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LAMBEK INVARIANTS IN A P-SEMI-ABELIAN CATEGORY

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Abstract. We consider the well-known invariants Ker and Im for commutative squares in P-semi-abelian categories. These invariants were introduced by Lambek for groups and then studied in a more general context by Hilton and Nomura. In this paper, P-semi-abelian analogs are proved for Lambek's isomorphism and acyclic sequences that include these invariants are found.

Keywords: P-semi-abelian category, commutative square, Lambek invariants.

1. Introduction

In 1964, Lambek introduced the following invariants for a commutative square

$$
\begin{array}{ccc}
C & \xrightarrow{\alpha} & D \\
{\scriptstyle g}\downarrow & S & \downarrow{\scriptstyle f} \\
A & \xrightarrow[\beta]{} & B
\end{array}
\tag{1}
$$

in the category of groups:

$$\operatorname{Im} S = (\operatorname{Im}\beta \cap \operatorname{Im} f)/\operatorname{Im}(f\alpha), \quad \operatorname{Ker} S = \operatorname{Ker}(f\alpha)/(\operatorname{Ker}\alpha \cdot \operatorname{Ker} g).$$

Since $\operatorname{Ker}\alpha$ and $\operatorname{Ker} g$ are normal subgroups in $\operatorname{Ker} f\alpha$, their product $\operatorname{Ker}\alpha \cdot \operatorname{Ker} g$ is the normal subgroup generated by them, and $\operatorname{Im}(f\alpha) = g \operatorname{Ker}\beta \trianglelefteq g(C) = \operatorname{Im} g$. In [18], Lambek proved the following assertion:

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Given a commutative diagram

$$(2)\qquad \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle a}\downarrow & S & \downarrow{\scriptstyle b} & T & \downarrow{\scriptstyle c} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' \end{array}$$

of groups and group homomorphisms with exact rows, there is a natural isomorphism

$$\Lambda : \operatorname{Im} S \xrightarrow{\sim} \operatorname{Ker} T.$$

Later Leicht [19] extended this theorem to the more abstract context of Puppe exact categories. In [22, 23], in a Puppe exact category, Nomura considered the case where the rows in (2) are not exact but only null sequences, constructed a canonical morphism $\Lambda : \operatorname{Im} S \to \operatorname{Ker} T$, and proved that there is an exact sequence

$$(3)\qquad 0 \to H(\operatorname{Ker}(bf) \to \operatorname{Ker} b \to \operatorname{Ker} c) \longrightarrow \operatorname{Ker}(H \to H') \longrightarrow \operatorname{Im} S \xrightarrow{\Lambda} \operatorname{Ker} T \longrightarrow \operatorname{Coker}(H \to H') \longrightarrow H(\operatorname{Coker} a \to \operatorname{Coker} b \to \operatorname{Coker}(g'b)) \to 0$$

where the arrows between the kernels and cokernels in parentheses are natural morphisms, $H(\cdot \to \cdot \to \cdot)$ stands for the homology of the null sequence in parentheses, $H = H(A \to B \to C)$, and $H' = H(A' \to B' \to C')$. Later Nomura's results were extended to the so-called "categorías hofmanianas" by Ubeda Bescansa [30, 31].

In [10], we considered these invariants in a quasi-abelian category and established Lambek's isomorphism provided that the morphism b in (2) is strict and proved generalizations of some assertions of Nomura. In this paper, we study the Lambek invariants in the weaker class of P-semi-abelian categories, i.e., in categories semi-abelian in the sense of Palamodov.

In contrast to abelian or even quasi-abelian categories, much of the usual homological-algebraic stuff fails in P-semi-abelian categories — for example, it is even impossible to define (co)homology for a general (co)chain complex. However, homological methods are possible and fruitful in this context. For instance, in [24], Palamodov proved some properties of functors from a P-semi-abelian category with enough injectives into an abelian category and showed that the category of spectra corrresponding to a P-semi-abelian category is itself P-semi-abelian. In [25], he developed a theory of splitting for complexes in a P-semi-abelian category. Also, the derived category of a P-semi-abelian category (and even of an additive category) can be defined in several different ways (see [5, 7, 21]). Under some conditions, it is possible to derive an exact couple in the sense of Massey and thus move towards constructing the spectral sequence of the couple (see [16]).

In a P-semi-abelian category, Nomura's morphism $\Lambda : \operatorname{Im} S \to \operatorname{Ker} T$ is defined only under certain conditions on the morphism b in (2), and Lambek's theorem holds provided that b is strict and its image or coimage is "semi-stable."

The structure of the paper is as follows. In Section 2, we recall some definitions and facts about preabelian and P-semi-abelian categories. In Section 3, we define the left and right homology objects. In Section 4, we deal with the Lambek invariants in P-semi-abelian categories, prove a P-semi-abelian version of Lambek's isomorphism, and generalize some assertions proved by Nomura [22] and Hilton [8] for abelian and Puppe exact categories.

2. P-semi-abelian categories

A *preabelian* category is an additive category satisfying the following axiom:

Axiom 1. Every morphism has a kernel and a cokernel.

We denote by $\ker\alpha$ ($\operatorname{coker}\alpha$) an arbitrary kernel (cokernel) of α and by $\operatorname{Ker}\alpha$ ($\operatorname{Coker}\alpha$) the corresponding object; the equality $a = \ker b$ ($a = \operatorname{coker} b$) means that a is a kernel of b (a is a cokernel of b).

In a preabelian category, every morphism α admits a canonical decomposition $\alpha = (\operatorname{im}\alpha)\overline{\alpha}(\operatorname{coim}\alpha)$, where $\operatorname{im}\alpha = \ker\operatorname{coker}\alpha$, $\operatorname{coim}\alpha = \operatorname{coker}\ker\alpha$. Below we sometimes use the notation $\widetilde{\alpha}$ for $\overline{\alpha}(\operatorname{coim}\alpha)$.

A morphism α is called *strict* if $\overline{\alpha}$ is an isomorphism.

We write $\alpha \,|\, \beta$ if $\alpha = \ker\beta$ and $\beta = \operatorname{coker}\alpha$.

Lemma 2.1 ([2, 3, 17, 33]). *The following assertions hold in a preabelian category*:

(i) *Strict monomorphisms = kernels, strict epimorphisms = cokernels.*

(ii) *α is a kernel $\Longleftrightarrow \alpha = \operatorname{im}\alpha$, α is a cokernel $\Longleftrightarrow \alpha = \operatorname{coim}\alpha$.*

(iii) *A morphism α is strict if and only if it is representable in the form $\alpha = \alpha_1\alpha_0$ with α_0 a cokernel and α_1 a kernel*; *in every such representation, $\alpha_0 = \operatorname{coim}\alpha$ and $\alpha_1 = \operatorname{im}\alpha$.*

(iv) *The relations* $\ker\alpha = \ker\operatorname{coim}\alpha$ *and* $\operatorname{coker}\alpha = \operatorname{coker}\operatorname{im}\alpha$ *hold for every morphism α.*

(v) *Suppose that a commutative square*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[\beta]{} & B \end{array} \tag{4}$$

is a pullback. Then $\ker f = \alpha\ker g$. *If* $f = \ker h$ *for some* h *then* $g = \ker(h\beta)$. *If f is a monic then g is monic; if f is a kernel then g is a kernel.*

Dually, suppose that (4) *is a pushout. Then* $\operatorname{coker} g = (\operatorname{coker} f)\beta$. *If* $g = \operatorname{coker} e$ *for some e then* $f = \operatorname{coker}(\alpha e)$. *If g is epic then f is epic. If g is a cokernel then f is a cokernel.*

A preabelian category is abelian if and only if $\overline{\alpha}$ is an isomorphism for every α, that is, if and only if every morphism is strict.

A preabelian category is called *P-semi-abelian* or *semi-abelian* (*in the sense of Palamodov*) [24, 26] if it satisfies

Axiom 2. For every morphism α, $\overline{\alpha}$ is a bimorphism, that is, a monomorphism and an epimorphism.

We call a sequence $\ldots \xrightarrow{a} B \xrightarrow{b} \ldots$ in an additive category a *null sequence* B if $ba = 0$. A sequence $\ldots \xrightarrow{a} B \xrightarrow{b} \ldots$ in a preabelian category is said to be *acyclic* (*at the term*) B if $\operatorname{im} a = \ker b$. Lemma 2.1(iv) implies that the sequence is acyclic at B if and only if $\operatorname{coker} a = \operatorname{coim} b$.

The following lemma is well known. For a detailed characterization of P-semi-abelianity, the reader is referred to [15].

Lemma 2.2. *Let $\mathcal{A}$ be a preabelian category. Then the following are equivalent:*

(i) *$\mathcal{A}$ is P-semi-abelian.*
(ii) *If $h \circ l$ is a kernel then so is l. If $h \circ l$ is a cokernel then so is h.*
(iii) *If (1) is a pushout and g is a kernel then f is a monomorphism. If (1) is a pullback and f is a kernel then g is an epimorphism.*
(iv) *If l and h are kernels (cokernels) and $h \circ l$ is defined then $h \circ l$ is a kernel (cokernel).*

If, for a cokernel f in a preabelian category, in every pullback (1), g is a cokernel (for a kernel g in a preabelian category, in every pushout (1), f is a kernel) then f is called a *semi-stable cokernel* (g is called a *semi-stable kernel*).

We recall some basic properties of semi-stable kernels and cokernels (following from [9, Propositions 5.11 and 5.12]).

Lemma 2.3. *The following hold in a preabelian category*:

(i) *if gf is a semi-stable kernel then so is f, if gf is a semi-stable cokernel then so is g*;

(ii) *if f and g are semi-stable kernels and gf is defined then gf is a semi-stable kernel; if f and g are semi-stable cokernels and gf is defined then gf is a semi-stable cokernel.*

(iii) *a pushout of a semi-stable kernel is a semi-stable kernel; a pullback of a semi-stable cokernel is a semi-stable cokernel.*

Examples of non-semi-stable cokernels in P-semi-abelian categories of functional analysis are not unusual (see [1, 27, 28, 29, 32]).

3. Homology objects

Suppose first that the ambient category is preabelian.

Given a sequence of the form

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \tag{5}$$

such that $\psi\varphi = 0$, there are a natural morphism $\sigma : A \to \operatorname{Ker}\psi$ such that $\varphi = (\ker\psi)\sigma$ and a natural morphism $\tau : \operatorname{Coker}\varphi \to C$ such that $\psi = \tau \operatorname{coker}\varphi$.

Call $H_-(B) = H_-(B,\varphi,\psi) = \operatorname{Coker}\sigma$ and $H_+(B) = H_+(B,\varphi,\psi) = \operatorname{Ker}\tau$ the *left* and *right homology objects of* (5) *at the term B*.

These two notions coincide for abelian categories (see, for example, [4, p. 124]), for quasi-abelian categories [13, Lemma 4], and even in the nonadditive setting of homological categories in the sense of Grandis [6, p. 155].

If the ambient category is P-semi-abelian then there is an equivalent description of the left and right homology objects. Consider the natural morphisms $r : \operatorname{Im}\varphi \to \operatorname{Ker}\psi$ and $r' : \operatorname{Coker}\varphi \to \operatorname{Coim}\psi$. Then $\operatorname{coker} r = \operatorname{coker}\sigma$ and $\ker r' = \ker\tau$, and hence $H_-(B,\varphi,\psi) = \operatorname{Coker} r$ and $H_+(B,\varphi,\psi) = \operatorname{Ker} r'$.

As we see from Lemma 2.1(iv), the acyclicity of sequence (5) at B in a P-semi-abelian category is equivalent to the nullity of either of the homologies $H_-(B,\varphi,\psi)$ or $H_+(B,\varphi,\psi)$.

As was shown in [13], in a preabelian category, there is a unique morphism $m : H_-(B) \to H_+(B)$ such that

$$(\ker\tau)m \operatorname{coker}\sigma = (\operatorname{coker}\varphi)(\ker\psi). \tag{6}$$

The following assertion holds ([11, Lemma 7], [12, Proposition 1]):

Lemma 3.1. *Let the ambient category be P-semi-abelian. The morphism $m : H_-(B) \to H_+(B)$ is a bimorphism. If $\ker\psi$ is a semi-stable kernel or $\operatorname{coker}\varphi$ is a semi-stable cokernel then m is an isomorphism.*

Examples of situations when the left and right homology objects do not coincide can be obtained from the following observation ([12, Lemma 4]):

Let

$$\begin{array}{ccc} P & \xrightarrow{u'} & F \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow[u]{} & G \end{array}$$

be a pullback in a P-semi-abelian category such that v is a kernel, u is a cokernel, and u' is not a cokernel. Let $H_-(E)$ and $H_+(E)$ be the left and right homology objects of the sequence

$$K \overset{\ker u}{\longrightarrow} E \overset{\operatorname{coker} v'}{\longrightarrow} L$$

at the term E. Then the canonical morphism $m : H_-(E) \to H_+(E)$ is not an isomorphism.

As was shown by Wengenroth (see [32]), such pullbacks are not unusual, for example, in the P-semi-abelian category of bornological locally convex spaces and arise when non-α-regular inductive limits in the sense of Makarov [20] are considered.

4. Lambek invariants

Given a commutative square (1), consider the pullback

$$\begin{array}{ccc} I & \xrightarrow{k} & \operatorname{Im} f \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \operatorname{im} f} \\ \operatorname{Im}\beta & \xrightarrow[\operatorname{im}\beta]{} & B \end{array} \tag{7}$$

There are obvious morphisms $k' : \operatorname{Im}(f\alpha) \to \operatorname{Im} f$ and $l' : \operatorname{Im}(f\alpha) \to \operatorname{Im}\beta$ with $\operatorname{im}(f\alpha) = (\operatorname{im} f)k' = (\operatorname{im}\beta)l'$. Since (7) is a pullback, there is a unique morphism $\rho : \operatorname{Im}(f\alpha) \to I$ such that $k' = k\rho$ and $l' = l\rho$. We put $\operatorname{Im} S = \operatorname{Coker}\rho$. If we denote by Φ the epimorphism $\widetilde{f\alpha} = \overline{f\alpha} \operatorname{coim} f\alpha$ then, obviously, $\operatorname{Im} S = \operatorname{Coker}(\rho\Phi)$.

Now, let $\mu : \operatorname{Ker} g \to \operatorname{Ker}(f\alpha)$ and $\nu : \operatorname{Ker}\alpha \to \operatorname{Ker}(f\alpha)$ be the natural inclusions. They form a morphism $\langle\mu,\nu\rangle : \operatorname{Ker} g \oplus \operatorname{Ker}\alpha \to \operatorname{Ker}(f\alpha)$. We put $\operatorname{Ker} S = \operatorname{Coker}\langle\mu,\nu\rangle$. Alternatively, $\operatorname{Ker} S$ can be described as follows (see, for example, [22]). Consider the pushout

$$\begin{array}{ccc} C & \xrightarrow{\operatorname{coim}\alpha} & \operatorname{Coim}\alpha \\ {\scriptstyle \operatorname{coim} g}\downarrow & & \downarrow{\scriptstyle j} \\ \operatorname{Coim} g & \xrightarrow[i]{} & J \end{array}$$

There is a unique morphism $\sigma : L \to B$ such that $\sigma j = f(\operatorname{im}\alpha)\bar{\alpha}$ and $\sigma i = \beta(\operatorname{im} g)\bar{g}$. Then $\operatorname{Ker} S$ is naturally isomorphic to $\operatorname{Ker}\sigma$. Thus, $\operatorname{Im} S$ and $\operatorname{Ker} S$ are dual notions.

In what follows, we endow all the morphisms and objects introduced above for a commutative square S with the subscript S when it becomes necessary to distinguish the corresponding morphisms of different squares.

Here is an easy example of commutative squares with zero kernel or image:

Example 4.1. As was observed by Hilton (see [8, Proposition 2.4]) and is easily checked, every composition $h = gf$ gives two commutative squares $\Delta' : h(\mathrm{id}) = gf$ and $\Delta'' : (\mathrm{id})h = gf$ such that $\operatorname{Im}\Delta' = 0$ and $\operatorname{Ker}\Delta'' = 0$.

As was noted in the introduction, for a sequence of the form (2) with acyclic rows, $\operatorname{Ker} S$ and $\operatorname{Im} T$ are naturally isomorphic (see [19] or [22]) in abelian categories and some of their "exact" nonabelian analogs. For this to hold even in a quasi-abelian category, one must have $\operatorname{Im} b = \operatorname{Coim} b$, that is, b must be strict (see [10]) As we will see below, this is indeed not enough for P-semi-abelian categories. But even in this generality, we have the following, literally the same, result as in the quasi-abelian case [10], which gives a mapping $\zeta : \operatorname{Ker} T \to \operatorname{Im} S$ "in the opposite direction".

Theorem 4.2. *Let $\mathfrak{A}$ be a P-semi-abelian category and consider diagram* (2) *with acyclic rows in $\mathfrak{A}$. Then there exist unique morphisms $\xi : \operatorname{Ker}(g'b) \to I_S$ and $\zeta : \operatorname{Ker} T \to \operatorname{Im} S$ such that*

$$(\operatorname{coker}\rho_s)\xi = \zeta \operatorname{coker}\langle \mu_T, \nu_T\rangle.$$

Proof. We must follow verbatim the proof of Theorem 3.2 in [10]. □

As a consequence of Theorem 4.2, by analogy with [10], we obtain Lambek's isomorphism, established for abelian or Puppe exact categories in [18, 19, 22] and for quasi-abelian categories in [10], provided that b is strict. In the P-semi-abelian case, we have to add some semi-stability conditions.

Corollary 4.3. *Suppose that, under the conditions of Theorem* 4.2*, b is strict. If* $\operatorname{coim} b$ *is a semi-stable cokernel or* $\operatorname{im} b$ *is a semi-stable kernel, then ζ is an isomorphism.*

Proof. We assume that b is strict and $\operatorname{coim} b$ is a semi-stable cokernel (the case of $\operatorname{im}\beta$ being a semi-stable kernel is treated by duality). Note that the square

$$\begin{array}{ccc} \operatorname{Ker}(g'b) & \xrightarrow{\ker(g'b)} & B \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \tilde{b}} \\ I & \xrightarrow[k_S]{} & \operatorname{Im} b \end{array} \tag{8}$$

is a pullback (see the proof of Theorem 2 in [10]).

Since $\ker b = \ker \tilde{b} = (\ker(g'b))\mu_T$, b is strict, $\operatorname{coim}\beta$ is a semi-stable cokernel, and (8) is a pullback, from Lemma 2.1(v) it follows that $\mu_T = \ker\xi$ and $\xi = \operatorname{coker}\mu_T$. Obviously, $(\operatorname{coker}\langle\mu_T,\nu_T\rangle)\mu_T = 0$, and so there exists a unique morphism

$$\tau : I \to \operatorname{Coker}\langle\mu_T,\nu_T\rangle$$

such that $\operatorname{coker}\langle\mu_T,\nu_T\rangle = \tau\xi$. We have $\zeta\tau\xi = (\operatorname{coker}\rho_S)\xi$, and the fact that ξ is a cokernel yields $\zeta\tau = \operatorname{coker}\rho_S$.

Denote by γ_S the unique morphism such that $\operatorname{im}(bf)\gamma_S = b(\operatorname{im} f)$ $(= b(\ker g)$ due to the acyclicity of the upper row in (2)). We have

$$\tau\rho_S\gamma_S = \tau\xi\nu_T = (\operatorname{coker}\langle\mu_T,\nu_T\rangle)\nu_T = 0. \tag{9}$$

Since

$$\operatorname{im}(bf)\gamma_S\tilde{f} = b(\operatorname{im} f)\tilde{f} = bf = \operatorname{im}\tilde{f}$$

and $\operatorname{im} bf$ is monic, γ_S is epic. So (9) implies that $\tau\rho_S = 0$. Thus there is a unique morphism $\Lambda_0 : \operatorname{Coker}\rho \to \operatorname{Coker}\langle\mu_T,\nu_T\rangle$ with the property $\tau = \Lambda_0(\operatorname{coker}\rho)$. It is easy to see that $\zeta\Lambda_0$ and $\Lambda_0\zeta$ are identities and, therefore, ζ and Λ_0 are mutually inverse isomorphisms. □

Note that the morphism Λ_0 in the proof above coincides with Nomura's morphism Λ up to the identification $\operatorname{Ker} T \cong \operatorname{Ker}\sigma_T$.

Corollary 4.4. *Let S be a commutative square of the form (1). The following hold:*

(i) *Suppose that S is a pullback and f is strict. If* $\operatorname{coim} f$ *is a semi-stable cokernel or* $\operatorname{im} f$ *is a semi-stable kernel then* $\operatorname{Ker} S = 0$.

(ii) *Suppose that S is a pushout and g is strict. If* $\operatorname{im} g$ *is a semi-stable kernel or* $\operatorname{coim} g$ *is a semi-stable cokernel then* $\operatorname{Im} S = 0$.

Proof. We prove (i). By Lemma 2.1(v), the natural morphism $\hat{f} : \operatorname{Ker}\alpha \to \operatorname{Ker}\beta$ is an isomorphism, and hence $\operatorname{Ker}\alpha = \operatorname{Ker}\beta$. Consider the commutative diagram with acyclic rows

$$\begin{array}{ccccc} \operatorname{Ker}\alpha & \xrightarrow{\ker\alpha} & C & \xrightarrow{\alpha} & D \\ \| & S_0 & \downarrow g & S & \downarrow f \\ \operatorname{Ker}\beta & \xrightarrow[\ker\beta]{} & A & \xrightarrow[\beta]{} & B \end{array}$$

Corollary 4.3 implies that $\operatorname{Im} S_0 = \operatorname{Ker} S$. By Example 4.1, $\operatorname{Im} S_0 = 0$. Thus, $\operatorname{Ker} S = 0$.

Assertion (ii) is dual to (i). □

We now pass to the more general case of a commutative diagram of the form (2) with null sequences as rows.

We prove the following P-semi-abelian version of Corollary A_2 of [22]:

Theorem 4.5. *Suppose that in diagram* (2) *the rows are null sequences. The following assertions hold.*

(1) *If the sequence $A' \to B' \to C'$ is acyclic and b is a kernel then there exists a canonical morphism $\theta : H_-(A \to B \to C) \to \operatorname{Im} S$ such that the sequence*

$$0 \to H_-(A \to B \to C) \xrightarrow{\theta} \operatorname{Im} S \xrightarrow{\Lambda} \operatorname{Ker} T \longrightarrow 0$$

is acyclic.

(2) *If the sequence* $A \to B \to C$ *is acyclic and* b *is a cokernel then there exists a canonical morphism* $\varkappa : \operatorname{Ker} T \to H_+(A' \to B' \to C')$ *such that the sequence*

$$0 \to \operatorname{Im} S \xrightarrow{\Lambda} \operatorname{Ker} T \xrightarrow{\varkappa} H_+(A' \to B' \to C') \longrightarrow 0$$

is acyclic.

Proof. We prove only item (1) since (2) is obtained from it by duality.

By the equivalent description of left homology in Section 3, the left homology object $H_-(A \to B \to C)$ is the cokernel of the unique morphism ε such that $\operatorname{im} f = (\ker g)\varepsilon$. We have $(\operatorname{coker} \rho_S)\xi\nu_T\varepsilon = 0$ and hence there exists a unique morphism θ with $(\operatorname{coker} \rho_S)\xi\nu_T = \theta(\operatorname{coker} \varepsilon)$.

Obviously, $g'b(\ker(g'b)) = 0$, whence there exists a unique morphism y with $b\ker(g'b)) = (\ker g')y_0 = (\operatorname{im} f')y_0$. Since (7) is a pullback, there exists a unique morphism $\xi : \operatorname{Ker}(g'b) \to \operatorname{Im} S$ such that $\tilde{b}(\ker(g'b)) = k_S\xi$ and $y = l_S\xi$. We infer

$$k_S\xi\mu_T = \tilde{b}(\ker(g'b))\mu_T = \tilde{b}(\ker b) = 0,$$

and hence $\xi\mu_T = 0$ because k_S is monic. Now, denote by $\gamma = \gamma_S$ the unique morphism for which $\operatorname{im}(bf)\gamma = b(\operatorname{im} f)$ $(= b(\ker g)\varepsilon)$. We obtain

$$\begin{aligned}(\operatorname{im} b)k_S\rho_S\gamma_S\tilde{f} &= (\operatorname{im}(bf))\gamma_S\tilde{f} = b(\operatorname{im} f)\tilde{f} = bf \\ &= (\operatorname{im} b)\tilde{b}(\ker g)\varepsilon = (\operatorname{im} b)\tilde{b}(\ker(g'b))\nu_T\varepsilon\tilde{f} \\ &= (\operatorname{im} b)k_S\xi\nu_T\varepsilon\tilde{f}.\end{aligned}$$

Since $(\operatorname{im} b)k_S$ is monic $\tilde{f}$ is epic, this gives $\xi\nu_T\varepsilon = \rho_S\gamma_S$.

Now, since $b(\operatorname{im} f) = (\operatorname{im}(bf))\gamma_S$, b is a kernel, and the proof of Corollary 4.3 implies that γ_S is epic, it follows that γ_S is in fact an isomorphism. In addition, ξ is an isomorphism, too. Indeed, as above, ξ is a part of pullback (8), where $\tilde{b}$ is an isomorphism, which implies that ξ is an isomorphism, too. Thus we may write $\rho_S = \nu_T\varepsilon$. Since we thus obtain a pullback $\rho_S \operatorname{id} = \nu_T\varepsilon$ and hence ρ is strict, the morphism of the cokernels $\theta : \operatorname{Coker} \varepsilon \to \operatorname{Coker} \rho_S$ is monic by Lemma 2.1(v). This gives acyclicity at $H(A \to B \to C)$.

Furthermore, since

$$\Lambda\theta(\operatorname{coker} \varepsilon) = \Lambda(\operatorname{coker} \rho_S)\xi\nu_T = (\operatorname{coker}\langle\mu_t, \nu_T\rangle)\nu_T = 0,$$

we infer $\Lambda\theta = 0$. Now, take a morphism y with $y\theta = 0$. Then $y(\operatorname{coker} \rho_S)\nu = y\theta(\operatorname{coker} \varepsilon) = 0$ and, obviously, $y(\operatorname{coker} \rho_S)\mu_T = 0$. Hence, there exists a unique morphism v with $y(\operatorname{coker} \rho_S) = v(\operatorname{coker}\langle\mu_T, \nu_T\rangle) = v\Lambda(\operatorname{coker} \rho_S)$. Since $\operatorname{coker} \rho_S$ is epic, $y = v\Lambda$. Thus, $\Lambda = \operatorname{coker} \theta$ and so we have the acyclicity at $\operatorname{Im} S$. □

As a consequence of Theorem 4.5, we will now prove an assertion generalizing [22, Lemma 4.1]:

Corollary 4.6. *Suppose that in* (1) β *is strict. Assume in addition that* $\operatorname{coim} \beta$ *is a semi-stable cokernel or* $\operatorname{im} \beta$ *is a semi-stable kernel. Then there exists a bimorphism*

$$\operatorname{Im} S \to H(\operatorname{Ker} \beta \to \operatorname{Coker} g \to \operatorname{Coker} f).$$

Suppose that in (1) α *is strict. Assume in addition that* $\operatorname{im} \alpha$ *is a semi-stable cokernel or* $\operatorname{coim} \alpha$ *is a semi-stable kernel. Then there exists a bimorphism*

$$H(\operatorname{Ker} g \to \operatorname{Ker} f \to \operatorname{Coker} \alpha) \to \operatorname{Ker} S.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
 & & C & \xrightarrow{\alpha} & D \\
 & & \downarrow{\scriptstyle g} & S & \downarrow{\scriptstyle f} \\
\operatorname{Ker}\beta & \xrightarrow{\ker\beta} & A & \xrightarrow[\beta]{} & B \\
\| & S' & \downarrow{\scriptstyle \operatorname{coker} g} & S'' & \downarrow{\scriptstyle \operatorname{coker} f} \\
\operatorname{Ker}\beta & \longrightarrow & \operatorname{Coker} g & \xrightarrow[\hat\beta]{} & \operatorname{Coker} f
\end{array}$$

Lambek's isomorphism (Corollary 4.3) implies the equality $\operatorname{Im} S = \operatorname{Ker} S''$. Furthermore, by Theorem 4.5, the sequence

$$0 \to \operatorname{Im} S' \xrightarrow{\Lambda} \operatorname{Ker} S'' \xrightarrow{\varkappa'} H(\operatorname{Ker}\beta \to \operatorname{Coker} g \to \operatorname{Coker} f) \to 0$$

is acyclic. By Example 4.1, $\operatorname{Im} S' = 0$. Consequently, $\varkappa'$ is a bimorphism. □

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