\mathbf{s}}) - \mathcal{F}_n(\vec{\mathbf{v}}_{k,j})| \le 2 \sup_{\vec{\mathbf{v}}_{k,j} \le \vec{\mathbf{r}} \le \vec{\mathbf{u}}_{k,j}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{v}}_{k,j})| \text{ a.s.} \tag{2.3}$$

Let's consider the events

$$A_{kj} := \left\{\omega : \sup_{\vec{\mathbf{v}}_{k,j} \le \vec{\mathbf{r}} \le \vec{\mathbf{u}}_{k,j}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{v}}_{k,j})| > \frac{\varepsilon}{2}\right\}.$$

Using the inequality (2.3), we obtain

$$\mathbf{P}\left(\sup_{\vec{\mathbf{s}}, \vec{\mathbf{t}} \in [0,1]^d:\ \vec{\mathbf{s}} < \vec{\mathbf{t}},\ |\vec{\mathbf{s}} - \vec{\mathbf{t}}| \le \delta} \left(\sup_{\vec{\mathbf{s}} \le \vec{\mathbf{r}} \le \vec{\mathbf{t}}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{s}})|\right) > \varepsilon\right)$$

$$\le \mathbf{P}\left(\bigcup_{2 \le k,j \le l(\delta)} A_{kj}\right) \le \sum_{2 \le k,j \le l(\delta)} \mathbf{P}(A_{kj}) \le (l(\delta) - 1)^2 \sup_{2 \le k,j \le l(\delta)} \mathbf{P}(A_{kj})$$

$$\le (l(\delta) - 1)^2 \sup_{\vec{\mathbf{s}}, \vec{\mathbf{t}} \in [0,1]^d:\ \vec{\mathbf{s}} < \vec{\mathbf{t}},\ |\vec{\mathbf{s}} - \vec{\mathbf{t}}| = \frac{2}{[\frac{1}{\delta}]}} \mathbf{P}\left(\sup_{\vec{\mathbf{s}} \le \vec{\mathbf{r}} \le \vec{\mathbf{t}}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{s}})| > \frac{\varepsilon}{2}\right). \tag{2.4}$$

Applying (2.1), (2.4), we get

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} g_n(\delta, \varepsilon)$$

$$\le \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \frac{1}{\psi(n)} \ln \left((l-1)^2 \sup_{\vec{\mathbf{s}}, \vec{\mathbf{t}} \in [0,1]^d:\ \vec{\mathbf{s}} < \vec{\mathbf{t}},\ |\vec{\mathbf{s}} - \vec{\mathbf{t}}| = \frac{2}{[\frac{1}{\delta}]}} \mathbf{P}\left(\sup_{\vec{\mathbf{s}} \le \vec{\mathbf{r}} \le \vec{\mathbf{t}}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{s}})| > \frac{\varepsilon}{2}\right)\right)$$

$$= \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \frac{1}{\psi(n)} \ln \sup_{\vec{\mathbf{s}}, \vec{\mathbf{t}} \in [0,1]^d:\ \vec{\mathbf{s}} < \vec{\mathbf{t}},\ |\vec{\mathbf{s}} - \vec{\mathbf{t}}| = \frac{2}{[\frac{1}{\delta}]}} \mathbf{P}\left(\sup_{\vec{\mathbf{s}} \le \vec{\mathbf{r}} \le \vec{\mathbf{t}}} |\mathcal{F}_n(\vec{\mathbf{r}}) - \mathcal{F}_n(\vec{\mathbf{s}})| > \frac{\varepsilon}{2}\right) = -\infty,$$

where the last equality follows from (2.1). Thus, lemma 2.1 is proved for the case $d = 2$. Obviously, the proof in the general case $d \ge 1$ is similar.

□

Let us prove one more auxiliary lemma.

Lemma 2.2. *Let the sequence of functions $f_n(\delta,\varepsilon)$, $n\in\mathbb{N}$ satisfies the conditions:*

1) for any $\varepsilon>0$

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} f_n(\delta,\varepsilon)=-\infty;$$

2) for any $n\in\mathbb{N}$, $\varepsilon>0$, $\delta\in(0,1)$, $h\in(0,\delta)$

$$f_n(\delta,\varepsilon)\geq f_n(h,\varepsilon).$$

Then for any $C>0$, $\varepsilon>0$ there exist $N(C,\varepsilon)\in\mathbb{N}$ and $\delta(\varepsilon,C)\in(0,1)$ such that for all $n\geq N(C,\varepsilon)$, $h\in(0,\delta(\varepsilon,C))$ the inequality

$$f_n(h,\varepsilon)\leq -C$$

holds.

The sense of lemma 2.2 is as follows: by condition 2) at every point $\varepsilon>0$ the repeated limit $\lim\limits_{\delta\downarrow 0}\limsup\limits_{n\to\infty} f_n(\delta,\varepsilon)$ coincides with the double limit.

P r o o f of lemma 2.2. Let us fix $\varepsilon>0$. It follows from the condition 1) that for any $C>0$ there exists $\delta(\varepsilon,C)\in(0,1)$ such that

$$\limsup_{n\to\infty} f_n(\delta(\varepsilon,C),\varepsilon)\leq -2C.$$

Therefore, there is $N(\varepsilon,C)$ such that the inequality

$$f_n(\delta(\varepsilon,C),\varepsilon)\leq -C \tag{2.5}$$

is met for all $n\geq N(\varepsilon,C)$. It follows from the inequality (2.5) and condition 2) that for all $h\in(0,\delta(\varepsilon,C))$ the inequality

$$f_n(h,\varepsilon)\leq f_n(\delta(\varepsilon,C),\varepsilon)\leq -C$$

is true.

□

Let us continue the proof of theorem 2.1. Now we show that for any $C>0$, $\varepsilon>0$ the inequality

$$\lim_{\delta\downarrow 0}\sup_{n\in\mathbb{N}} g_n(\delta,\varepsilon)\leq -C \tag{2.6}$$

is true. Indeed, it follows from (2.2) and lemma 2.2 (for $f_n(\delta,\varepsilon)=g_n(\delta,\varepsilon)$) that for any $C>0$, $\varepsilon>0$ there exist $N(C,\varepsilon)<\infty$ and $\delta(\varepsilon,C)\in(0,1)$ such that for all $n\geq N(C,\varepsilon)$, $h\in(0,\delta(\varepsilon,C))$ we have

$$g_n(h,\varepsilon)\leq -C; \tag{2.7}$$

it follows from the continuity of the field $\mathcal{F}_n$ that it is uniformly continuous on compact space $[0,1]^d$, so for any $n\in\mathbb{N}$

$$\lim_{h\downarrow 0} g_n(h,\varepsilon)=-\infty. \tag{2.8}$$

It follows from (2.7), (2.8) that one can choose such $\tilde\delta(\varepsilon,C)\in(0,1)$ that the inequality (2.7) holds for all $h\in(0,\tilde\delta(\varepsilon,C))$ and $n\in\mathbb{N}$, i.e. for all $h\in(0,\tilde\delta(\varepsilon,C))$ the inequality

$$\sup_{n\in\mathbb{N}} g_n(h,\varepsilon)\leq -C. \tag{2.9}$$

is true. Inequality (2.9) is equivalent to inequality (2.6).

Let us show that it follows from the inequality (2.6) (i.e. from (2.9)) that the sequence of the fields $\mathcal{F}_n$ is ET. We consider the sequence $\varepsilon_r\downarrow 0$ as $r\to\infty$. It follows from (2.9) that for any $Q > 0$ and $\varepsilon_r \in (0,1)$ there is $h_r \in (0, h_{r-1})$ such that

$$\sup_{n\in\mathbb{N}} g_n(h_r, \varepsilon_r) \le -rQ. \tag{2.10}$$

It follows from (2.10) that for any $n \in \mathbb{N}$, $r \in \mathbb{N}$ we have

$$\mathbf{P}\left(\sup_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in[0,1]^d:\ \vec{\mathbf{s}}<\vec{\mathbf{t}},\ |\vec{\mathbf{s}}-\vec{\mathbf{t}}|\le h_r}\left(\sup_{\vec{\mathbf{s}}\le\vec{\mathbf{r}}\le\vec{\mathbf{t}}}|\mathcal{F}_n(\vec{\mathbf{r}})-\mathcal{F}_n(\vec{\mathbf{s}})|\right) > \varepsilon_r\right)$$
$$= e^{\psi(n)g_n(h_r,\varepsilon_r)} \le e^{-Qr\psi(n)}. \tag{2.11}$$

For the sequence (ε_r, h_r) defined above we denote by $K(Q)$ the set of functions $\mathcal{H}(\cdot) \in \mathbb{C}_0[0,1]^d$ such that *for any* $r \in \mathbb{N}$ *and any* $\vec{\mathbf{t}}, \vec{\mathbf{s}} \in [0,1]^d$*:* $\vec{\mathbf{s}} < \vec{\mathbf{t}}$, $|\vec{\mathbf{t}} - \vec{\mathbf{s}}| \le h_r$ *the inequality*

$$\sup_{\vec{\mathbf{s}}\le\vec{\mathbf{r}}\le\vec{\mathbf{t}}}|\mathcal{H}(\vec{\mathbf{r}})-\mathcal{H}(\vec{\mathbf{s}})| \le \varepsilon_r.$$

holds. By the Arzela – Ascoli theorem (for functions of several variables see, e.g. [5, p. 236, theorem 21]) it follows that $K(Q)$ is a compact subset of $\mathbb{C}_0[0,1]^d$.

Using the inequality (2.11), we obtain

$$\limsup_{n\to\infty}\frac{1}{\psi(n)}\ln\mathbf{P}\Big(\mathcal{F}_n \notin K(Q)\Big) \le \limsup_{n\to\infty}\frac{1}{\psi(n)}\ln\left(\sum_{r=1}^{\infty}e^{-Qr\psi(n)}\right)$$
$$= \limsup_{n\to\infty}\frac{1}{\psi(n)}\ln\left(\frac{e^{-Q\psi(n)}}{1-e^{-Q\psi(n)}}\right) = -Q.$$

The ET of the sequence $\mathcal{F}_n$ is proved.

□

It is easy to see that the following result follows from theorem 2.1, and it is a convenient tool to prove the ET for integral – type functionals of independent random variables.

Corollary 2.1. *Let for any* $n \in \mathbb{N}$ *a random field* $\mathcal{F}_n = \mathcal{F}_n(\vec{\mathbf{t}})$ *belongs to the space* $\mathbb{C}_0[0,1]^d$ *with probability 1. Let for any* $\delta \in (0,1)$, $\varepsilon \in (0,1)$ *there exist an integer* $N(\delta,\varepsilon)$*, positives* $C_1 = C_1(\delta,\varepsilon)$ *and* $C_2 = C_2(\varepsilon)$ *such that for all* $n \ge N(\delta,\varepsilon)$ *the inequality*

(2.12)
$$\sup_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in[0,1]^d:\ \vec{\mathbf{s}}<\vec{\mathbf{t}},\ |\vec{\mathbf{s}}-\vec{\mathbf{t}}|=\delta}\mathbf{P}\left(\sup_{\vec{\mathbf{s}}\le\vec{\mathbf{r}}\le\vec{\mathbf{t}}}|\mathcal{F}_n(\vec{\mathbf{r}})-\mathcal{F}_n(\vec{\mathbf{s}})| > \varepsilon\right) \le C_1\exp\left\{-\psi(n)\frac{C_2}{\delta}\right\}$$

holds, where $\lim_{n\to\infty}\psi(n) = \infty$*. Then the sequence* $\mathcal{F}_n$ *is ET in the space* $\mathbb{C}[0,1]^d$ *with NF* $\psi(n)$*.*

Remark 1. *In section 3 we will use the condition (2.12) where the constant* C_1 *does not depend on the parameters* δ *and* ε*;* $C_2(\varepsilon) = \frac{\varepsilon^2}{C_3}$ *for some* $C_3 > 0$*.*

Remark 2. *The sufficient conditions for the ET of the sequence* $\mathcal{F}_n$ *in the case* $d = 1$ *(in a form close to the one considered in theorem 2.1) were obtained earlier in* [6]*, see also* [7, theorem 4.1]*. Let's note also the paper* [8] *(see bibliography therein), where another metric is considered and the sufficient conditions for ET are obtained in the case when the sequence* $\mathcal{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n}\xi_i(t)$ *is a normalized sum* *of an independent equally distributed continuous random fields, and the NF has the form $\psi(n) = n$. Since different problems are solved in present paper and* [8], *the results of these papers are difficult to compare.*

3. The proof of theorem 1.1

P r o o f of theorem 1.1. For simplicity, we prove this theorem for the case $d = 2$. For any $\vec{\mathbf{0}} < \vec{\mathbf{s}} < \vec{\mathbf{r}} < \vec{\mathbf{1}}$ the inequality

$$|\mathcal{G}_n(\vec{\mathbf{r}}) - \mathcal{G}_n(\vec{\mathbf{s}})| = \left| \int_0^{r_1} \int_0^{r_2} \mathcal{P}_n(u_1,u_2)du_1du_2 - \int_0^{s_1} \int_0^{s_2} \mathcal{P}_n(u_1,u_2)du_1du_2 \right|$$
$$\leq \left| \int_{s_1}^{r_1} \int_0^{r_2} \mathcal{P}_n(u_1,u_2)du_1du_2 \right| + \left| \int_0^{s_1} \int_{s_2}^{r_2} \mathcal{P}_n(u_1,u_2)du_1du_2 \right| \text{ a.s.}$$

is true. Thus, for arbitrary $\vec{\mathbf{0}} < \vec{\mathbf{s}} < \vec{\mathbf{r}} < \vec{\mathbf{t}} < \vec{\mathbf{1}}$ we have

$$\sup_{\vec{\mathbf{s}} \leq \vec{\mathbf{r}} \leq \vec{\mathbf{t}}} |\mathcal{G}_n(\vec{\mathbf{r}}) - \mathcal{G}_n(\vec{\mathbf{s}})| \leq \sup_{s_1 \leq r_1 \leq t_1, 0 \leq r_2 \leq t_2} \left| \int_{s_1}^{r_1} \int_0^{r_2} \mathcal{P}_n(u_1,u_2)du_1du_2 \right|$$
$$+ \sup_{s_2 \leq r_2 \leq t_2} \left| \int_0^{s_1} \int_{s_2}^{r_2} \mathcal{P}_n(u_1,u_2)du_1du_2 \right| \text{ a.s.}$$

Therefore, for any $\varepsilon > 0$ we obtain

$$\mathbf{P}\left(\sup_{\vec{\mathbf{s}} \leq \vec{\mathbf{r}} \leq \vec{\mathbf{t}}} |\mathcal{G}_n(\vec{\mathbf{r}}) - \mathcal{G}_n(\vec{\mathbf{s}})| > \varepsilon \right)$$
$$\leq \mathbf{P}\left(\sup_{s_1 \leq r_1 \leq t_1, 0 \leq r_2 \leq t_2} \left| \int_{s_1}^{r_1} \int_0^{r_2} \mathcal{P}_n(u_1,u_2)du_1du_2 \right| > \frac{\varepsilon}{2} \right)$$
$$+\mathbf{P}\left(\sup_{s_2 \leq r_2 \leq t_2} \left| \int_0^{s_1} \int_{s_2}^{r_2} \mathcal{P}_n(u_1,u_2)du_1du_2 \right| > \frac{\varepsilon}{2} \right) =: \mathbf{P}_1 + \mathbf{P}_2. \tag{3.1}$$

Let us estimate $\mathbf{P}_1$ from above. We denote

$$A_{s_1,r_1,r_2} := \left\{ i_1, i_2 : \Delta_n\left(\frac{i_1}{n}, \frac{i_2}{n}\right] \subseteq [s_1,r_1] \times [0,r_2] \right\},$$

$$B_{s_1,r_1,r_2} := \left\{ i_1, i_2 : \Delta_n\left(\frac{i_1}{n}, \frac{i_2}{n}\right] \not\subseteq [s_1,r_1] \times [0,r_2] \text{ and } \Delta_n\left(\frac{i_1}{n}, \frac{i_2}{n}\right] \cap [s_1,r_1] \times [0,r_2] \neq \varnothing \right\}$$

It is easy to see that

$$\int_{s_1}^{r_1} \int_0^{r_2} \mathcal{P}_n(u_1,u_2)du_1du_2$$
$$= \frac{1}{x} \sum_{(i_1,i_2) \in A_{s_1,r_1,r_2}} X_{i_1,i_2,n} + \frac{n^2}{x} \sum_{(i_1,i_2) \in B_{s_1,r_1,r_2}} X_{i_1,i_2,n}\mu(i_1,i_2), \tag{3.2}$$

where $\mu(i_1,i_2)$ is Lebesgue measure of the set

$$\Delta_n\left(\frac{i_1}{n}, \frac{i_2}{n}\right] \cap [s_1,r_1] \times [0,r_2].$$

Using equality (3.2), we obtain

$$\mathbf{P}_1 \leq \mathbf{P}\left(\sup_{s_1 \leq r_1 \leq t_1, 0 \leq r_2 \leq t_2} \left| \frac{1}{x} \sum_{(i_1,i_2) \in A_{s_1,r_1,r_2}} X_{i_1,i_2,n} \right| > \frac{\varepsilon}{4} \right)$$

$$+\mathbf{P}\biggl(\sup_{s_1\le r_1\le t_1, 0\le r_2\le t_2}\biggl|\frac{1}{x}\sum_{(i_1,i_2)\in B_{s_1,r_1,r_2}} X_{i_1,i_2,n}n^2\mu(i_1,i_2)\biggr| > \frac{\varepsilon}{4}\biggr) = \mathbf{P}_{11}+\mathbf{P}_{12}.$$

Let's estimate $\mathbf{P}_{11}$ from above. It is easy to see that the sum $\sum\limits_{(i_1,i_2)\in A_{s_1,r_1,r_2}} X_{i_1,i_2,n}$ contains no more then $[n^2(t_1-s_1)t_2]$ independent terms, all of which satisfies the condition $[\mathbf{C}_0(\alpha)]$. Therefore, it follows from lemma 4.1 (see section 4) that for sufficiently large n we have

$$\mathbf{P}_{11} \le 3\exp\biggl\{-\frac{\varepsilon^2x^2}{64\sigma^2_{\max}n^2(t_1-s_1)t_2}\biggr\} \le 3\exp\biggl\{-\frac{\varepsilon^2x^2}{64\sigma^2_{\max}n^2(t_1-s_1)}\biggr\}. \tag{3.3}$$

Let's estimate $\mathbf{P}_{12}$ from above. The sum $\sum\limits_{(i_1,i_2)\in B_{s_1,r_1,r_2}} X_{i_1,i_2,n}n^2\mu(i_1,i_2)$ contains no more then $O(n)$ independent terms. It follows from the inequality

$$0\le n^2\mu(i_1,i_2)\le 1$$

that all these terms satisfy the condition $[\mathbf{C}_0(\alpha)]$. Therefore, it follows from lemma 4.1 (see section 4) that the estimate (3.3) is also true for $\mathbf{P}_{12}$.

Thus, we have

$$\mathbf{P}_1 \le 6\exp\biggl\{-\frac{\varepsilon^2x^2}{64\sigma^2_{\max}n^2(t_1-s_1)}\biggr\}. \tag{3.4}$$

The following inequality

$$\mathbf{P}_2 \le 6\exp\biggl\{-\frac{\varepsilon^2x^2}{64\sigma^2_{\max}n^2(t_2-s_2)}\biggr\} \tag{3.5}$$

can be obtained by completely similar arguments. It follows from the inequalities (3.1), (3.4) and (3.5) that for sufficiently large n we have

$$\mathbf{P}\biggl(\sup_{\vec{\mathbf{s}}\le\vec{\mathbf{r}}\le\vec{\mathbf{t}}}|\mathcal{G}_n(\vec{\mathbf{r}})-\mathcal{G}_n(\vec{\mathbf{s}})|>\varepsilon\biggr) \le 12\exp\biggl\{-\frac{\varepsilon^2x^2}{64\sigma^2_{\max}n^2|\vec{\mathbf{t}}-\vec{\mathbf{s}}|}\biggr\}.$$

Therefore, the sequence of the fields $\mathcal{G}_n$ satisfies the conditions of corollary 2.1 with NF $\psi(n)=\frac{x^2}{n^2}$, and hence it is ET.

□

4. Auxiliary results

Let's formulate and prove the auxiliary lemma.

Lemma 4.1. *Let the random variables $Y_1,\dots,Y_k,\dots$ are independent, have zero mean and satisfy the uniform condition $[\mathbf{C}_0(\alpha)]$. Then*

$$\sigma^2_{\max} := \sup_{k\in\mathbb{N}}\mathbf{E}Y_k^2<\infty,$$

and for any $r>0$, $\varepsilon\in(0,1)$ there exists $N(r,\varepsilon)<\infty$ such that for all $n\ge N(r,\varepsilon)$ the inequality

$$\mathbf{P}\biggl(\sup_{1\le u\le r}\frac{1}{x}\biggl|\sum_{k=1}^{[n^du]}Y_k\biggr|>\varepsilon\biggr) \le 3\exp\Bigl\{-\frac{\varepsilon^2x^2}{4\sigma^2_{\max}n^dr}\Bigr\}. \tag{4.1}$$

holds.

P r o o f. Let's estimate from above the left – hand side of the inequality (4.1)

$$\mathbf{P}\bigg(\sup_{1\le u\le r}\frac{1}{x}\bigg|\sum_{k=1}^{[n^d u]}Y_k\bigg|>\varepsilon\bigg)=\mathbf{P}\bigg(\sup_{1\le u\le r}\frac{1}{x}\bigg|\sum_{k=1}^{[n^d u]}Y_k\bigg|>\varepsilon,\ \sup_{1\le k\le[n^d r]}|Y_k|\le x\bigg)$$

$$+\mathbf{P}\bigg(\sup_{1\le u\le r}\frac{1}{x}\bigg|\sum_{k=1}^{[n^d u]}Y_k\bigg|>\varepsilon,\ \sup_{1\le k\le[n^d r]}|Y_k|>x\bigg)$$

$$\le\mathbf{P}\bigg(\sup_{1\le u\le r}\frac{1}{x}\bigg|\sum_{k=1}^{[n^d u]}Y_k\mathbf{I}(|Y_k|\le x)\bigg|>\varepsilon\bigg)+\sum_{k=1}^{[n^d r]}\mathbf{P}\big(|Y_k|>x\big)$$

$$\le\mathbf{P}\bigg(\sup_{1\le u\le r}\frac{1}{x}\sum_{k=1}^{[n^d u]}Y_k\mathbf{I}(|Y_k|\le x)>\varepsilon\bigg)+\mathbf{P}\bigg(\sup_{1\le u\le r}\frac{1}{x}\sum_{k=1}^{[n^d u]}(-Y_k)\mathbf{I}(|Y_k|\le x)>\varepsilon\bigg)$$

$$+\sum_{k=1}^{[n^d r]}\mathbf{P}\big(|Y_k|>x\big)=:\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3, \tag{4.2}$$

where $\mathbf{I}(\cdot)$ is an indicator of an event $\{\cdot\}$.

Let us estimate $\mathbf{P}_3$ from above. Using the Chebyshev inequality and the condition $[\mathbf{C}_0(\alpha)]$, we obtain

$$\mathbf{P}\big(|Y_k|>x\big)\le\frac{\mathbf{E}e^{\lambda|Y_k|^\alpha}}{e^{\lambda x^\alpha}}\le Me^{-\lambda x^\alpha}. \tag{4.3}$$

It follows from the inequality (4.3) and condition (1.2) that there exists $N(r)\in\mathbb{N}$ such that for all $n\ge N(r)$

$$\mathbf{P}_3\le M[n^d r]e^{-\lambda x^\alpha}\le e^{-\frac{\lambda}{2}x^\alpha}. \tag{4.4}$$

Let us estimate $\mathbf{P}_1$ from above. It follows from the condition $[\mathbf{C}_0(\alpha)]$ that for all $k\in\mathbb{N}$

$$\begin{aligned}\sigma_k^2=\mathbf{E}Y_k^2&=\mathbf{E}Y_k^2\mathbf{I}(|Y_k|\le z(\alpha,\lambda))+\mathbf{E}Y_k^2\mathbf{I}(|Y_k|>z(\alpha,\lambda))\\&\le z^2(\alpha,\lambda)+\mathbf{E}e^{\lambda|Y_k|^\alpha}\le z^2(\alpha,\lambda)+M,\end{aligned}$$

where $z(\alpha,\lambda):=\min\{z>0:\ \text{for all } u\ge z \text{ the inequality } e^{\lambda u^\alpha}\ge u^2 \text{ holds}\}$.

Hence, we have

$$\sigma_{\max}^2:=\sup_{k\in\mathbb{N}}\mathbf{E}Y_k^2<\infty.$$

Applying the Cauchy – Bunyakovsky – Schwarz inequality and (4.3), we obtain

$$\mathbf{E}|Y_k|\mathbf{I}(|Y_k|>x)\le(\mathbb{E}Y_k^2)^{\frac{1}{2}}(\mathbf{P}(|Y_k|>x))^{\frac{1}{2}}\le\sqrt{M}\sigma_{\max}e^{-\frac{\lambda}{2}x^\alpha}. \tag{4.5}$$

Using the inequality (4.5) and the fact that $\mathbf{E}Y_k=0$, for any $k\in\mathbb{N}$, $c>0$ we get

$$\mathbf{E}e^{\frac{c}{x}Y_k\mathbf{I}(|Y_k|\le x)}\le\mathbf{E}\bigg(1+\frac{c}{x}Y_k+\sum_{v=2}^{\infty}\frac{|cY_k|^v}{x^v v!}\bigg)\mathbf{I}(|Y_k|\le x)$$

$$\le1+\frac{c}{x}\mathbf{E}|Y_k|\mathbf{I}(|Y_k|>x)+\mathbf{E}\frac{c^2}{2x^2}Y_k^2\exp\bigg\{\frac{c}{x}|Y_k|\bigg\}\mathbf{I}(|Y_k|\le x)$$

$$\le1+\frac{c}{x}\sqrt{M}\sigma_{\max}e^{-\frac{\lambda}{2}x^\alpha}+\mathbf{E}\frac{c^2}{2x^2}Y_k^2\exp\bigg\{\frac{c}{x}|Y_k|^\alpha|Y_k|^{1-\alpha}\bigg\}\mathbf{I}(|Y_k|\le x)$$

$$\leq 1 + \frac{c}{x}\sqrt{M}\sigma_{\max} e^{-\frac{\lambda}{2}x^\alpha} + \mathbf{E}\frac{c^2}{2x^2}Y_k^2 \exp\left\{\frac{c}{x}|Y_k|^\alpha x^{1-\alpha}\right\}.$$

Let $c = o(x^\alpha)$ as $n \to \infty$, then it follows from the condition $[\mathbf{C}_0(\alpha)]$ and Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \mathbf{E}Y_k^2 \exp\left\{\frac{c}{x^\alpha}|Y_k|^\alpha\right\} \leq \sigma_{\max}^2,$$

and the convergence is uniform over all $k \in \mathbb{N}$.

Therefore, for sufficiently large n we get

$$\mathbf{E}e^{\frac{c}{x}Y_k \mathbf{I}(|Y_k|\leq x)} \leq 1 + \frac{c}{x}\sqrt{M}\sigma_{\max} e^{-\frac{\lambda}{2}x^\alpha} + \frac{3c^2}{4x^2}\sigma_{\max}^2$$

$$\leq 1 + \frac{c^2}{x^2}\sigma_{\max}^2. \tag{4.6}$$

It follows from inequality (4.6) that for sufficiently large n we have

$$\sup_{1\leq u\leq r} \prod_{k=1}^{[n^d u]} \mathbf{E}e^{\frac{c}{x}Y_k \mathbf{I}(|Y_k|\leq x)} \leq \left(1 + \frac{c^2}{x^2}\sigma_{\max}^2\right)^{[n^d u]} \leq \exp\left\{\frac{c^2\sigma_{\max}^2[n^d r]}{x^2}\right\}. \tag{4.7}$$

It is easy to see that for any $c > 0$

$$V(v) := \frac{e^{\frac{c}{x}\sum\limits_{k=1}^{v} Y_k \mathbf{I}(|Y_k|\leq x)}}{\prod\limits_{k=1}^{v} \mathbf{E}e^{\frac{c}{x}Y_k \mathbf{I}(|Y_k|\leq x)}}, \quad v \in \mathbb{N}$$

is martingale and $\mathbf{E}V(v) = 1$. Therefore, using Doob's martingale inequality, we obtain

(4.8)

$$\mathbf{P}_1 = \mathbf{P}\left(\sup_{1\leq u\leq r} \frac{1}{x}\sum_{k=1}^{[n^d u]} Y_k \geq \varepsilon\right) \leq \mathbf{P}\left(\sup_{1\leq u\leq r} V([n^d u]) \geq \frac{e^{c\varepsilon}}{\Pi(r,x)}\right) \leq \frac{\Pi(r,x)}{e^{c\varepsilon}},$$

where

$$\Pi(r,x) := \sup_{1\leq u\leq r} \prod_{k=1}^{[n^d u]} \mathbf{E}e^{\frac{c}{x}Y_k \mathbf{I}(|Y_k|\leq x)}.$$

Let's choose

$$c := \frac{\varepsilon x^2}{2\sigma_{\max}^2 n^d r},$$

it follows from the condition (1.2) that $c = o(x^\alpha)$ as $n \to \infty$. Thus, using the inequalities (4.7) and (4.8), for sufficiently large n we have

$$\mathbf{P}_1 \leq \exp\left\{-\frac{\varepsilon^2 x^2}{4\sigma_{\max}^2 n^d r}\right\}. \tag{4.9}$$

Completely similarly, we can obtain the same estimate from above for $\mathbf{P}_2$.

Therefore, it follows from the inequalities (4.2), (4.4), (4.9) and condition (1.2) that for sufficiently large n we obtain

$$\mathbf{P}\left(\sup_{1\leq u\leq r} \frac{1}{x}\left|\sum_{k=1}^{[n^d u]} Y_k\right| > \varepsilon\right) \leq 2\exp\left\{-\frac{\varepsilon^2 x^2}{4\sigma_{\max}^2 n^d r}\right\} + e^{-\frac{\lambda}{2}x^\alpha} \leq 3\exp\left\{-\frac{\varepsilon^2 x^2}{4\sigma_{\max}^2 n^d r}\right\}.$$

$\square$

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