S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 19, №1, стр. 285–291 (2022) УДК 512.54
DOI 10.33048/semi.2022.19.022 MSC 20F28, 20D06, 20E22, 20J06

AUTOMORPHISMS OF NONSPLIT COVERINGS OF $PSL_2(q)$ IN ODD CHARACTERISTIC DIVIDING $q-1$

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Abstract. We classify the nonsplit extensions of elementary abelian p-groups by $\mathrm{PSL}_2(q)$, with odd p dividing $q-1$, for an irreducible induced action, calculate the relevant low-dimensional cohomology groups, and describe the automorphism groups of such extensions.

Keywords: Automorphism group, nonsplit extension, cohomology.

1. Introduction

Given a short exact sequence of groups

$$0 \to V \to G \to L \to 1, \tag{1}$$

where V is abelian (written additively), we say that G is *an extension* of V by L, or *a covering* of L with kernel V. Such extensions arise naturally in inductive arguments or when constructing minimal examples and counterexamples. We will be interested in the case where G is finite and nonsplit and V acquires the structure of an irreducible FL-module (for a suitable finite field F of characteristic p) from the conjugation in G. Such extensions can only exist if p divides $|L|$. We also restrict ourselves to the case $L \cong \mathrm{PSL}_2(q)$. Extensions of this form for $p=2$ and q odd were explicitly constructed in [1], and their automorphism groups were described [11]. Some results in the case where q is a power of p were obtained in [2]. The aim of this paper is to classify such extensions in the case $2 \neq p \mid (q-1)$ and describe their automorphism groups. In this case, we can use the fact that the natural permutation FL-module arising from the action of L on the projective line

Zavarnitsine, A.V., Automorphisms of nonsplit coverings of $PSL_2(q)$ in odd characteristic dividing $q-1$.

Supported by RAS Fundamental Research Program, project FWNF-2022-0002.
Received Murch, 31, 2022, published May, 17, 2022.

over $\mathbb{F}_q$ is completely reducible. This is not so if $2 \neq p \mid (q+1)$ which case will be the subject of a future research. We now state the main results.

Theorem 1. *Up to isomorphism there is a unique nonsplit extension of an elementary abelian p-group V by $L = \mathrm{PSL}_2(q)$ with irreducible induced action of L on V, where $2 \neq p \mid (q-1)$. In this extension, $|V| = p^q$.*

The group V from Theorem 1 can be identified with the unique nonprincipal irreducible $\mathbb{F}_pL$-module in the principal p-block of L. The low-dimensional cohomology of V is as follows.

Theorem 2. *In the above notation, we have $H^1(L,V) \cong H^2(L,V) \cong \mathbb{F}_p$.*

Recall that $P\Gamma L_2(q)$ denotes the extension of $PGL_2(q)$ by its field automorphisms. The automorphism group of the nonsplit extension from Theorem 1 is described by

Theorem 3. *Let G fit in the nonsplit exact sequence (1), where V is an irreducible $\mathbb{F}_pL$-module for $L = \mathrm{PSL}_2(q)$ and $2 \neq p \mid (q-1)$. Then there is a short exact sequence*

$$0 \to W \to \mathrm{Aut}(G) \to P\Gamma L_2(q) \to 1, \tag{2}$$

where W is elementary abelian of order p^{q+1}.

2. Auxiliary facts

Basic notation and facts of homological algebra can be found in [7, 13]. For abelian groups A and B, we denote $\mathrm{Hom}(A,B) = \mathrm{Hom}_{\mathbb{Z}}(A,B)$ and $\mathrm{Ext}(A,B) = \mathrm{Ext}^1_{\mathbb{Z}}(A,B)$.

Lemma 4 (The Universal Coefficient Theorem for Cohomology)**.** [7, Ch. 3, Theorem 3] *For all $i \geqslant 1$, every group G, and every trivial G-module A,*

$$H^i(G,A) \cong \mathrm{Hom}(H_i(G,\mathbb{Z}),A) \oplus \mathrm{Ext}(H_{i-1}(G,\mathbb{Z}),A).$$

Lemma 5. [7, §3.5] *For a trivial G-module A, we have*

(*i*) $H^1(G,A) \cong \mathrm{Hom}(G/G', A)$.

(*ii*) $H_1(G,A) \cong G/G' \otimes_{\mathbb{Z}} A$.

Lemma 6 (Shapiro's lemma)**.** [13, §6.3] *Let $H \leqslant G$ with $|G:H|$ finite. If V is an H-module and $i \geqslant 0$ then $H^i(G,V^G) \cong H^i(H,V)$, where V^G is the induced G-module.*

Lemma 7. [5, p. 322] $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$, *where $d = (m,n)$.*

Lemma 8. [13, Proposition 3.3.4]. $\mathrm{Ext}^i_R(A, B_1 \oplus B_2) \cong \mathrm{Ext}^i_R(A,B_1) \oplus \mathrm{Ext}^i_R(A,B_2)$ *for all rings R, R-modules A, B_1, B_2, and all $i \geqslant 0$.*

The Schur multiplier of a group G is denoted by $\mathrm{Sch}(G)$. If A is a finite abelian group and p a prime then $A_{(p)}$ denotes the p-primary component of A. Henceforth, we assume that G is finite.

Lemma 9. [9, Theorem 25.1] *Let p be a prime and let $S \in \mathrm{Syl}_p(G)$. Then $\mathrm{Sch}(G)_{(p)}$ is isomorphic to a subgroup of $\mathrm{Sch}(S)$.*

Lemma 10. [6] *Let F be a field of characteristic $p > 0$ and let V be an irreducible FG-module that does not belong to the principal p-block of G. Then $H^n(G,V) = 0$ for all $n \geqslant 0$.*

Let θ be an irreducible character of G. If $\mathrm{Z}(G) = 1$ then $G \trianglelefteq \mathrm{Aut}(G)$ and we may speak of the inertia group $I_{\mathrm{Aut}(G)}(\theta) = \{g \in \mathrm{Aut}(G) \mid \theta^g = \theta\}$.

Proposition 1. [11, Proposition 4] *Let F be a field and $\mathscr{X}$ a faithful irreducible F-representation of a group G with Brauer character $\theta \in \mathrm{iBr}_F(G)$ of degree n. Suppose that $\mathrm{Z}(G) = 1$ and denote*

$$N = N_{\mathrm{GL}_n(F)}(\mathscr{X}(G)) \quad \textit{and} \quad Z = C_{\mathrm{GL}_n(F)}(\mathscr{X}(G)).$$

Then $N/Z \cong I_{\mathrm{Aut}(G)}(\theta)$.

3. Isomorphic extensions

Let Q be a group, K a commutative ring with 1, and M a right KQ-module. The pair $(\nu, \mu) \in \mathrm{Aut}(Q) \times \mathrm{Aut}_K(M)$ is *compatible* if

$$(mg)\mu = (m\mu)(g\nu)$$

for all $m \in M$, $g \in Q$. The set of all compatible pairs forms a group $\mathrm{Comp}(Q, M)$ under composition. Given $\tau \in Z^2(Q, M)$, one can define

$$\tau^{(\nu,\mu)}(g, h) = \tau(g\nu^{-1}, h\nu^{-1})\mu \tag{3}$$

for all $g, h \in Q$. Then the map $\tau \mapsto \tau^{(\nu,\mu)}$ is an action of $\mathrm{Comp}(Q, M)$ on $Z^2(Q, M)$ which preserves $B^2(Q, M)$ and so yields an action on $H^2(Q, M)$.

A *KQ-module extension* on M by Q is a group E that fits in the short exact sequence

$$0 \to M \xrightarrow{\iota} E \xrightarrow{\pi} Q \to 1 \tag{4}$$

so that the conjugation of M (identified with $M\iota$) by elements of E agrees with the KQ-module structure of M, i. e. $m^e = m(e\pi)$ for all $m \in M$, $e \in E$.

Proposition 2. [8, §2.7.4] *Classes of those isomorphisms of KQ-module extensions of M by Q that leave M invariant as a K-module are in a one-to-one correspondence with the orbits of* $\mathrm{Comp}(Q, M)$ *on* $H^2(Q, M)$.

In Proposition 2, an isomorphism leaving M invariant as a K-module means one that induces on M an element of $\mathrm{Aut}_K(M)$.

The KQ-module structure on M gives rise to the representation homomorphism $\mathscr{C} : Q \to \mathrm{Aut}_K(M)$ by the rule $\mathscr{C}(g) : m \mapsto mg$ for all $m \in M$, $g \in Q$. Let C be the centraliser of $\mathscr{C}(Q)$ in $\mathrm{Aut}_K(M)$. Then $(1, \gamma) \in \mathrm{Comp}(Q, M)$ for every $\gamma \in C$, because

$$(mg)\gamma = m\mathscr{C}(g)\gamma = m\gamma\mathscr{C}(g) = (m\gamma)g$$

for all $m \in M$, $g \in Q$. Hence, we also have an action of C on both $Z^2(Q, M)$ and $H^2(Q, M)$ by setting $\tau^\gamma = \tau^{(1,\gamma)}$ for $\tau \in Z^2(Q, M)$, $\gamma \in C$, i. e. $\tau^\gamma(g, h) = \tau(g, h)\gamma$. By Proposition 2, this yields the following:

Lemma 11. *The elements of $H^2(Q, M)$ that are in the same C-orbit correspond to isomorphic KQ-module extensions.*

In particular, we have the following fact, where elements of $H^2(Q, M)$ are called scalar multiples if they differ by a factor in $K^\times$.

Corollary 12. *KQ-module extensions of M by Q corresponding to scalar multiples in $H^2(Q, M)$ are isomorphic.*

4. Automorphisms of extensions

Fix an extension

$$\boldsymbol{e}: \qquad 0 \to M \xrightarrow{\iota} E \to Q \to 1 \tag{5}$$

with abelian kernel M. Let $\mathscr{C} : Q \to \mathrm{Aut}(M)$ be the induced representation and let $\overline{\varphi} \in H^2(Q, M)$ be the element that corresponds to $\boldsymbol{e}$. We assume that $\mathscr{C}$ is faithful. In particular, $Q \cong \mathscr{C}(Q)$ and the conjugation of $\mathscr{C}(Q)$ by any $\mu \in N_{\mathrm{Aut}(M)}(\mathscr{C}(Q))$ induces an element $\mu' \in \mathrm{Aut}(Q)$, i. e. $\mathscr{C}(g)^\mu = \mathscr{C}(g\mu')$ for all $g \in Q$. One defines an action of $N_{\mathrm{Aut}(M)}(\mathscr{C}(Q))$ on $H^2(Q, M)$ given by

$$\overline{\psi} \mapsto (\mu')^{-1}\overline{\psi}\mu \tag{6}$$

for every $\mu \in N_{\mathrm{Aut}(M)}(\mathscr{C}(Q))$ and $\overline{\psi} \in H^2(Q, M)$, which should be understood modulo $B^2(Q, M)$ for representative cocycles, see [12] for details. We denote by $N^{\overline{\varphi}}_{\mathrm{Aut}(M)}(\mathscr{C}(Q))$ the stabiliser of $\overline{\varphi}$ with respect to this action. Let $\mathrm{Aut}(\boldsymbol{e})$ denote the group of those automorphisms of E that leave $M\iota$ invariant as a set.

Proposition 3. [12, Statements (4.4),(4.5)] *Let the extension* (5) *have abelian kernel M and let it determine an element $\overline{\varphi} \in H^2(Q, M)$ and an injective induced representation $\mathscr{C} : Q \to \mathrm{Aut}(M)$. Then there exists a short exact sequence of groups*

$$0 \to Z^1(Q, M) \to \mathrm{Aut}(\boldsymbol{e}) \to N^{\overline{\varphi}}_{\mathrm{Aut}(M)}(\mathscr{C}(Q)) \to 1. \tag{7}$$

Remark. It is easy to see that, in the notation above, there is an embedding $N_{\mathrm{Aut}(M)}(\mathscr{C}(Q)) \to \mathrm{Comp}(Q, M)$, $\mu \mapsto (\mu', \mu)$, where we view M as a $\mathbb{Z}Q$-module, under which action (6) becomes a particular case of (3), and that this embedding is in fact an isomorphism in case $\mathscr{C}$ is faithful (which we assume).

5. Cohomology of $PSL_2(q)$ in characteristic dividing $q-1$

The aim of this section is to classify up to group isomorphism nonsplit extensions (1), where $L = \mathrm{PSL}_2(q)$, V is an elementary abelian p-group with irreducible induced action of L, and $p \neq 2$ is a divisor of $q-1$.

By Lemma 10, V must belong to the the principal p-block of L. This block contains only one nonprincipal module with Brauer character χ, see [3]. The values of characters in the principal block are shown in Table 1.

Table 1. Brauer p-modular characters of $L = PSL_2(q)$ in the principal block, where $2 \neq p \mid (q-1)$. Notation: $q = l^m$, l prime, $d = (2, q-1)$, $x, y \in L$, $|x| = \frac{1}{d}(q-1)_{p'}$, $|y| = \frac{1}{d}(q+1)$.

q odd	$1a$	$2a$	la	lb	$(x^r)^L$	$(y^t)^L$
1_L	1	1	1	1	1	1
χ	q	-1	0	0	1	-1

q even	$1a$	$2a$	$(x^r)^L$	$(y^t)^L$
1_L	1	1	1	1
χ	q	0	1	-1

We first note that V is not the principal module. Indeed, extension (1) would otherwise be central, but $\mathrm{Sch}(L)$ has no p-torsion, because

$$\mathrm{Sch}(L) = \begin{cases} \mathbb{Z}_2, & q \neq 9 \text{ odd or } q = 4; \\ \mathbb{Z}_6, & q = 9; \\ 1, & q \neq 4 \text{ even} \end{cases} \tag{8}$$

as follows from [4]. Therefore, V must be the $\mathbb{F}_pL$-module with character χ.

We can now prove Theorem 2 stated in the introduction.

Proof. Let P be the permutation $\mathbb{F}_pL$-module of dimension $q+1$ that corresponds to the natural permutation action of L on the projective line over $\mathbb{F}_q$. We have $P = I_L \oplus V$, where I_L is the principal $\mathbb{F}_pL$-module. This can be deduced either by considering the Brauer character χ of V or from [10, Table 1]. In particular, by Lemma 8, we have

$$H^i(L,P) \cong H^i(L,I_L) \oplus H^i(L,V) \tag{9}$$

for $i = 1, 2$, since $H^i(L,B) \cong \mathrm{Ext}^i_{\mathbb{F}_pL}(\mathbb{F}_p, B)$ for every $\mathbb{F}_pL$-module B, see [13, Exercise 6.1.2]. Since P is a permutation module, we have $P \cong (I_H)^L$, where I_H is the principal $\mathbb{F}_pH$-module for a point stabiliser $H \leqslant L$. Hence, Lemma 6 implies

$$H^i(L,P) \cong H^i(H,I_H) \tag{10}$$

for $i = 1, 2$. By Lemma 5(i), we have

$$H^1(L,I_L) \cong \mathrm{Hom}(L/L', I_L) = 0, \tag{11}$$

since $L = L'$. Also,

$$H^1(H,I_H) \cong \mathrm{Hom}(H/H', I_H) \cong \mathbb{F}_p, \tag{12}$$

since $I_H \cong \mathbb{F}_p$, $H \cong \mathbb{F}_q \rtimes \mathbb{Z}_{(q-1)/(2,q-1)}$, and $p \mid (q-1)$. By Lemma 4, we have

$$H^2(L,I_L) \cong \mathrm{Hom}(H_2(L,\mathbb{Z}), I_L) \oplus \mathrm{Ext}(H_1(L,\mathbb{Z}), I_L),$$

where the first summand vanishes, since $H_2(L,\mathbb{Z}) \cong \mathrm{Sch}(L)$ has no p-torsion by (8), and the second summand vanishes by Lemma 5(ii), since $L/L' = 1$. Thus

$$H^2(L,I_L) = 0. \tag{13}$$

Finally, Lemma 4 also yields

$$H^2(H,I_H) \cong \mathrm{Hom}(H_2(H,\mathbb{Z}), I_H) \oplus \mathrm{Ext}(H_1(H,\mathbb{Z}), I_H). \tag{14}$$

By Lemma 9, the p-part of $H_2(H,\mathbb{Z}) \cong \mathrm{Sch}(H)$ is isomorphic to a subgroup of $\mathrm{Sch}(S)$ for a p-Sylow subgroup S of H. However, S is cyclic and cyclic groups have trivial Schur multiplier. Thus, the first summand in (14) vanishes, because $I_H \cong \mathbb{F}_p$. Since $H_1(H,\mathbb{Z}) \cong H/H' \cong \mathbb{Z}_{(q-1)/d}$ and $\mathrm{Ext}(\mathbb{Z}_{(q-1)/d}, \mathbb{F}_p) \cong \mathbb{F}_p$ by Lemma 7, we have

$$H^2(H,I_H) \cong \mathbb{F}_p. \tag{15}$$

The claim follows by combining (9) through (15). □

We can now prove Theorem 1 stated in the introduction.

Proof. As we explained in the beginning of this section, V viewed as an $\mathbb{F}_pL$-module must be the unique nonprincipal module in the principal p-block of L. This module has dimension q and can be written over $\mathbb{F}_p$, since it is a direct summand of a permutation module. Therefore, $|V| = p^q$. By Theorem 2, we have $H^2(V,L) \cong \mathbb{F}_p$ and so all nonzero elements of $H^2(V,L)$ are scalar multiples of one another. By Corollary 12, they correspond to isomorphic nonsplit extensions. The claim follows. □

6. The automorphism group

In this section, we prove that the structure of the automorphism group of the unique nonsplit extension from Theorem 1 is as stated in Theorem 3.

Proof. Consider the extension $\boldsymbol{e}$ given by (1). Theorem 1 implies that G is unique up to isomorphism and V has order p^q. Moreover, viewed as an $\mathbb{F}_pL$-module, V has Brauer character χ from Table 1. By Proposition 3, we have the short exact sequence

$$0 \to Z^1(L,V) \to \operatorname{Aut}(\boldsymbol{e}) \to N^{\overline{\varphi}}_{\operatorname{Aut}(V)}(\mathscr{X}(L)) \to 1, \tag{16}$$

where the representation $\mathscr{X} : L \to \operatorname{Aut}(V)$ and the element $\overline{\varphi} \in H^2(L,V)$ are determined by (1). First, note that $\operatorname{Aut}(\boldsymbol{e}) = \operatorname{Aut}(G)$ as V is characteristic in G. Denote $W = Z^1(L,V)$. Since $B^1(L,V) \cong V/C_V(L)$ and L acts on V irreducibly and nontrivially, we have $C_V(L) = 0$ and $B^1(L,V) \cong V$. Now, since $H^1(L,V) = Z^1(L,V)/B^1(L,V)$, we have $|Z^1(L,V)| = p^{q+1}$ in view of Theorem 2.

Denote $N = N_{\mathrm{GL}(V)}(\mathscr{X}(L))$ and $Z = C_{\mathrm{GL}(V)}(\mathscr{X}(L))$. By Proposition 1, we have $N/Z \cong I_{\operatorname{Aut}(L)}(\chi)$. Since χ is the only irreducible character of L of dimension q, it must be invariant under any automorphism; in particular, $I_{\operatorname{Aut}(L)}(\chi) = \operatorname{Aut}(L)$. By [4], $\operatorname{Aut}(L) \cong P\Gamma L_2(q)$. Since V is absolutely irreducible as an $\mathbb{F}_pL$-module, by Schur's lemma, we see that $Z \cong \mathbb{F}_p^\times \cong \mathbb{Z}_{p-1}$ consists of scalars.

In order to determine the structure of the stabiliser $N_0 = N^{\overline{\varphi}}_{\operatorname{Aut}(V)}(\mathscr{X}(L))$, we consider the action of N on $H^2(L,V)$ as explained in Section 4. Let $H^\times$ denote the set of $p-1$ nonzero elements of $H^2(L,V)$. The elements of $H^\times$ correspond to nonsplit extensions and so we have an action homomorphism $\alpha : N \to \operatorname{Sym}(H^\times)$ to the symmetric group on $H^\times$. Since all nonsplit extension of V by L are isomorphic by Theorem 1, we may assume that $\overline{\varphi}$ is an arbitrary element of $H^\times$. The subgroup $Z \leqslant N$ acts on $H^\times$ by scalar multiplication, cf. Corollary 12, and so the image $\alpha(Z)$ is a cyclic subgroup of $\operatorname{Sym}(H^\times)$ generated by a full cycle of length $p-1$. Since Z is central in N, $\alpha(N)$ must centralise $\alpha(Z)$. However, a full cyclic subgroup is self-centralising in $\operatorname{Sym}(H^\times)$ and so $\alpha(Z)$ must be the entire image $\alpha(N)$. Thus, $\operatorname{Ker}(\alpha)$ is a normal subgroup of N of index $p-1$ which intersects trivially with Z and is thus isomorphic to $N/Z \cong P\Gamma L_2(q)$. Furthermore, $\operatorname{Ker}(\alpha)$ coincides with the stabiliser of every element of $H^\times$ which yields $N = N_0 \times Z$ and $N_0 \cong P\Gamma L_2(q)$ as claimed. ☐

It also follows from this proof that the representation $\mathscr{X} : L \to \operatorname{Aut}(V)$ with character χ extends to a representation of $I_{\operatorname{Aut}(L)}(\chi) \cong P\Gamma L_2(q)$. This fact does not hold in general for a simple group L and its irreducible character χ, see [11, Example 1].

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