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DESCRIPTION OF MODAL LOGICS WHICH ENJOY CO-COVER PROPERTY

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Abstract. Here we use admissible rules to determine whenever modal logic satisfies weak co-cover property. We prove that logic λ over $S4$ satisfies such property iff the given set of rules are admissible in λ.

Keywords: modal logic, inference rule, Kripke frame and model, admissible rule.

1. Introduction

For modern application of logic in Computer Science and Artificial Intelligence it is often required that the capable language of representing the knowledge about dynamic systems. Distinct non-standard logics (e.g. modal and temporal, logic for multi-agent reasoning) efficiently serve these applications. Firstly they describe a statements by formulas which are peculiar to studied a models in general, and do not take to consideration a variable conditions and a changing assumptions. These conditions and assumptions can be modeled by distinct variations of the notion of logical consequence. The problem of such adequate modeling is one of extremely important problems originated from mathematical logic and mathematics in general. Most important component of our approach consists of the fact that we study logical consequence in terms of inference rules, clauses, but not only the formulas or statements. The formalism concerning description of properties by formulas is well-developed, widely spread and well represented in a scientific literature. It is a representing basis of human reasoning. But formulas describe only a stable, static events; the statement only fix the fact, and isn't able to handle a changing conditions.

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Therefore study of (structural) inference rules (or sequents), expressions which have a premise - a given collection of assumptions - and a conclusion, brings us more flexibility and more expressive power to model human reasoning and computing. Premises gives us a current informations collected as assumptions and conclusions represent a knowledge, i.e. facts which we can obtain from our assumptions. The intelligence of reasoning (as a part of AI) again requires the understanding of what are the consistent consequences of observable facts. Within Computer Science these aspects are involved in the analysis of correct instructions for computations, verification of programs and many other areas. Such rules allow us to model a standard situation in a study of logical consequence: given certain assumptions, what does follow from them?

Evidently the notion of inference rule generalizes the notion of formulas, and any formula can be viewed as a structural inference rule without premise and assumption. But admissible inference rules are actually stronger then the clauses, since Harrop (1960, [2]) we know that even the intuitionistic logic H is not structurally complete: it has the admissible inference rules which cannot be represented by formulas, which are not inferred. The same is valid for a broad range of basic modal logics since examples of G. Mints [3] and J. Port [4].

Notion of a admissible inference rule goes back to Lorenzen ([5], 1955). For arbitrary logic admissible rules are exactly those, under which the logic is closed. Clearly any derivable rule is admissible, but, in general, not vice versa. Also, directly from the definition, we see that the set of all rules admissible in a logic λ is the *greatest class* of inference rules by which we can extend an axiomatic system of the logic λ preserving the theorems of λ. Derivable rules may replace some fragments of the fixed length in derivations, thereby linearly shortening them. Admissible rules, which are not derivable, basically may reduce a derivations even more drastically.

The history of study an admissible rules could be dated since H. Fridman's question ([6], 1975) about an existence of an algorithm which could distinguish rules admissible in intuitionistic logic H. In the middle of 70-th G. Mints [3] found the strong sufficient conditions for derivability in H admissible rules in special form. The Fridman's question about the existence of an algorithm recognizing admissibility of inference rule was answered affirmatively by V. Rybakov (1984, cf. [7]) for the intuitionistic logic H and a broad class of modal logics (e.g. $K4$, $S4$, GL, cf. [1]).

In 2000-2010 a few results on describing of explicit bases for admissible inference rules for nonstandard logics (S4, K4, H etc.) appeared (see for example [9]). The key condition of these results was weak co-cover property. Possible, weak co-cover property, disjunction property and FMP of logic allow us to describe such basis. Beside the improvement of deductive power in logic, an admissible rule are able to describe some semantic property of given logic. One of the first attempt was the description of intuitionistic logic H by the set of admissible inference rules (cf. R. Iemhoff [8]). That's why this property is interest of article. Here we use admissible rules to determine whenever given modal logic satisfy weak co-cover property. We prove that FMP logic λ over $S4$ satisfy such property iff the given set of rules are admissible in λ.

2. Denotation, preliminary facts

We assume the reader to be familiar with the algebraic and Kripke semantics for modal logics and to have basic knowledge concerning inference rules and their admissibility (though we briefly recall all necessary facts below). Among modern literature we recommend Rybakov [1] as a good entry point to the subject: it contains both basic and advanced techniques concerning modal logics and inference rules. Following modern trends by a *logic* we understand the set of all theorems provable in a given axiomatic system, or the set of valid formulas for a certain class of Kripke frames. In particular, a normal modal logic λ is a set of modal formulas which is closed under substitution, modus ponens and necessitation rule $A\ /\ \Box A$, and contains all theorems of the minimal propositional modal logic K. In what follows by a modal logic we understand an algebraic propositional logic extending $S4$ that satisfies the finite model property.

A frame $\mathcal{F} := \langle F, R\rangle$ is a pair, where F is a non-empty set and R is a binary relation on F. The frame and its underlying set are often denoted by the same letter for simplicity. Further we consider only frames where R is a transitive and reflexive relation.

A model is a triple $\mathcal{M} = \langle W,\ R,\ V\rangle$, where $\mathcal{F} := \langle F, R\rangle$ is a frame and V is a valuation of a set of propositional letters P in the frame $\mathcal{F}$ that is $V : P \to 2^W$. $Dom(V) = P$ is called the domain of V.

A frame $\mathcal{F} = \langle F,\ R\rangle$ is called an open subframe of frame $\mathcal{G} = \langle G,\ R\rangle$ (denoted $\mathcal{F} \sqsubseteq \mathcal{G}$) if $F \subseteq G$ and $\forall a \in F\, \forall b \in G\, (aRb \Longrightarrow b \in F)$ holds. If $\mathcal{M}_1 = \langle W_1,\ R_1,\ V_1\rangle$, $\mathcal{M}_2 = \langle W_2,\ R_2,\ V_2\rangle$ are models then we call $\mathcal{M}_1$ an open submodel of $\mathcal{M}_2$ (denoted $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$) if : 1) $\langle W_1,\ R_1\rangle$ is open subframe of $\langle W_2,\ R_2\rangle$; 2) $Dom(V_1) = Dom(V_2)$ and $\forall p \in Dom(V_1)\ V_1(p) = V_2(p) \cap W_1$.

A mapping $f : \langle F,\ R\rangle \to \langle G,\ S\rangle$ is called p-morphism if (1) $aRb \Longrightarrow f(a)Sf(b)$; (2) $f(x)Sz \Longrightarrow \exists y \in F : f(y) = z\, \&\, xRy$.

We say a mapping $f\ :\ \mathcal{M}_1 = \langle W_1,\ R_1,\ V_1\rangle \to \mathcal{M}_2 = \langle W_2,\ R_2,\ V_2\rangle$ is a p-morphism of the model $\mathcal{M}_1$ into the $\mathcal{M}_2$ if 1) f is a p-morphism of the frame $\mathcal{F}_1 = \langle W_1,\ R_1\rangle$ into the frame $\mathcal{F}_2 = \langle W_2,\ R_2\rangle$; 2) the valuations $V_1,\ V_2$ are defined on the same set of propositional letters; 3) $\forall\, p \in Dom(V_1), \forall a \in W_1 (a \models_{V_1} p \iff f(a) \models_{V_2} p)$.

The primary property of open submodels and p-morphisms consist of the fact that they preserve the truth of formulas:

Proposition 1 (cf. [1]). 1) *If* $\mathcal{M}_1$ *is an open submodel of a model* $\mathcal{M}_2$ *then for every formula* α, $\mathcal{M}_2 \models \alpha$ *implies* $\mathcal{M}_1 \models \alpha$*;*

2) *If* f *is a p-morphism of model* $\mathcal{M}_1 = \langle W_1,\ R_1,\ V_1\rangle$ *onto model* $\mathcal{M}_2 = \langle W_2,\ R_2,\ V_2\rangle$ *then for any formula* α *which is built out of letters from the domain* $Dom(V_1)$*, then* $\forall a \in W_1 (a \models_{V_1} \alpha \iff f(a) \models_{V_2} \alpha)$.

Let $\mathcal{F}_i = \langle W_i,\ R_i\rangle,\ i \in I$ be a family of pairwise disjoint frames, i.e. $W_i \cap W_j = \emptyset$ for $i \neq j \in I$. The disjoin union of this family is the frame $\sqcup_{i\in I}\mathcal{F}_i = \langle W,\ R\rangle$, where $W = \cup_{i\in I} W_i,\ R = \cup_{i\in I} R_i$. Disjoint union of models is defined analogously.

By Lemma 2.5.26 [1] disjoint union of frames (or models) preserves the truth of formulas: $\sqcup_{i\in I}\mathcal{F}_i \models \alpha \iff \forall i(\mathcal{F}_i \models \alpha)$. A disjoint union of λ-frames is λ-frame.

Any subset C of a frame F, which is a set C satisfying the following properties: (1) $\forall\, x, y \in C(xRy\, \&\, yRx)$; (2) $\forall x \in C \forall y \in W(xRy\, \&\, yRx \Longrightarrow y \in C)$, is called a *cluster* C *of the frame* F. A cluster is *proper* if $|C| > 1$, otherwise that cluster is called degenerated, singleton. For any element $a \in F$ a cluster generated by a is denoted by $C(a)$. Any set of clusters of F which are non comparable by R is called an antichain. The antichain $\mathcal{A}$ is non-trivial if it consists of at least two clusters; otherwise $\mathcal{A}$ is trivial.

The maximal number of clusters in ascending chains of starting with the given element (or cluster) we call the depth of this element (or cluster). For any frame F (or a Kripke model M) *n-slice* $S_n(F)$ ($S_n(M)$) is the set of all elements of depth n from F (M respectively). And $S_{\leq n}(F)$ is the set of all elements from F with a depth not more than n.

We say a frame F is an λ–frame for a logic λ if all theorems of λ are valid at F, and $\lambda(F)$ – the set of all formulas valid in F – is the logic generated by F. A frame F is rooted if $\exists\, a \in F$ such that $\forall\, b \in F\ aRb$. Then we say $C(a)$ is the root of F.

We put $b^R := \{x | \exists y \in C(b) : yRx\}$; $b^{<R} := b^R \setminus C(b)$. For any subset $X \subseteq F$ X^R is $\cup\{x^R | x \in X\}$. That is subframe b^R of F is upwards cone generated by b, and X^R is subframe of F generated by X respectively. An element $b \in F$ or cluster $C(b)$ is *a co-cover* for a set $X \subseteq F$, if $b^R \setminus C(b) = X^R$. We understand a cluster C as λ-co-cover for X if it generate λ-frame $C^R := X^R \cup C$ as the root. We'll identify one-element cluster $C(a)$ and reflexive element a (generating this cluster) as co-cover since the set of R-accessible elements is the same for both.

A subset $\mathcal{X}$ of the given model $\mathcal{M}$ is definable if there is a formula α such $\forall\, x \in \mathcal{M}\, [\, x \in \mathcal{X} \Longleftrightarrow x \models_V \alpha]$. And a valuation V is definable in a model $\mathcal{M}$ if for any letter p, the set $V(p)$ is definable.

Let $\alpha_1, \ldots, \alpha_n, \beta$ to be some formulas. By a (structural) inference rule r we understand an expression where

$$r := \frac{\alpha_1(x_1, \ldots, x_n), ..., \alpha_n(x_1, \ldots, x_n)}{\beta(x_1, \ldots, x_n)},$$

which derives $s(\beta)$ from $s(\alpha_1), ..., s(\alpha_n)$ for every substitution s. We say r is *derivable* in a logic λ if there is a derivation β in λ from the set of assumptions $\{\alpha_1, ..., \alpha_n\}$.

An *inference rule* $r = \{\alpha_1(x_1, \ldots, x_n), \ldots, \alpha_k(x_1, \ldots, x_n)/\beta(x_1, \ldots, x_n)\}$ is called *admissible in a logic* λ, if for any formulas $\delta_1, \ldots, \delta_n$ the assertion

$$((\alpha_1(\delta_1, \ldots, \delta_n) \in \lambda) \cdots (\alpha_k(\delta_1, \ldots, \delta_n) \in \lambda) \implies \beta(\delta_1, \ldots, \delta_n) \in \lambda)$$

holds.

The admissible inference rules have the following algebraic description

Proposition 2 ([1]). *An inference rule $r = \{\alpha_1, \ldots, \alpha_k/\beta\}$ is admissible in a logic λ iff quasi-identity $r^* = \{\alpha_1 = 1\ \&\ldots\&\ \alpha_k = 1 \Longrightarrow \beta = 1\}$ is true on free algebra of countable rank $\mathcal{F}_w(\lambda)$ from the variety $Var(\lambda)$ generated by λ.*

Any derivable rule is admissible, but the inverse does not necessarily hold. Also directly from the definition we see that the set of all rules admissible in a logic λ is the *largest* class of inference rules by which we can extend axiomatic system of the logic λ preserving the theorems of λ. Derivable rules may replace some fragments of the fixed length in derivations, thereby shortening them linearly. Admissible rules, which are not derivable, in principle may reduce derivations even more drastically.

For a given frame $\mathcal{F}$, valuation V and inference rule $r := \alpha_1, ..., \alpha_n/\beta$, we say r *is valid on $\mathcal{F}$ wrt V*, and write $\mathcal{F} \models_V r$, if as soon as $\forall x \in \mathcal{F}$ and $\forall i$ $(x \models_V \alpha_i)$ holds, we have $\forall x \in \mathcal{F}(x \models_V \beta)$. A rule r *is valid on a frame* $\mathcal{F}$ if r is valid at $\mathcal{F}$ under any valuation, we write then $\mathcal{F} \models r$.

A logic λ satisfies the finite model property (FMP) if for any $\alpha \notin \lambda$ there exists a finite λ-model on which α is not valid.

A Kripke Model $\langle F, R, V\rangle$, where $V : P_n \to 2^F$, $P_n = \{p_1, p_2, \dots, p_n\}$, is *$n$-characteristic for a logic λ* iff for any formula α which is built up on $p_1, \dots, p_n$ $\alpha \in \lambda$ iff $\langle F, R, V\rangle \models \alpha$.

The admissibility of inference rules in modal logic λ over $S4$ can be described via their validity in certain special n-characteristic Kripke models. The description of these models $Ch_n(\lambda)$ and criteria for recognizing admissibility in λ by means of them are given, for instance, in [1]. As we will make heavy use of these techniques in the sequel, we briefly recall the construction of $Ch_n(\lambda)$ for FMP logic λ over $S4$ and the semantic criterion for recognizing admissibility.

Given a FMP logic λ over $S4$, a set $P_n := \{p_1, ..., p_n\}$ of propositional letters, we construct the first slice $S_1(Ch_n(\lambda))$ as that follows. It consists of the collection of all clusters with all possible valuations V of letters from P_n which does not have doubling – no two elements within the same cluster should be evaluated the way, and no clusters which are isomorphic as Kripke models.

Assuming $S_{\le m}(Ch_n(\lambda))$ to be constructed, we put in $S_{m+1}(Ch_n(\lambda))$ the clusters as that follows. For any antichain $\mathcal{X}$ from $S_{\le m}(Ch_n(\lambda))$ with at least one cluster of depth m and any cluster C in $S_1(Ch_n(\lambda))$ put a copy of C as a co-cover of $\mathcal{X}$, provided

(i) C^R is a λ–frame and

(ii) $\mathcal{X}$ is not an open submodel of C in case $\mathcal{X}$ is a trivial antichain.

Iterating this procedure we get the model $Ch_n(\lambda)$ as the result. We need the following facts:

Theorem 3 (cf. [1]). *For any FMP logic λ over $S4$ the model $Ch_n(\lambda)$ is n-characteristic for λ.*

Theorem 4 (cf. [1]). *For any inference rule r, r is admissible in FMP logic λ over $S4$ iff r is valid in the frame of $Ch_n(\lambda)$ wrt any definable valuation for any given n.*

3. Main results

We say that a logic λ, extending logic $S4$, has *weak co-cover property* (WCP for short) whenever for every finite rooted λ-frame $\mathcal{F}$ and an arbitrary non-trivial antichain $\mathcal{X}$ of clusters from $\mathcal{F}$, the frame $\mathcal{F}_1$ which is result of adjoining a singleton reflexive co-covering as the root to the frame $\bigcup_{c\in\mathcal{X}^R} c^R$ is a λ-frame as well.

Given $n \in N$ with $n > 1$, define the formulas:

$$\pi_i := p_i \wedge \bigwedge_{j\neq i} \neg p_j;\ 1 \le i, j \le n, \quad A_n := \bigwedge_{1\le i\le n} \Diamond\pi_i;$$

$$A_{n,1} := \Box\Big[\bigwedge_{1\le i\le n} (p_i \to \neg\Diamond q)\Big]; \quad B := q \vee \neg\Diamond q.$$

Also we define the rules:

$$\mathcal{R}_n := \frac{\Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big)}{\Box\neg A_n};\ n = 2, 3, \dots$$

Note, these rules are special case of those from [9] which gives an explicit basis for admissible rules of logic S4. Next theorem is almost the same as in Lemma 3.1 [9]. The WCP of logic is a key condition in proof of this statement. Let's reproduce common part of that proof.

Theorem 1. *The rules $\mathcal{R}_n,\ n > 1$, are admissible in every FMP logic λ over $S4$ that enjoys the weak co-cover property.*

Proof. Assume not. Let for some n the rule

$$\mathcal{R}_n := \frac{\Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big)}{\Box\neg A_n}$$

is not admissible in λ. Hence there is a definable valuation V of variables from $\mathcal{R}_n$ in a certain constructive k–characteristic model $Ch_\lambda(k)$. Therefore

$$Ch_\lambda(k) \models_V \Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big) \ \&\ Ch_\lambda(k) \not\models_V \Box\neg A_n. \qquad (\mathbf{1})$$

Consequently there exists element $a \in Ch_\lambda(k)$ such that $a \not\models_V \Box\neg A_n$. Then there are elements $b_1, \dots, b_n \in Ch_\lambda(k)$ such that aRb_i & $b_i \models_V p_i$. Note if all elements $b_1, \dots, b_n \in Ch_\lambda(k)$ belong to the same cluster (say $C(b_1)$ for example) then $b_1 \models_V A_n$ holds and by $(\mathbf{1})$ $b_1 \models_V A_{n,1}$. Therefore $b_1 \models_V \neg\Diamond q$, e.g. we obtain $b_1 \models_V A_n \wedge B$ which contradicts to $b_1 \models_V \Box\neg(A_n \wedge B)$ by the assumption $(\mathbf{1})$. So, elements $b_1, \dots, b_n \in Ch_\lambda(k)$ generate non-trivial antichain and there are $i, j \leq n$: $C(b_i) \neq C(b_j)$.

By the weak co-cover property there exists a reflexive element $b \in Ch_\lambda(k)$ which is a co-cover for the set of R-minimal clusters from the set $\{C(b_1), \dots, C(b_n)\}$, that is:

$$\{b\}^R := \{b\} \cup \bigcup_{1 \leq i \leq n} (b_i)^R.$$

By $(\mathbf{1})$ it follows that $b \models_V A_{n,1}$ and $b \models_V A_n$. Since b is a co-cover for $\{b_1, \dots, b_n\}$ that's clear that $b \models_V B$. Indeed $b_i \models_V p_i$ and $b \models_V A_{n,1}$ holds, therefore $\forall i \leq n\ b_i \models_V \neg\Diamond q$. From this we conclude $b \models_V q$ or $b \models_V \neg q$ and hence $b \models_V \neg\Diamond q$. Therefore we obtain $b \models_V A_n \wedge B$ which contradicts the fact that $b \models_V \Box\neg(A_n \wedge B)$ by the assumption $(\mathbf{1})$. $\Box$

Theorem 2. *If $\forall n$ the rules $\{\mathcal{R}_n, n > 1\}$ are admissible in FMP logic λ, over $S4$ then logic λ enjoys weak co-cover property.*

Proof. Suppose all rules $\mathcal{R}_n,\ n > 1, n \in N$, are admissible in FMP logic λ over $S4$, but λ does not enjoy weak co-cover property. By definition there exists *a finite* rooted λ-frame $G = b^R$ and a non-trivial antichain of clusters $\mathcal{X} \subset G$ such that frame $\varepsilon^R := \bigcup_{c \in \mathcal{X}^R} c^R \cup \{\varepsilon\}$ which is obtained by adjoining a singleton reflexive co-cover ε as a root to the frame $\bigcup_{c \in \mathcal{X}^R} c^R$, is not a λ-frame. Let's fix this non-trivial antichain of clusters $\mathcal{X} \subset G$. We'll prove that in such case at least one rule $\mathcal{R}_n$ is not admissible in the logic λ. To do this we construct a λ-frame $\mathcal{M}$ containing the frame G as an open subframe and refuting $\mathcal{R}_n,\ n > 1$, under some valuation. Then we define p-morphism from the frame of k-characteristic model $Ch_k(\lambda)$ for some k on $\mathcal{M}$. Transferring a valuation from $\mathcal{M}$ onto $Ch_k(\lambda)$ we will then refute $\mathcal{R}_n,\ n > 1$, on $Ch_k(\lambda)$, which will contradict the admissibility of $\mathcal{R}_n$

Let's take a frame $G \sqcup \{e\}$ where $\{e\}$ is reflexive singleton which is not R-comparable to any element in G. The frame $\{e\}$ is λ-frame as p-morfic image of G. So the frame $G \sqcup \{e\}$ is λ-frame as disjoint union of λ-frames.

We define λ-successor $\mathcal{M}$ of $G \sqcup \{e\}$ as follows. Let's fix the non-trivial antichaine $\mathcal{X}$ from G and define λ-frame $\mathcal{M}_0 = G \sqcup \{e\}$. Then we choose all non-trivial antichaines of clusters $\{X_t \subseteq S_1(\mathcal{M}_0)\}$ which don't have singleton co-cover in $\mathcal{M}_0$ and add such co-cover t whenever it generates λ-frame $t^R = \{t\} \cup X_t^R$ as a root. So, we obtain $\mathcal{M}_1$. Notice by construction $\mathcal{M}_1$ is λ-frame (we add elements generating λ-frame as a root to λ-frame $G \sqcup \{e\}$) and $G \sqsubseteq G \sqcup \{e\} \sqsubseteq \mathcal{M}_1$ as we add new elements as co-covers to $G \sqcup \{e\}$.

Assuming that $\mathcal{M}_k$ has already been constructed and $G \sqcup \{e\} \sqsubseteq \mathcal{M}_k$, we obtain $\mathcal{M}_{k+1}$ as follows: we choose in $S_{\leq(k+1)}(\mathcal{M}_k)$ all non-trivial antichaines $\{X_t \subseteq S_{\leq(k+1)}(\mathcal{M}_k)\}$ which don't have singleton co-cover and have at least one cluster of depth $k+1$. Then we add singleton reflexive element t as co-cover to each chosen antichain whenever it generates λ-frame $t^R = \{t\} \cup X_t^R$ as a root. Notice by construction $\mathcal{M}_{k+1}$ is λ-frame and $G \sqcup \{e\} \sqsubseteq \mathcal{M}_{k+1}$ as we add new elements as λ-co-covers to $\mathcal{M}_k$ and $G \sqcup \{e\} \sqsubseteq \mathcal{M}_k$.

Continuing this process we obtain λ-successor $\mathcal{M} = \cup_{k \in N} \mathcal{M}_k$ of $G \sqcup \{e\}$. This frame is potentially infinite.

By construction, the frame $\mathcal{M}$ has following properties:

- first slice $S_1(\mathcal{M})$ of $\mathcal{M}$ contains at least one singleton cluster $C(e) = \{e\}$;
- fixed antichain $\mathcal{X} \subset G$ does not have co-cover in $\mathcal{M}$ as on each step of construction we adjoin only co-cover generating λ-frame as a root;
- the frame G is open subframe of $\mathcal{M}$;
- every non-trivial antichain of clusters of $\mathcal{M}$ (differ from $\mathcal{X}$) has singleton λ-co-cover in $\mathcal{M}$ (whenever it generates as a root λ-frame).

Proposition 3. *$\mathcal{M} \not\models_V \mathcal{R}_n$ holds for some n and valuation V.*

Proof. We define a valuation V on $\mathcal{M}$ as follows. Let's suppose that fixed antichain $\mathcal{X} \subset b^R$ consists of clusters $\{C_1,\ C_2,\ \ldots, C_n\}$.

Now we define $\mathcal{X}^{-R} = \{x\ :\ xRC_1 \& xRC_2 \& \ldots xRC_n\}$ and

$$V(q) := \{y \in \mathcal{M} \setminus \mathcal{X}^R : y \notin \mathcal{X}^{-R}\ \& \exists x \in \mathcal{X}^{-R}(xRy)\},\ V(p_i) := C_i,$$

e.g. $\forall x \in C_i\ x \models_V p_i$. Let's prove that premise of $\mathcal{R}_n$ is valued on $\mathcal{M}$ under V while conclusion is not.

By definition of V we obtain:

$$\forall x \in \mathcal{X}^R\ x \models_V \neg\Diamond q; \quad \forall x \in \mathcal{M}\ x \models_V p_i \iff x \in C_i.$$

If some element $z \in \mathcal{M}$ R-sees fixed antichain $\mathcal{X} = \{C_1,\ C_2,\ \ldots, C_n\}$ and $z \models_V A_n$ holds, then $z \in \mathcal{X}^{-R}$. Therefore $z \models_V \neg q$ holds by definition of $V(q)$. Since the cluster $C(z)$ can't be a co-cover for antichain $\mathcal{X}$ by construction of $\mathcal{M}$, there should be some y such that $y \notin \mathcal{X}^{-R}$, $y \notin \mathcal{X}^R$ and zRy, in which case $y \models_V q$ and $z \models_V \neg q \wedge \Diamond q$ holds too, that infer $z \models_V \neg(A_n \wedge B)$.

Indeed if the cluster $C(z)$ is immediate R-predecessor for $\mathcal{X}$, but not co-cover for $\mathcal{X}$ (that is the depth $d(C(z))$ of $C(z)$ is $\max_{i \in \mathcal{X}} d(i)+1$), then there should be at least one element y such that zRy and $C(y) \cup \mathcal{X}$ form antichain for which $C(z)$ is co-cover (there can be a few elements $y_1, \ldots, y_k$ with such property – $C(y_1) \cup \cdots \cup C(y_k) \cup \mathcal{X}$ form antichain). Then for this y holds $y \notin \mathcal{X}^{-R}$, $y \notin \mathcal{X}^R$ and zRy.

If the cluster $C(z)$ is not immediate R-predecessor for $\mathcal{X}$, then from it is R-accessible some cluster $C(z_1)$ which is immediate R-predecessor for $\mathcal{X}$ or R-accessible some elements $z_1, z_2, \ldots, z_k$ which are immediate R-predecessor for some subsets of $\mathcal{X}$ and $\mathcal{X} \subseteq z_1^R \cup \cdots \cup z_k^R$. In first case (as before) we obtain some element y such that $y \notin \mathcal{X}^{-R}$, $y \notin \mathcal{X}^R$ and z_1Ry (and hence zRy be transitivity of R). In second case we can take as y element z_1 wich has desirable property.

So, we conclude $\forall x \in \mathcal{M}\ x \models_V \neg(A_n \wedge B)$.

By the definition of valuation V, $\forall x \in \mathcal{M}\ x \not\models_V \bigvee_{1\leq i\leq n} p_i \wedge \Diamond q$ is true obviously. In force of

$$\Box A_{n,1} \equiv \Box \bigwedge_{1,n}(p_i \to \neg\Diamond q) \equiv \Box \bigwedge_{1,n}(\neg p_i \vee \neg\Diamond q) \equiv \Box\neg[\bigvee_{1,n}(p_i \wedge \Diamond q)],$$

we have $\forall x \in \mathcal{M}\ x \models_V \Box A_{n,1}$. Consequently we proved that the premise of the rule is valued on arbitrary elements of $\mathcal{M}$.

As element b is R-predecessor for fixed antichain $\mathcal{X} = \{C_1,\ C_2,\ \dots, C_n\}$ and $\forall x \in C_i\ x \models_V p_i$ holds, we have $b \models_V \bigwedge_{1\leq i\leq n} \Diamond p_i$, e.g. $b \models_V A_n$. From this we infer $b \not\models_V \Box\neg A_n$. $\Box$

Proposition 4. *The rule $\mathcal{R}_n$ is not admissible in logic λ, where $n > 1$ is a number of clusters of antichain $\mathcal{X} = \{C_1, C_2, \dots C_n\}$.*

Proof. Let's choose some (the least) k so that $\mathcal{M} \sqsubseteq Ch_k(\lambda)$. Indeed, we can start with model on the *finite* frame $\mathcal{M}_0 = G \sqcup \{e\}$ and valuation S which compare to each element $c_i \in \mathcal{M}_0,\ i \in I, |I| = |\mathcal{M}_0| < w$, an individual variable p_i:

$$\forall i \in I\ c_i \models_S p_i;\ c_i \not\models p_j, i \neq j.$$

Under such valuation S all clusters of $\mathcal{M}_0$ are not isomorphic as models and all elements of any cluster have different valuation.

It's easy to see that model $\langle \mathcal{M}_0, S\rangle$ is open submodel of $Ch_k(\lambda)$ for $k = |\mathcal{M}_0|$:

(1) For each cluster $C \in S_1(\langle \mathcal{M}_0, S\rangle)$ we can find cluster $K \in S_1(Ch_k(\lambda))$ which is isomorphic to cluster C as model. So we have $S_1(\langle \mathcal{M}_0, S\rangle) \sqsubseteq S_1(Ch_k(\lambda))$ as models.

(2) Let's take some cluster $C_1 \in S_2(\langle \mathcal{M}_0, S\rangle)$ which is co-cover for antichain $A \subseteq S_1(\langle \mathcal{M}_0, S\rangle)$. By (1) there is antichain $B \subseteq S_1(Ch_k(\lambda)) : \langle A, S\rangle \cong B$ as models. Since $C_1^R = C_1 \cup A$ is λ-frame by construction of $Ch_k(\lambda)$ there is cluster $K_1 \in S_2(Ch_k(\lambda))$ such that $\langle C_1^R, S\rangle \cong K_1^R \sqsubseteq Ch_k(\lambda)$ as models. So, we have $\langle S_{\leq 2}(\mathcal{M}), S\rangle \sqsubseteq Ch_k(\lambda)$.

(3) Continuing slice by slice this reasoning we conclude $\langle \mathcal{M}_0, S\rangle \sqsubseteq Ch_k(\lambda)$. Therefore the frame $\mathcal{M}_0$ is open subframe of the frame of $Ch_k(\lambda)$.

It's easy to see the process of construction of $\mathcal{M}$ is the part of one of the frame of $Ch_k(\lambda)$ and the resulting frame is open subframe of the frame $Ch_k(\lambda)$.

(a) At first step of construction of $\mathcal{M}$ we take non-trivial antichain $X_t \subseteq S_1(\mathcal{M}_0)$ which don't have singleton co-cover in $\mathcal{M}_0$ and add such co-cover whenever it generates λ-frame $t^R = \{t\} \cup X_t$ as a root. Since $\langle \mathcal{M}_0, S\rangle \sqsubseteq Ch_k(\lambda)$ there is antichain $B \subseteq S_1(Ch_k(\lambda)) : \langle X_t, S\rangle \cong B$ as models. As t^R is λ-frame by construction of model $Ch_k(\lambda)$ we can find co-cover for B – cluster C_B such that $t^R \cong C_B^R$ as frames. So we can transfer valuation from C_B on t and obtain $\langle t^R, S\rangle \cong C_B^R$ as models. Hence we have $\langle \mathcal{M}_1, S\rangle \sqsubseteq Ch_k(\lambda)$.

(b) Assume $\langle \mathcal{M}_k, S\rangle \sqsubseteq Ch_k(\lambda)$. By construction of $\mathcal{M}_{k+1}$ we take non-trivial antichain $X_t \subseteq S_{\leq(k+1)}(\mathcal{M}_k)$ which don't have singleton co-cover in $\mathcal{M}_k$ and have at least one cluster of depth $k+1$. Then we add such co-cover whenever it generates λ-frame $t^R = \{t\} \cup X_t$ as a root. For this antichain X_t there is antichain $B \subseteq S_{k+1}(Ch_k(\lambda)) : \langle X_t^R, S\rangle \cong B^R$ as models. As t^R is λ-frame by construction of model $Ch_k(\lambda)$ we can find co-cover for antichain B – cluster C_B such that $t^R \cong C_B^R$ as frames. So we can transfer valuation from C_B on t and obtain $\langle t^R, S\rangle \cong C_B^R$ as models. Hence we have $\langle \mathcal{M}_{k+1}, S\rangle \sqsubseteq Ch_k(\lambda)$.

(c) Continuing slice by slice this reasoning we conclude $\langle\mathcal{M}, S\rangle \sqsubseteq Ch_k(\lambda)$. Therefore the frame $\mathcal{M}$ is open subframe of the frame of $Ch_k(\lambda)$.

Note as all elements of model $Ch_k(\lambda)$ are definable and $\langle\mathcal{M}, S\rangle \sqsubseteq Ch_k(\lambda)$ holds then all elements of $\langle\mathcal{M}, S\rangle$ definable too (the definition and structure of this formula β one can find in the proof of Theorem 3.3.7 [1]):

$$\forall x \in \mathcal{M} \exists \beta_x : x \models_S \beta_x \ \& \ (y \neq x \Longrightarrow y \not\models_S \beta_x).$$

So we conclude that any (finite) subset $A \subseteq \mathcal{M}$ is definable too under valuation S: $\beta(A) = \bigvee_{a\in A} \beta_a$. Hence, now we can define :

$$\mathbf{V(p_i)} = \bigvee_{x\in C_i} \beta_x, 1 \leq i \leq n;\ \mathbf{V(q)} = \{y \in \mathcal{F} \setminus X^R : y \notin X^{-R} \& \exists z \in X^{-R}(zRy)\} =$$

$$= V\Big(\neg \bigwedge_{\beta_j\in\mathcal{X}} \Diamond\beta_j \wedge \neg\Box\big(\bigvee_{\beta_j\in\mathcal{X}^R} \beta_i\big) \wedge \bigvee\{\beta_y : \exists\beta_z(z \models_S \bigwedge_{\beta_j\in\mathcal{X}} \Diamond\beta_j \Longrightarrow z \models_S \Diamond\beta_y\}\Big).$$

So the sets $V(p_i), 1 \leq i \leq n$, and $V(q)$ are definable in model $\langle\mathcal{M}, S\rangle$.

Now we define a p-morphism from the frame of k-characteristic model $Ch_k(\lambda)$ onto frame $\mathcal{M}$ as follows.

- for elements of frame $\mathcal{M} \sqsubseteq C_k(\lambda)$ we define the p-morphism f as identical, e.g $\forall\ x \in \mathcal{M}\ f(x) := x$.
- for all elements $x \in S_1(C_k(\lambda) \setminus \mathcal{M})$ we set $f(x) := e$, where $\{e\}$ is singleton cluster of first slice of $G \sqcup \{e\}$.
- Let's suppose that for all elements $x \in S_{\leq t}(Ch_k(\lambda) \setminus \mathcal{M})$ of the depth no more than t required p-morphism is defined already. Let's choose arbitrary element $y \in S_{t+1}(Ch_k(\lambda) \setminus \mathcal{M})$ which is co-cover for antichain (possible trivial) $\mathcal{A} \subset S_{\leq t}(Ch_k(\lambda))$. By the construction of k-characteristic model $Ch_k(\lambda)$ the frames $y^R = C(y) \cup \mathcal{A}^R$ and $\mathcal{A}^R$ are λ-frames.

  By inductive conjecture p-morphism $f(\mathcal{A}^R)$ is defined already. If $f(\mathcal{A}) = C$ is trivial antichain and is a cluster C we define $f(y) := \varepsilon \in f(\mathcal{A})$.

  Assume $f(\mathcal{A}^R)$ is not trivial. As p-morphism preserves the truth of formulas the subframe $f(\mathcal{A}^R)$ is λ-frame. And the result of adding co-cover (root) to $f(\mathcal{A}^R)$ would be a p-morphic image of y^R and thus also a λ-frame. From this by construction of λ-successor $\mathcal{M}$ we infer that the antichain of R-minimal clusters from $f(\mathcal{A}^R)$ has singleton reflexive co-cover ε in $\mathcal{M}$ and such element actually exists in $\mathcal{M}$. So we can define $f(y) := \varepsilon$. As thus co-cover is unique in $\mathcal{M}$ so all elements of $C(y)$ will be mapped to it.

  Accordingly to the arbitrariness of y in such way we define p-morphism f on whole slice $S_{t+1}(Ch_k(\lambda)$.
- Continue this process we obtain p-morphism f from the frame of k-characteristic model $Ch_k(\lambda)$ onto $\mathcal{M}$. Note the mapping f is defined in a way that preserves the property of being a h-morphism.

Then we transfer the valuation V from $\mathcal{M}$ on $Ch_k(\lambda)$ as

$$\forall x \in Ch_k(\lambda)\ x \models_{f^{-1}(V)} p \iff f(x) \models_V p.$$

This gives us the p-morphism of models:

$$\langle Ch_k(\lambda),\ f^{-1}(V)\rangle \longrightarrow_f \langle\mathcal{M},\ V\rangle.$$

Note as the sets $V(p_i), 1 \le i \le n$, and $V(q)$ are definable in model $\langle \mathcal{M}, S\rangle$ and p-morphism preserves the truth of formulas, hence the sets $f^{-1}(V(p_i)), 1 \le i \le n$, and $f^{-1}(V(q))$ are definable in model $Ch_k(\lambda)$ too.

As p-morphism f preserves the truth of formulas and the rule $\mathcal{R}_n$ is refuted on $\langle \mathcal{M},\ V\rangle$ we refute this rule on k-characteristic model $\langle Ch_k(\lambda),\ f^{-1}(V)\rangle$ under definable valuation. Consequently the rule $\mathcal{R}_n$ is not admissible in logic λ. □

Taking into account Proposition 3 and Proposition 4, Theorem 2 is proved. □

So from theorems 1 and 2 we obtain

Theorem 5. *Let FMP logic λ extends $S4$. All rules $\mathcal{R}_n$, $n > 1$, $n \in N$, are admissible in λ $\iff$ logic λ enjoys the weak co-cover property.*

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