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## REGULARITY CRITERION FOR WEAK SOLUTIONS TO THE NAVIER-STOKES INVOLVING ONE VELOCITY AND ONE VORTICITY COMPONENTS

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**ABSTRACT.** In this note, we are devoted to study the conditional regularity for the three dimensional Navier-Stokes in terms of the Morrey and *BMO* spaces. More precisely, we show that if  $u$  is a weak solution and  $u_3 \in L^2(0, T; BMO(\mathbb{R}^3))$  and  $\omega_3 \in L^{\frac{2}{2-r}}(0, T; \mathcal{M}_{2, \frac{3}{r}}(\mathbb{R}^3))$  with  $0 < r < 1$ , then  $u$  is regular on  $(0, T]$ . This improves the available result by Zhang (2018) with  $u_3 \in L^2(0, T; L^\infty(\mathbb{R}^3))$  and  $\omega_3 \in L^{\frac{2}{2-r}}(0, T; L^{\frac{3}{r}}(\mathbb{R}^3))$  with  $0 < r < 1$ .

**Keywords:** Navier-Stokes equations, regularity criteria, Morrey space.

### 1. INTRODUCTION

In this note, we study the regularity condition of weak solutions to the Navier-Stokes equations in  $\mathbb{R}^3 \times (0, T)$  :

$$(1) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla \pi = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

Here,  $u$  is the unknown velocity vector and  $\pi$  is the unknown scalar pressure.

For  $u_0 \in L^2(\mathbb{R}^3)$  with  $\nabla \cdot u_0 = 0$  in  $\mathbb{R}^3$ , Leray [19] constructed global weak solutions. The smoothness of Leray's weak solutions is unknown. While the existence of regular solutions is still an open problem, there are many interesting sufficient

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conditions which guarantee that a given weak solution is smooth. A well-known condition states that if

$$(2) \quad u \in L^s(0, T; L^r(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{s} + \frac{3}{r} = 1, \quad 3 \leq r \leq \infty,$$

then the solution  $u$  is actually regular [6, 9, 14, 20, 22, 23, 25, 26]. A similar condition

$$(3) \quad \nabla u \in L^s(0, T; L^r(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{s} + \frac{3}{r} = 2, \quad \frac{3}{2} \leq r \leq \infty,$$

also implies the regularity as shown by Beirão da Veiga [1].

Many regularity results on the weak solutions to the three-dimensional Navier-Stokes equations have been well studied, for example see [2, 3, 4, 8, 10, 11, 12, 13, 15, 16, 17, 24, 27, 29, 31, 32, 33, 34] and the related references therein, where they have proved that the solution is a smooth one if the velocity, or vorticity, or the gradient of velocity, or their components are regular. In particular, Penel and Pokorný in [21], proposed the following regularity criterion:

$$\begin{aligned} u_3 &\in L^s(0, T; L^r(\mathbb{R}^3)); \quad \partial_1 u_2, \partial_2 u_1 \in L^p(0, T; L^q(\mathbb{R}^3)) \\ \text{with } \frac{2}{s} + \frac{3}{r} &= 1, \quad 3 < r \leq \infty \quad \text{and} \quad \frac{2}{p} + \frac{3}{q} = 2, \quad 2 \leq q \leq 3, \end{aligned}$$

and remarked that it is an interesting open problem whether we could make assumptions on  $\omega_3 = \partial_1 u_2 - \partial_2 u_1$  instead of  $\partial_1 u_2$  and  $\partial_2 u_1$ , where  $\omega_3$  is the third component of the vorticity  $\omega = (\omega_1, \omega_2, \omega_3) = \text{rot} u \equiv (\partial_j u^k - \partial_k u^j)_{1 \leq j, k \leq 3}$ .

After that Zhang et al. in [30], proposed the following criterion under the condition

$$\begin{aligned} \partial_3 u_3 &\in L^s(0, T; L^r(\mathbb{R}^3)); \quad \omega_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \\ \text{with } \frac{2}{s} + \frac{3}{r} &= 2, \quad \frac{3}{2} < r \leq \infty \quad \text{and} \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \leq \infty. \end{aligned}$$

Recently, Zhang [28] studied the regularity criterion of the weak solutions involving one velocity and one vorticity component in Lebesgue space

$$(4) \quad \begin{aligned} u_3 &\in L^s(0, T; L^r(\mathbb{R}^3)); \quad \omega_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \\ \text{with } \frac{2}{s} + \frac{3}{r} &= 1, \quad 3 < r \leq \infty \quad \text{and} \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \leq \infty. \end{aligned}$$

The aim of this study is to refine (4) when  $r = \infty$  and to improve the above regularity criterion (4) from Lebesgue space framework to critical Morrey space framework.

Recall that for  $1 < p \leq q < \infty$ , the homogeneous Morrey space  $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)$  is defined as the function  $f \in L^p_{\text{loc}}(\mathbb{R}^3)$  such that

$$\|f\|_{\dot{\mathcal{M}}_{p,q}} = \sup_{x \in \mathbb{R}^3, 0 < R < \infty} R^{\frac{3}{q} - \frac{3}{p}} \left( \int_{|x-y| < R} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

This space is an homogeneous space of degree  $-\frac{3}{q}$  and implies the following relation ship :

$$L^q(\mathbb{R}^3) = \dot{\mathcal{M}}_{q,q}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{p,q}(\mathbb{R}^3) \subset L^p_{\text{loc}}(\mathbb{R}^3), \quad 1 < p \leq q < \infty.$$

Moreover, we also need the predual of  $\dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3)$ , which is investigated by Lemarié-Rieusset [18] will be used in the following section.

**Lemma 1.** For  $0 \leq r < \frac{3}{2}$ , let the space  $\mathcal{M}(\dot{B}_{r,1}^2 \mapsto L^2)$  as the space of functions which are locally square integrable on  $\mathbb{R}^3$  and such that pointwise multiplication with these functions maps boundedly the Besov space  $\dot{B}_{2,1}^r(\mathbb{R}^3)$  to  $L^2(\mathbb{R}^3)$ . The norm in  $\mathcal{M}(\dot{B}_{2,1}^r \mapsto L^2)$  is given by the operator norm of pointwise multiplication:

$$\|f\|_{\mathcal{M}(\dot{B}_{2,1}^r \mapsto L^2)} = \sup_{\|g\|_{\dot{B}_{2,1}^r} \leq 1} \|fg\|_{L^2}.$$

Then,  $f$  belongs to  $\mathcal{M}(\dot{B}_{2,1}^r \mapsto L^2)$  if and only if  $f$  belongs to  $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$  (with equivalence of norms).

Before stating our result, let us recall the definitions of weak solutions to (1).

**Definition 1.** Let  $u_0 \in L^2(\mathbb{R}^3)$  with  $\nabla \cdot u_0 = 0$  in  $\mathbb{R}^3$ . The function  $u$  is called a Leray weak solution of (1) in  $(0, T)$  if  $u$  satisfies the following properties:

- (i):  $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$ ;
- (ii):  $\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla \pi = 0$  in  $\mathcal{D}'((0, T) \times \mathbb{R}^3)$ ;
- (iii):  $\nabla \cdot u = 0$  in  $\mathcal{D}'((0, T) \times \mathbb{R}^3)$ ;
- (iv): the energy inequality holds

$$(5) \quad \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2, \quad \text{for all } t \in [0, T].$$

Our main result reads as follows.

**Theorem 1.** Let  $u_0 \in L^2(\mathbb{R}^3)$  with  $\nabla \cdot u_0 = 0$  and  $T > 0$ . Suppose that  $u$  is a Leray weak solution of (1) in  $(0, T)$ . If

$$u_3 \in L^2(0, T; BMO(\mathbb{R}^3)) \quad \text{and} \quad \omega_3 \in L^{\frac{2}{2-r}}(0, T; \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)) \quad \text{with} \quad 0 < r < 1,$$

then  $u$  is smooth in  $(0, T]$ . Here,  $BMO$  is the space of functions of bounded mean oscillations.

**Remark 1.** It is easy to see that the Theorem 1 is a refined improvement of that Theorem 1 in [28] due to the well-known embedding  $L^{\frac{3}{r}}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$  for  $0 \leq r < \frac{3}{2}$ . Furthermore, our purpose is to extend these results to the marginal space  $BMO$ , which is larger than  $L^\infty$ . It is a natural way to extend the space widely and improve the previous results.

## 2. PROOF OF THEOREM 1.

*Proof.* Taking inner product of the first equation of (1) with  $-\Delta u$  and integrating by parts, we see that

$$(6) \quad \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int_{\mathbb{R}^3} (u \cdot \nabla)u \cdot \Delta u dx = I.$$

Firstly, with the aid of the divergence free condition  $\sum_{j=1}^3 \partial_j u_j = 0$  and integration by parts, we decompose  $I$  as :

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \sum_{i,j,k=1}^3 u_j \partial_j u_i \partial_{kk}^2 u_i dx = - \int_{\mathbb{R}^3} \sum_{i,j,k=1}^3 \partial_k (u_j \partial_j u_i) \partial_k u_i dx \\ &= - \int_{\mathbb{R}^3} \sum_{i,j,k=1}^3 \partial_k u_j \partial_j u_i \partial_k u_i dx - \frac{1}{2} \int_{\mathbb{R}^3} \sum_{i,j,k=1}^3 u_j \partial_j (\partial_k u_i \partial_k u_i) dx \\ &= - \int_{\mathbb{R}^3} \sum_{i,j,k=1}^3 \partial_k u_j \partial_j u_i \partial_k u_i dx. \end{aligned}$$

Following [28], we classify the terms  $\partial_k u_j \partial_j u_i \partial_k u_i$ ,  $1 \leq i, j, k \leq 3$  as

(1) If  $k = j = 3$  or  $j = i = 3$  or  $k = i = 3$ , we then invoke the divergence free condition to replace  $\partial_3 u_3$  by  $-(\partial_1 u_1 + \partial_2 u_2)$ ;

(2) Otherwise, at least two indices belong to  $\{1, 2\}$ . Thus  $I$  will be

$$\begin{aligned} I &= \sum_{i,j,kl=1}^3 \int_{\mathbb{R}^3} \eta_{11ijkl} \partial_1 u_1 \partial_i u_j \partial_k u_l dx + \sum_{i,j,kl=1}^3 \int_{\mathbb{R}^3} \eta_{12ijkl} \partial_1 u_2 \partial_i u_j \partial_k u_l dx \\ &+ \sum_{i,j,kl=1}^3 \int_{\mathbb{R}^3} \eta_{21ijkl} \partial_2 u_1 \partial_i u_j \partial_k u_l dx + \sum_{i,j,kl=1}^3 \int_{\mathbb{R}^3} \eta_{22ijkl} \partial_2 u_2 \partial_i u_j \partial_k u_l dx \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where  $\eta_{mni jkl}$  ( $1 \leq m, n \leq 2$ ), are suitable integers. Next, we want to represent  $\partial_m u_n$  ( $1 \leq m, n \leq 2$ ) by  $u_3$  and  $\omega_3$ . Denoting by  $\Delta_h = \partial_1 \partial_1 + \partial_2 \partial_2$  the horizontal Laplacian, we have

$$\begin{aligned} \Delta_h u_1 &= \partial_1 \partial_1 u_1 + \partial_2 \partial_2 u_1 = \partial_1 (-\partial_2 u_2 - \partial_3 u_3) + \partial_2 \partial_2 u_1 \\ &= -\partial_2 (\partial_1 u_2 - \partial_2 u_1) - \partial_1 \partial_3 u_3 \\ &= -\partial_2 \omega_3 - \partial_1 \partial_3 u_3 \end{aligned}$$

and

$$\begin{aligned} \Delta_h u_2 &= \partial_1 \partial_1 u_2 + \partial_2 \partial_2 u_2 = \partial_1 \partial_1 u_2 + \partial_2 (-\partial_1 u_1 - \partial_3 u_3) \\ &= \partial_1 (\partial_1 u_2 - \partial_2 u_1) - \partial_2 \partial_3 u_3 \\ &= \partial_1 \omega_3 - \partial_2 \partial_3 u_3. \end{aligned}$$

Based on the computations above, we can use the two-dimension Riesz transformation  $\mathcal{R}_m = \frac{\partial_m}{\sqrt{-\Delta_h}}$  to denote the term  $\partial_m u_n$  ( $1 \leq m, n \leq 2$ ),

$$\begin{aligned} \partial_m u_1 &= \frac{\partial_m}{\sqrt{-\Delta_h}} \frac{\partial_2}{\sqrt{-\Delta_h}} \omega_3 + \frac{\partial_m}{\sqrt{-\Delta_h}} \frac{\partial_1}{\sqrt{-\Delta_h}} \partial_3 u_3 \\ (7) \quad &= \mathcal{R}_m \mathcal{R}_2 \omega_3 + \mathcal{R}_m \mathcal{R}_1 \partial_3 u_3, \end{aligned}$$

and

$$\begin{aligned} \partial_m u_2 &= -\frac{\partial_m}{\sqrt{-\Delta_h}} \frac{\partial_1}{\sqrt{-\Delta_h}} \omega_3 + \frac{\partial_m}{\sqrt{-\Delta_h}} \frac{\partial_2}{\sqrt{-\Delta_h}} \partial_3 u_3 \\ (8) \quad &= -\mathcal{R}_m \mathcal{R}_1 \omega_3 + \mathcal{R}_m \mathcal{R}_2 \partial_3 u_3. \end{aligned}$$

By (7), the term  $I_1$  could be turned into

$$\begin{aligned}
I_1 &= \sum_{i,j,k,l=1}^3 \int_{\mathbb{R}^3} \eta_{11ijkl} \partial_1 u_1 \partial_i u_j \partial_k u_l dx \\
&= \sum_{i,j,k,l=1}^3 \int_{\mathbb{R}^3} \eta_{11ijkl} (\mathcal{R}_1 \mathcal{R}_2 \omega_3 + \mathcal{R}_1 \mathcal{R}_1 \partial_3 u_3) \partial_i u_j \partial_k u_l dx \\
&= \sum_{i,j,k,l=1}^3 \int_{\mathbb{R}^3} \eta_{11ijkl} \mathcal{R}_1 \mathcal{R}_2 \omega_3 \partial_i u_j \partial_k u_l dx - \sum_{i,j,k,l=1}^3 \int_{\mathbb{R}^3} \eta_{11ijkl} \mathcal{R}_1 \mathcal{R}_1 u_3 \partial_3 (\partial_i u_j \partial_k u_l) dx \\
&= \sum_{i,j,k,l=1}^3 \int_{\mathbb{R}^3} \eta_{11ijkl} \mathcal{R}_1 \mathcal{R}_2 \omega_3 \partial_i u_j \partial_k u_l dx \\
&\quad - \sum_{i,j,k,l=1}^3 \int_{\mathbb{R}^3} \eta_{11ijkl} \mathcal{R}_1 \mathcal{R}_1 u_3 (\partial_3 \partial_i u_j \partial_k u_l + \partial_i u_j \partial_3 \partial_k u_l) dx = I_{11} + I_{12}.
\end{aligned}$$

Now we estimate  $I_{11}$  firstly. By using Hölder's inequality, the interpolation inequality  $\|f\|_{\dot{B}_{2,1}^r} \leq C \|f\|_{L^2}^{1-r} \|\nabla f\|_{L^2}^r$  for  $0 < r < 1$ , and the Young inequality, we can estimate  $I_{11}$  as

$$\begin{aligned}
|I_{11}| &\leq \|\mathcal{R}_2 \mathcal{R}_1 \omega_3 \cdot \nabla u\|_{L^2} \|\nabla u\|_{L^2} \\
&\leq C \|\omega_3\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} \|\nabla u\|_{\dot{B}_{2,1}^r} \|\nabla u\|_{L^2} \\
&\leq C \|\omega_3\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} \|\nabla u\|_{L^2}^{2-r} \|\Delta u\|_{L^2}^r \\
(9) \quad &\leq C \|\omega_3\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{2-r}} \|\nabla u\|_{L^2}^2 + \frac{1}{4} \|\Delta u\|_{L^2}^2
\end{aligned}$$

where we have used the continuity of the operator  $\mathcal{R}_i \mathcal{R}_j$  on Morrey spaces  $\dot{\mathcal{M}}_{p,\frac{3}{r}}(\mathbb{R}^2)$  for the values  $p \geq 2$  and  $0 < r < 1$  (see e.g. [7]).

In order to handle  $I_{12}$ , we recall the following property of Hardy space  $\mathcal{H}^1$  and  $BMO$  [5]:

$$(10) \quad \int_{\mathbb{R}^3} fgh dx \leq C \|f\|_{BMO} \|gh\|_{\mathcal{H}^1} \leq C \|f\|_{BMO} \|g\|_{L^2} \|h\|_{L^2},$$

for any  $\nabla \cdot g = 0$  and  $\nabla \times h = 0$ . Because of the Riesz transformation being bounded in  $BMO(\mathbb{R}^2)$  to  $BMO(\mathbb{R}^2)$ , and according to above inequality (10) and Young inequality, we can show

$$\begin{aligned}
|I_{12}| &\leq \|u_3\|_{BMO} \|\partial_i u \cdot \nabla \partial_i u\|_{\mathcal{H}^1} \\
&\leq C \|u_3\|_{BMO} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \\
&\leq \frac{1}{4} \|\Delta u\|_{L^2}^2 + C \|u_3\|_{BMO}^2 \|\nabla u\|_{L^2}^2.
\end{aligned}$$

Inserting (9) and (10) into (6), we see that

$$\frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq C \left( \|u_3\|_{BMO}^2 + \|\omega_3\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{2-r}} \right) \|\nabla u\|_{L^2}^2.$$

Taking the Gronwall inequality into consideration, one shows that

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\Delta u(\tau)\|_{L^2}^2 d\tau \\ \leq \|\nabla u_0\|_{L^2}^2 \exp\left(C \int_0^T \|u_3(t)\|_{BMO}^2 + \|\omega_3(t)\|_{\mathcal{M}_{2,\frac{3}{r}}}^{\frac{2}{2-r}} dt\right) < \infty, \end{aligned}$$

for any  $t \in [0, T)$ , with constant  $C$  independent of  $T$ . Thus,

$$u \in L^\infty(0, T; H^1(\mathbb{R}^3)) \subseteq L^\infty(0, T; L^6(\mathbb{R}^3)),$$

which implies that  $u$  is smooth. This completes the proof of Theorem 1.  $\square$

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