S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

*Том 19, №1, стр. 66–80 (2022)* УДК 517.95
DOI 10.33048/semi.2022.19.006 MSC 31C12

# ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE POISSON EQUATION ON MODEL RIEMANNIAN MANIFOLDS

A.G. LOSEV, E.A. MAZEPA

Abstract. The paper is devoted to estimating the speed of approximation of solutions of the Dirichlet problem for the Poisson equation on non-compact model Riemannian manifolds to their boundary data at "infinity". Quantitative characteristics that estimate the speed of the approximation are found in terms of the metric of the manifold and the smoothness of the inhomogeneity in the Poisson equation.

**Keywords:** Dirichlet problem, Poisson equation, model Riemannian manifold, asymptotic behavior.

## 1. Introduction

The work is devoted to studying of asymptotic behaviour of solutions of the Dirichlet problem for the Poisson equation on non-compact model Riemannian manifolds.

Active studies on solutions to partial differential equations on non-compact Riemannian manifolds started in the middle of the 1970s and continue until the present time. The following statements of problems in this area of mathematics are historically established.

1. Find the conditions that ensure that every solution of an equation from the given class is trivial (Liouville-type theorems).

2. Find the conditions that ensure unique solvability of boundary value and external boundary value problems.

---

Losev, A.G., Mazepa, E.A., Asymptotic behavior of solutions of the Dirichlet problem for the Poisson equation on model Riemannian manifolds.



The work was carried out with partial financial support of Ministry of Science and Higher Education of the Russian Federation (gov. assignment No. 0633-2020-0003).

*Received February, 9, 2021, published January, 21, 2022.*

One of the sources of the mentioned issues is the classification theory of two-dimensional non-compact Riemannian surfaces. A distinctive feature of two-dimensional surfaces of parabolic type is that Liouville-type theorems, stating that every positive supergharmonic function on the given surface is the identity constant, hold for them (see, for example, [1, 2]). This property has served as the basis for extending of the notions of parabolicity and hyperbolicity of the type onto arbitrary non-compact Riemannian manifolds.

In particular, non-compact Riemannian manifolds, on which every superharmonic function bounded from below is constant, are called manifolds of a *parabolic* type (see [3]). In the opposite case, a manifold is said to be of a hyperbolic, or in other words, non-parabolic type.

The first results in defining types of a Riemannian manifold by using geometrical characteristics include a theorem by S.Y. Cheng and S.T. Yau [4], stating that a complete manifold is parabolic if the volume of a geodesic ball of radius $R$ grows not faster than $R^2$ for $R \to \infty$.

Studies dedicated to finding the conditions of parabolicity of the type of non-compact Riemannian manifolds have a sufficiently extended history. In recent years, a number of conditions ensuring parabolicity (non-parabolicity) of the type of non-compact Riemannian manifolds in terms of volume growth, cross-section area, changes of different curvatures (sectional, Ricci, and others), capacity characteristics, etc. (see, for example, [3, 4, 5, 6, 7]). A general idea on present state of research on this issue can be obtained in survey [3].

Questions on existence of non-trivial harmonic and superharmonic functions naturally lead to Liouville-type theorems. In traditional formulation, the Liouville theorem states that every bounded harmonic function in Euclidean space is the identity constant. During the last decades, dozens of different conditions have been found for geometric structure of non-compact Riemannian manifolds that ensure triviality of some classes of solutions to elliptic equations. Most often, bounded, positive, summable, having a finite Dirichlet integral, and some other spaces of harmonic functions and solutions of other elliptic equations and inequalities are studied (see, for example, [3, 4, 8, 9, 10, 11, 12]).

At the same time, the class of non-compact Riemannian manifolds, on which there exist non-trivial bounded harmonic functions, is quite large. In recent time, the approach to Liouville-type theorems tends to be more general. In particular, the dimension of different spaces of harmonic functions on non-compact Riemannian manifolds is estimated (see, for example, [9, 13, 14, 15]).

Equally interesting are the questions of solubility of different boundary value problems, for example, the Dirichlet problem on reconstruction of a harmonic function by continuous boundary data on "infinity".

Generally speaking, on arbitrary non-compact Riemannian manifold, the common statement of the Dirichlet problem is quite challenging. However, in some cases geometric compactification of a manifold allows to do it using the classic formulation. One class of Riemannian manifolds for which the statement of boundary value problems has a natural geometric interpretation is the class of manifolds with a negative sectional curvature. For example, M. Anderson and D. Sullivan (see [16, 17]) showed that on a complete simply connected manifold with a negative sectional curvature $\operatorname{sect} M$, satisfying the conditions $-b^2 \le \operatorname{sect} M \le -a^2 < 0$, the

Dirichlet problem on reconstruction of a harmonic function on such manifold by continuous boundary data is uniquely solvable.

The exact conditions for unique solvability of the Dirichlet problem for harmonic functions and solutions of the stationary Schrödinger equation on non-compact model manifolds generalizing spherically symmetric ones are obtained in works [6, 11, 13, 18].

Note that the majority of articles in the framework of the topic is devoted to studying of solutions of homogeneous elliptic equations. However, first works were suggested in recent years dedicated to studies of asymptotic behavior of solutions to nonhomogeneous elliptic equation on non-compact Riemannian manifolds (see, for example, [19, 20, 21]).

In studies on solvability of boundary value problems, traditionally a lot of attention is paid to the following question: in what sense, for example, in which metrics, we should understand the proximity of the solution to the boundary data (see [18, 22, 23]). At the same time, it is of interest to obtain quantitative characteristics estimating the speed of approximation of the solution to the boundary data. Among others, here we can mention the renowned "lemma on increasing"(see [23]). Our work is carried out exactly in this direction. In particular, we obtained significantly accurate estimates of the speed of approximation of solutions to the Dirichlet problem to their boundary data on non-compact Riemannian manifolds of some special form.

## 2. Statement of the problem

One class of Riemannian manifolds on which the statement of boundary value problems has a natural geometric interpretation is the set of manifolds of the following form (referred to as model ones): $M_g = B \cup D$, where $B$ is some precompact with a non-empty interior, and $D$ is isometric to the direct product $[r_0; +\infty) \times S$ ($r_0 > 0$ and $S$ is a compact Riemannian manifold without an edge) with the metric

$$ds^2 = dr^2 + g^2(r)\, d\theta^2.$$

Here $g(r)$ is a positive and smooth function on $[r_0; +\infty)$, and $d\theta^2$ is a metric on $S$. Examples of such manifolds include the Euclidean space $\mathbb{R}^n$, hyperbolic space $\mathbb{H}^n$, surfaces of revolution and others. Manifolds of type $D$ are sometimes referred to as "metric horns".

In recent years, a number of papers dedicated to studying of the behavior of solutions to distinct elliptic equations and inequalities on manifolds of this type and some on their generalizations were published (see, for example, [6, 9, 11, 13, 18, 19]). In particular, work [19] studies the solutions of the Poisson equation $u \in C^2(M_g)$

$$\Delta u = f(x), \tag{1}$$

where $f \in C^{0,\alpha}(M_g) \bigcap G^{\left[\frac{3n}{2}\right]}(D)$. Here $n = \dim M_g$, and

$$G^p(D) = \{(r,\theta) :\, \forall r \in [r_0; +\infty)\, f(r,\theta) \in C^p(S)\}$$

is a subset of the space of Hölder functions on $M_g$, that are $p$ times continuously differentiable at the second argument on $D$ for every fixed first argument. We introduce the notation

$$\varphi_0(r) = \|f(r,\theta)\|_{L^1(S)}, \qquad \varphi_m(r) = \|\Delta_\theta^m f(r,\theta)\|_{L^2(S)},$$

$$h_m(r) = \int\limits_r^\infty g^{1-n}(t)\left(\int\limits_{r_0}^t \left(\frac{1}{g^2(\xi)} + \varphi_0(\xi) + \varphi_m(\xi)\right)g^{n-1}(\xi)d\xi\right)dt. \tag{2}$$

In [19], the following statement was proved.

**Theorem 1** ([19]). *Let the manifold $M_g$ be such that $h_m(r_0) < \infty$, where $m = [\frac{3n}{4}]$. Then for every function $\Phi(\theta) \in C(S)$ on $M_g$ there exists a unique solution $u(x)$ of equation (1) such that on $D$ the following equality holds:*

$$\lim_{r\to\infty} u(r,\theta) = \Phi(\theta).$$

Also note that the passage to the limit in boundary conditions at "infinity" is understood in the sense of uniform convergence. In particular, it is implied that the following equality holds:

$$\lim_{r\to\infty} ||u(r,\theta) - \Phi(\theta)||_{C(M_g\setminus B(r))} = 0,$$

where $B(r)$ is a geodesic ball of radius $r$ with the center at some fixed point.

Therefore, solvability of the Dirichlet problem for the Poisson equation on model Riemannian manifolds was proved. However, a quite interesting question remains: what exactly is the speed of convergence of the solution to its boundary data at "infinity". This work is devoted to studying of asymptotic behavior at "infinity" of solutions to the Poisson equation on such manifolds, and, in particular, estimating the asymptotic convergence speed of the solution to the boundary data.

## 3. Asymptotic behavior of solutions of the exterior boundary value Dirichlet problem

In this paragraph, we will consider the Poisson equation on a manifold $D$

$$\Delta u(r,\theta) = f(r,\theta),$$

where $f \in C^{0,\alpha}(D)\bigcap G^{[\frac{3n}{2}]+2}(D)$.

We will define the exterior boundary value Dirichlet problem for the Poisson equation on $D$ in the following way: find a function $u(r,\theta)$, such that

$$\begin{cases}\Delta u(r,\theta) = f(r,\theta),\\ u(r_0,\theta) = \Psi(\theta),\\ \lim\limits_{r\to\infty} ||u(r,\theta) - \Phi(\theta)||_{C(D\setminus B(r))} = 0,\end{cases} \tag{3}$$

where $\Psi(\theta)$ and $\Phi(\theta)$ are some functions given on $D$.

Traditionally, the solution of a nonhomogeneous problem is searched in the form

$$u(r,\theta) = u_1(r,\theta) + u_2(r,\theta),$$

where $u_1(r,\theta)$ is the solution of the Dirichlet problem for the homogeneous equation

$$\begin{cases}\Delta u_1(r,\theta) = 0,\\ u_1(r_0,\theta) = \Psi(\theta),\\ \lim\limits_{r\to\infty} ||u_1(r,\theta) - \Phi(\theta)||_{C(D\setminus B(r))} = 0,\end{cases} \tag{4}$$

and $u_2(r,\theta)$ is the solution of the homogeneous boundary value problem for the nonhomogeneous equation

$$(5) \qquad \begin{cases} \Delta u_2(r,\theta) = f(r,\theta), \\ u_2(r_0,\theta) = 0, \\ \lim\limits_{r\to\infty} ||u_2(r,\theta)||_{C(D\setminus B(r))} = 0. \end{cases}$$

In works [13] and [19], the solvability conditions for the corresponding boundary value problems were obtained.

Now, we introduce the notation

$$q\,(r) = \int\limits_r^\infty g^{1-n}(t) \left( \int\limits_{r_0}^t g^{n-3}(\xi) d\xi \right) dt.$$

It is clear that for all $r \geq r_0$ the condition $h_m(r) \geq q(r)$ is fulfilled.

We put $m_1 = [\frac{3n}{4}] + 1$ and formulate the first main result of this paper.

**Theorem 2.** *Let the Riemannian manifold $M_g$ and the right-hand side of the Poisson equation $f$ be such that $h_{m_1}(r_0) < \infty$. Then for all functions $\Psi\,(\theta) \in C^{2m_1}(S)$ and $\Phi\,(\theta) \in C^{2m_1}(S)$ on $D$ there exists a unique solution $u(r,\theta)$ of the Poisson equation, such that*

$$u(r_0,\theta) = \Psi(\theta) \quad \textit{and} \quad |u(r,\theta) - \Phi(\theta)| \leq C h_{m_1}(r)$$

*for all $(r,\theta)$ given $r > r_0$, where the constant $C > 0$ does not depend on $(r,\theta)$.*

*Доказательство.* We will search for the solution of the nonhomogeneous problem in the form $u(r,\theta) = u_1(r,\theta)+u_2(r,\theta)$, where $u_1(r,\theta)$ and $u_2(r,\theta)$ are the solutions of the corresponding problems (4) and (5). As it was shown in [13], due to convergence of the integral $h_{m_1}(r_0) < \infty$, the solution $u_1(r,\theta)$ on $D$ exists and is unique. We will show that the inequality

$$|u_1(r,\theta) - \Phi(\theta)| \leq C h_{m_1}(r)$$

holds for all $(r,\theta)$ given $r > r_0$, where the constant $C > 0$ does not depend on $(r,\theta)$.

Let $\{w_k(\theta)\}$ be an orthonormal basis in $L^2(S)$ formed of eigenfunctions of the Laplace operator $-\Delta_\theta$, and $\lambda_k$ be the corresponding eigenvalue $(0 = \lambda_0 < \lambda_1 \leq \ldots)$, that is,

$$\Delta_\theta w_k(\theta) + \lambda_k w_k(\theta) = 0.$$

Then the following expansions are true:

$$\Phi\,(\theta) = \sum_{k=0}^{\infty} c_k w_k(\theta), \qquad \Psi\,(\theta) = \sum_{k=0}^{\infty} z_k w_k(\theta),$$

where

$$c_k = \int\limits_S \Phi\,(\theta)\, w_k(\theta) d\theta, \qquad z_k = \int\limits_S \Psi\,(\theta)\, w_k(\theta) d\theta.$$

Also, for every fixed $r$, the following representation for the harmonic function $u_1(r,\theta)$ is true:

$$u_1\,(r,\theta) = \sum_{k=0}^{\infty} v_k\,(r)\, w_k(\theta), \qquad \text{where} \qquad v_k\,(r) = \int\limits_S u_1\,(r,\theta)\, w_k(\theta) d\theta.$$

In work [9], it was shown that for every $k \geq 0$, the Fourier coefficients $v_k(r)$ satisfy the ordinary differential equation

$$v_k^{''}(r) + (n-1)\frac{g^{'}(r)}{g(r)}v_k^{'}(r) - \frac{\lambda_k}{g^2(r)}v_k(r) = 0, \tag{6}$$

to which we will refer as spectral. In this case, $v_k(r)$ also satisfies the boundary conditions $v_k(r_0) = z_k$, $\lim_{r\to\infty} v_k(r) = c_k$. Moreover, in [13] it is shown that for all $r \geq r_0$ the following inequality holds $|v_k(r)| \leq |c_k| + |z_k|$, along with the absolute and uniform convergence of the series

$$\sum_{k=1}^{\infty} v_k(r)\, w_k(\theta), \quad \sum_{k=0}^{\infty} c_k w_k(\theta), \quad \sum_{k=0}^{\infty} z_k w_k(\theta).$$

Using the triangle inequality, we estimate the following difference:

$$| u_1(r,\theta) - \Phi(\theta) | = \left| \sum_{k=0}^{\infty} v_k(r)\omega_k(\theta) - \sum_{k=0}^{\infty} c_k\omega_k(\theta) \right| =$$

$$\leq |(v_0(r) - c_0)\omega_0(\theta)| + \sum_{k=1}^{\infty} |v_k(r) - c_k|\, |\omega_k(\theta)|\,. \tag{7}$$

Further, we estimate the first summand from the above. Integrating twice the spectral equation (6) for $k = 0$, as in [13], we obtain

$$v_0(r) = v_0'(r_0)g^{n-1}(r_0)\int_{r_0}^{r} \frac{dt}{g^{n-1}(t)} + v_0(r_0).$$

Since $\lim_{r\to\infty} v_0(r) = c_0$, we have that

$$c_0 = v_0'(r_0)g^{n-1}(r_0)\int_{r_0}^{\infty} \frac{dt}{g^{n-1}(t)} + v_0(r_0).$$

From the definition of eigenfunction it follows that $w_0(\theta) = \frac{1}{\sqrt{|S|}}$, where $|S|$ is the volume of $S$. Then the following equality holds:

$$|(v_0(r) - c_0)\omega_0(\theta)| = \frac{1}{\sqrt{|S|}}\, |v_0(r) - c_0| =$$

$$= \frac{1}{\sqrt{|S|}} \left| v_0'(r_0)g^{n-1}(r_0)\int_{r_0}^{r} \frac{dt}{g^{n-1}(t)} + v_0(r_0) - v_0'(r_0)g^{n-1}(r_0)\int_{r_0}^{\infty} \frac{dt}{g^{n-1}(t)} - v_0(r_0) \right| =$$

$$= \frac{1}{\sqrt{|S|}}\, |v_0'(r_0)|\, g^{n-1}(r_0)\int_{r}^{\infty} \frac{dt}{g^{n-1}(t)} \leq \frac{1}{\sqrt{|S|}}\, |v_0'(r_0)|\, g^{n-1}(r_0)q(r) = Cq(r)\,.$$

Here $C$ is a constant that does not depend on $r$.

Now, we will estimate from above every member of the series in the inequality (7). Integrating twice the spectral equation (6) for $k > 0$, we obtain

$$v_k(r) = \lambda_k \int_{r_0}^{r} \frac{dt}{g^{n-1}(t)} \int_{r_0}^{t} g^{n-3}(z)v_k(z)dz + v_k'(r_0)g^{n-1}(r_0)\int_{r_0}^{r} \frac{dt}{g^{n-1}(t)} + v_k(r_0). \tag{8}$$

Taking into account the boundary conditions $v_k(r_0) = z_k$ and $\lim\limits_{r\to\infty} v_k(r) = c_k$, we have

$$c_k = \lambda_k \int\limits_{r_0}^{\infty} \frac{dt}{g^{n-1}(t)} \int\limits_{r_0}^{t} g^{n-3}(z) v_k(z) dz + v_k'(r_0) g^{n-1}(r_0) \int\limits_{r_0}^{\infty} \frac{dt}{g^{n-1}(t)} + z_k.$$

From here, we express $z_k$

$$z_k = c_k - \lambda_k \int\limits_{r_0}^{\infty} \frac{dt}{g^{n-1}(t)} \int\limits_{r_0}^{t} g^{n-3}(z) v_k(z) dz - v_k'(r_0) g^{n-1}(r_0) \int\limits_{r_0}^{\infty} \frac{dt}{g^{n-1}(t)}.$$

Substituting the obtained expression into (8), we obtain

$$v_k(r) = \lambda_k \int\limits_{r_0}^{r} \frac{dt}{g^{n-1}(t)} \int\limits_{r_0}^{t} g^{n-3}(z) v_k(z) dz + v_k'(r_0) g^{n-1}(r_0) \int\limits_{r_0}^{r} \frac{dt}{g^{n-1}(t)} +$$

$$+ c_k - \lambda_k \int\limits_{r_0}^{\infty} \frac{dt}{g^{n-1}(t)} \int\limits_{r_0}^{t} g^{n-3}(z) v_k(z) dz - v_k'(r_0) g^{n-1}(r_0) \int\limits_{r_0}^{\infty} \frac{dt}{g^{n-1}(t)} =$$

$$= c_k - \lambda_k \int\limits_{r}^{\infty} \frac{dt}{g^{n-1}(t)} \int\limits_{r_0}^{t} g^{n-3}(z) v_k(z) dz - v_k'(r_0) g^{n-1}(r_0) \int\limits_{r}^{\infty} \frac{dt}{g^{n-1}(t)}.$$

Therefore, the following inequalities hold:

$$|v_k(r) - c_k| = \left| -\lambda_k \int\limits_{r}^{\infty} \frac{dt}{g^{n-1}(t)} \int\limits_{r_0}^{t} g^{n-3}(z) v_k(z) dz - v_k'(r_0) g^{n-1}(r_0) \int\limits_{r}^{\infty} \frac{dt}{g^{n-1}(t)} \right| \le$$

$$\le \lambda_k \int\limits_{r}^{\infty} \frac{dt}{g^{n-1}(t)} \int\limits_{r_0}^{t} g^{n-3}(z) \left| v_k(z) \right| dz + \left| v_k'(r_0) \right| g^{n-1}(r_0) \int\limits_{r}^{\infty} \frac{dt}{g^{n-1}(t)}.$$

Recall that $|v_k(r)| \le |c_k| + |z_k|$. Then the last inequality can be continued in the following way:

$$|v_k(r) - c_k| \le \lambda_k \left( |c_k| + |z_k| \right) \int\limits_{r}^{\infty} \frac{dt}{g^{n-1}(t)} \int\limits_{r_0}^{t} g^{n-3}(z) dz + \left| v_k'(r_0) \right| g^{n-1}(r_0) \int\limits_{r}^{\infty} \frac{dt}{g^{n-1}(t)} \le$$

$$\le \lambda_k \left( |c_k| + |z_k| \right) q(r) + \left| v_k'(r_0) \right| g^{n-1}(r_0) q(r).$$

Taking into account what was said above, we continue the estimate (7)

$$\sum_{k=1}^{\infty} |v_k(r) - c_k| \left| \omega_k(\theta) \right| \le q(r) \sum_{k=1}^{\infty} \left( \lambda_k \left( |c_k| + |z_k| \right) + g^{n-1}(r_0) \left| v_k'(r_0) \right| \right) \left| \omega_k(\theta) \right| \le$$

$$\le q(r) \sum_{k=1}^{\infty} \left( \lambda_k |c_k| \left| \omega_k(\theta) \right| + \lambda_k |z_k| \left| \omega_k(\theta) \right| + g^{n-1}(r_0) \left| v_k'(r_0) \right| \left| \omega_k(\theta) \right| \right) \le$$

$$\le q(r) \left( \sum_{k=1}^{\infty} \lambda_k |c_k| \left| \omega_k(\theta) \right| + \sum_{k=1}^{\infty} \lambda_k |z_k| \left| \omega_k(\theta) \right| + \sum_{k=1}^{\infty} g^{n-1}(r_0) \left| v_k'(r_0) \right| \left| \omega_k(\theta) \right| \right).$$

Next, we will show the convergence of the series

$$\sum_{k=1}^{\infty} \lambda_k |c_k| |\omega_k(\theta)|.$$

Using Green's formula and the definition of eigenfunction, we obtain

$$|c_k| = \left| \int\limits_S \Phi(\theta) w_k(\theta) d\theta \right| = \frac{1}{\lambda_k} \left| \int\limits_S \Phi(\theta) \Delta_\theta w_k(\theta) d\theta \right| = \frac{1}{\lambda_k} \left| \int\limits_S \Delta_\theta \Phi(\theta) w_k(\theta) d\theta \right|.$$

We apply Green's formula $m_1 = \left[\frac{3n}{4}\right] + 1$ times

$$|c_k| = \frac{1}{\lambda_k^{m_1}} \left| \int\limits_S \Delta_\theta^{m_1} \Phi(\theta) w_k(\theta) d\theta \right|.$$

Now, we use the Cauchy-Schwarz inequality

$$|c_k| \le \frac{1}{\lambda_k^{m_1}} \left( \int\limits_S \left(\Delta_\theta^{m_1} \Phi(\theta)\right)^2 d\theta \right)^{\frac{1}{2}} \left( \int\limits_S w_k^2(\theta) d\theta \right)^{\frac{1}{2}} = \frac{C}{\lambda_k^{m_1}},$$

where $C$ is a constant that depends on the choice of boundary conditions, and also on $S$ and $m_1$.

The following estimates of eigenfunctions and eigenvalues (Weyl law) of the Laplace operator in $L^2(S)$ are well known (see, for example, [24])

$$\|w_k\|_{C(S)} \le C_1 \lambda_k^{\frac{n-2}{4}}, \qquad C_2 k^{\frac{2}{n-1}} \le \lambda_k \le C_3 k^{\frac{2}{n-1}}.$$

Taking into account the above-mentioned estimates, we obtain that the following inequalities hold:

$$\sum_{k=1}^{\infty} \lambda_k |c_k| |\omega_k(\theta)| \le \sum_{k=1}^{\infty} C_3 k^{\frac{2}{n-1}} \frac{C}{\lambda_k^{m_1}} C_1 \lambda_k^{\frac{n-2}{4}} \le \sum_{k=1}^{\infty} C_3 k^{\frac{2}{n-1}} \frac{C^*}{k^{\frac{2m_1}{n-1}}} C_1^* k^{\frac{2}{n-1} \frac{n-2}{4}} \le$$

$$\le C_3^* \sum_{k=1}^{\infty} k^{\frac{2}{n-1} - \frac{2m_1}{n-1} + \frac{n-2}{2n-2}} = C_3^* \sum_{k=1}^{\infty} k^{\frac{4-4m_1+n-2}{2n-2}} = C_3^* \sum_{k=1}^{\infty} k^{\frac{2-4m_1+n}{2n-2}} < \infty.$$

The last series converges due to the fact that, given $m_1 = \left[\frac{3n}{4}\right] + 1$, the following inequality is true:

$$\frac{2 - 4m_1 + n}{2n - 2} < -1.$$

The convergence of the following series is proved in the similar way:

$$\sum_{k=1}^{\infty} \lambda_k |z_k| |\omega_k(\theta)| \qquad \text{and} \qquad \sum_{k=1}^{\infty} |v_k'(r_0)| |\omega_k(\theta)|.$$

We will take a closer look at the study of the last series. First, we separately estimate $v_k'(r)$. By definition, we have

$$v_k(r) = \int\limits_S u_1(r, \theta) w_k(\theta) d\theta.$$

Differentiating both sides of the equality and also using the definition of eigenfunctions $w_k(\theta)$, we obtain

$$|v_k'(r)| = \left| \int\limits_S \frac{\partial u_1}{\partial r}(r,\theta)\, w_k(\theta) d\theta \right| = \frac{1}{\lambda_k} \left| \int\limits_S \frac{\partial u_1}{\partial r}(r,\theta)\, \Delta_\theta w_k d\theta \right|.$$

We apply Green's formula $m_1$ times and use the Cauchy-Schwarz inequality

$$|v_k'(r_0)| = \frac{1}{\lambda_k^{m_1}} \left| \int\limits_S \Delta_\theta^{m_1} \frac{\partial u_1}{\partial r}(r_0,\theta)\, w_k(\theta) d\theta \right| \leq$$

$$\leq \frac{1}{\lambda_k^{m_1}} \sqrt{\int\limits_S \left( \Delta_\theta^{m_1} \frac{\partial u_1}{\partial r}(r_0,\theta) \right)^2 d\theta} \sqrt{\int\limits_S w_k^2(\theta) d\theta} = \frac{C}{\lambda_k^{m_1}},$$

where $C > 0$ is some constant depending on $S$ and $m_1$. From Weyl law, it follows that

$$|v_k'(r_0)| \leq \frac{C}{\lambda_k^{m_1}} \leq \frac{C^*}{\left(k^{\frac{2}{n-1}}\right)^{m_1}}.$$

Further proof of the convergence of this series literally coincides with the study of the convergence of the first series.

Taking into account the convergence of the series, we obtain that the following inequalities hold:

$$|\, u_1(r,\theta) - \Phi(\theta)\, | \leq |v_0(r)\omega_0 - c_0\omega_0| + \sum_{k=1}^{\infty} |v_k(r) - c_k|\, |\omega_k(\theta)| \leq q(r) \Bigg( C +$$

$$+ \sum_{k=1}^{\infty} \lambda_k\, |c_k|\, |\omega_k(\theta)| + \sum_{k=1}^{\infty} \lambda_k\, |z_k|\, |\omega_k(\theta)| + \sum_{k=1}^{\infty} g^{n-1}(r_0)\, |v_k'(r_0)|\, |\omega_k(\theta)| \Bigg) \leq C^{**} q(r)$$

for all $(r,\theta)$ given $r > r_0$, where $C^{**} > 0$ is some constant that depends on $S$ and $m_1$ and on the choice of boundary conditions, and does not depend on $(r,\theta)$.

Next, we consider the solution $u_2(r,\theta)$ of the homogeneous boundary value problem for the Poisson equation (5). Solvability of this problem was obtained in [19]. We will prove that the estimate

$$|u_2(r,\theta)| \leq Q h_{m_1}(r)$$

is true for all $(r,\theta)$ when $r > r_0$ with some constant $Q > 0$ that does not depend on $(r,\theta)$.

In work [19], the solution of the boundary value problem (5) was constructed in the form

$$u_2(r,\theta) = \sum_{k=0}^{\infty} v_k(r)\omega_k(\theta).$$

Here $v_k(r)$ is the solution of the nonhomogeneous spectral equation

$$v_k''(r) + (n-1)\frac{g'(r)}{g(r)} v_k'(r) - \frac{\lambda_k}{g^2(r)} v_k(r) = p_k(r), \tag{9}$$

where $\lambda_k$ are eigenvalues of the Laplace operator on $S$, and $p_k(r)$ are the coefficients of expansion into Fourier series by eigenfunctions of the Laplace operator of the right-hand side of the Poisson equation $f(r,\theta)$, that is,

$$f(r,\theta)=\sum_{k=0}^{\infty}p_k(r)w_k(\theta).$$

Integrating twice the equation (9) within the limits $[r_0,r]$, we obtain the following integral equation equivalent to the spectral equation (9):

$$v_k\left(r\right)=\lambda_k\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}\left(t\right)}\int\limits_{r_0}^{r}g^{n-3}\left(z\right)v_k\left(z\right)dz+\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}\left(r\right)}\int\limits_{r_0}^{r}g^{n-1}\left(z\right)p_k\left(z\right)dz+$$

$$+g^{n-1}\left(r_0\right)v_k^{'}\left(r_0\right)\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}\left(t\right)}+v_k\left(r_0\right). \tag{10}$$

For $k=0$, taking into account $\lambda_0=0$, we obtain

$$v_0(r)=\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}p_0(z)g^{n-1}(z)dz+v_0'(r_0)g^{n-1}(r_0)\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}+v_0(r_0). \tag{11}$$

In both cases, $v_k(r)$ also satisfies the boundary conditions $v_k\left(r_0\right)=0,\ \lim\limits_{r\to\infty}v_k\left(r\right)=0$.

In the mentioned paper, also the following estimates of the coefficients $p_k(r)$ and the solutions of the nonhomogeneous spectral equation (9) were obtained:

$$|p_0(r)|\le\frac{1}{\sqrt{|\,S\,|}}\varphi_0(r),\qquad |p_k(r)|\le\frac{\varphi_{m_1}(r)}{\lambda_k^{m_1}},\qquad |v_k(r)|\le\frac{h_{m_1}(r_0)}{\lambda_k^{m_1}}. \tag{12}$$

Then we have

$$|u_2(r,\theta)|=\left|\sum_{k=0}^{\infty}v_k(r)\omega_k(\theta)\right|=\left|v_0(r)\omega_0(\theta)+\sum_{k=1}^{\infty}v_k(r)\omega_k(\theta)\right|\le$$

$$\le|v_0(r)\omega_0(\theta)|+\left|\sum_{k=1}^{\infty}v_k(r)\omega_k(\theta)\right|\le|v_0(r)|\,|\omega_0(\theta)|+\sum_{k=1}^{\infty}|v_k(r)|\,|\omega_k(\theta)|\,.$$

We will estimate each of the summands in the last expression. Taking into account that $\omega_0(\theta)=\frac{1}{\sqrt{|S|}}$, the boundary conditions and (11), for the first summand we have:

$$|v_0(r)|\,|\omega_0(\theta)|\le\frac{1}{\sqrt{|\,S\,|}}\left|\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}p_0(z)g^{n-1}(z)dz+v_0^{'}(r_0)g^{n-1}(r_1)\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\right|\le$$

$$\le\frac{1}{\sqrt{|\,S\,|}}\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}|p_0(z)|\,g^{n-1}(z)dz+\frac{\left|v_0^{'}(r_0)\right|}{\sqrt{|\,S\,|}}g^{n-1}(r_0)\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}.$$

Taking into account the estimate (12), we obtain

$$|v_0(r)|\,|\omega_0(\theta)| \leq \frac{1}{|\,S\,|}\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}\varphi_0(z)g^{n-1}(z)dz+\frac{\left|v_0^{'}(r_0)\right|}{\sqrt{|\,S\,|}}g^{n-1}(r_0)\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)} \leq$$

$$\leq Q_1\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}\varphi_0(z)g^{n-1}(z)dz + Q_2\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}.$$

Each summand separately does not exceed $Q_i h_{m_1}(r)$, which yields the estimate

$$|v_0(r)|\,|\omega_0(\theta)| \leq Q h_{m_1}(r),$$

where the constant $Q$ depends on the compact $S$, and also on the boundary conditions of the problem.

Similarly, we will estimate $v_k(r)$, taking into account (10)

$$|v_k(r)| = \left|\lambda_k\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}v_k(z)g^{n-3}(z)dz + \int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}p_k(z)g^{n-1}(z)dz+\right.$$

$$\left.+v_k^{'}(r_0)g^{n-1}(r_0)\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\right| \leq \lambda_k\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}|v_k(z)|\,g^{n-3}(z)dz+$$

$$+\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}|p_k(z)|\,g^{n-1}(z)dz + \left|v_k^{'}(r_0)\right|g^{n-1}(r_0)\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}.$$

Applying the second estimate in (12), we obtain

$$|v_k(r)| \leq \lambda_k\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}|v_k(z)|\,g^{n-3}(z)dz+$$

$$+\frac{1}{\lambda_k^{m_1}}\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}\varphi_{m_1}(z)g^{n-1}(z)dz + \left|v_k^{'}(r_0)\right|g^{n-1}(r_0)\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}.$$

Taking into account the above-mentioned considerations and the estimates of the eigenvalues and eigenfunctions of the Laplace operator, in a way similar to the above, we estimate the series

$$\sum_{k=1}^{\infty}|v_k(r)|\,|\omega_k(\theta)| \leq C\sum_{k=1}^{\infty}\left(\lambda_k\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}|v_k(z)|\,g^{n-3}(z)dz+\right.$$

$$\left.+\frac{1}{\lambda_k^{m_1}}\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}\varphi_{m_1}(z)g^{n-1}(z)dz + \left|v_k^{'}(r_0)\right|g^{n-1}(r_0)\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\right)k^{\frac{n-2}{2(n-1)}}.$$

Next, we substitute the third estimate from (12) and obtain

$$\sum_{k=1}^{\infty}|v_k(r)|\,|\omega_k(\theta)| \leq C\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}g^{n-3}(z)dz\sum_{k=1}^{\infty}\lambda_k\frac{1}{\lambda_k^{m_1}}k^{\frac{n-2}{2(n-1)}}+$$

(13)

$$+C\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}\varphi_{m_1}(z)g^{n-1}(z)dz\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{m_1}}k^{\frac{n-2}{2(n-1)}}+C\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\sum_{k=1}^{\infty}k^{\frac{n-2}{2(n-1)}}\left|v_k'(r_0)\right|.$$

In a way similar to the above, we find the estimate $|v_k'(r)|\le\frac{C^*}{\left(k^{\frac{2}{n-1}}\right)^{m_1}}$ for the solutions of the nonhomogeneous spectral equation, and from (13), we obtain

$$\sum_{k=1}^{\infty}|v_k(r)|\,|\omega_k(\theta)|\le C_1\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}g^{n-3}(z)dz\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{m_1-1}}k^{\frac{n-2}{2(n-1)}}+$$

$$+C_2\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}\varphi_{m_1}(z)g^{n-1}(z)dz\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{m_1}}k^{\frac{n-2}{2(n-1)}}+C_3\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\sum_{k=1}^{\infty}k^{-\frac{2m_1}{n-1}}k^{\frac{n-2}{2(n-1)}}.$$

We will prove the convergence of every series on the right-hand side of the last inequality. We will estimate the series from the first summand. Taking into account that $m_1=\left[\frac{3n}{4}\right]+1$, we obtain the convergence of this series

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{m_1-1}}k^{\frac{n-2}{2(n-1)}}\le C\sum_{k=1}^{\infty}k^{\frac{2-2m_1}{n-1}}k^{\frac{n-2}{2(n-1)}}=C\sum_{k=1}^{\infty}k^{\frac{n-4m_1+2}{2(n-1)}}<\infty.$$

Similarly, the convergence of the remaining series on the right-hand side of the inequality is proved:

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{m_1}}k^{\frac{n-2}{2(n-1)}}\le C\sum_{k=1}^{\infty}k^{-\frac{2m_1}{n-1}}k^{\frac{n-2}{2(n-1)}}=C\sum_{k=1}^{\infty}k^{\frac{n-4m_1-2}{2(n-1)}}<\infty.$$

Taking into account the convergence of the series and the definition of the integral $h_{m_1}(r)$, we obtain that the following estimate is true:

$$\sum_{k=1}^{\infty}|v_k(r)|\,|\omega_k(\theta)|\le C_1^*\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}g^{n-3}(z)dz+C_2^*\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\int\limits_{r_0}^{t}\varphi_m(z)g^{n-1}(z)dz+$$

$$+C_3^*\int\limits_{r_0}^{r}\frac{dt}{g^{n-1}(t)}\le(C_1^*+C_2^*+C_3^*)h_{m_1}(r).$$

Combining the previous inequality and the estimate of the first summand in the expansion for the function $u_2(r,\theta)$, we obtain

$$|u_2(r,\theta)|\le|v_0(r)\omega_0|+\sum_{k=1}^{\infty}|v_k(r)|\,|\omega_k(\theta)|\le Q^*h_{m_1}(r)$$

for all $(r,\theta)$ given $r>r_0$, where $Q^*>0$ is some constant that does not depend on $(r,\theta)$.

Now, we will prove the main statement

$$|u(r,\theta)-\Phi(\theta)|=|u_1(r,\theta)+u_2(r,\theta)-\Phi(\theta)|\le|u_1(r,\theta)-\Phi(\theta)|+|u_2(r,\theta)|\le$$

$$\le(C^{**}+Q^*)h_{m_1}(r)\le Ch_{m_1}(r)$$

for all $(r,\theta)$ given $r>r_0$, where $C>0$ is some constant that does not depend on $(r,\theta)$. The theorem is proved. □

## 4. Asymptotic behaviour of solutions of the Dirichlet problem on a model manifold

In this section, we will demonstrate a possibility of continuing the solution of the exterior boundary value problem for the Poisson equation in the area $D$ onto the whole model manifold preserving the asymptotic inequality. We will formulate the main result.

**Theorem 3.** *Let the Riemannian manifold $M_g$ and the right-hand side of the Poisson equation $f$ be such that $h_{m_1}(r_0) < \infty$. Then for every function $\Phi(\theta) \in C^{2m_1}(S)$ on $M_g$ there exists a unique solution $u(x)$ of the Poisson equation, such that on $D$ the inequality*

$$|u(r,\theta) - \Phi(\theta)| \le C h_{m_1}(r)$$

*holds for all $(r,\theta)$, given $r > r_0$, where the constant $C > 0$ does not depend on $(r,\theta)$.*

*Доказательство.* Let $u_0$ be the solution of the exterior boundary value problem (3), whose existence in the conditions of this theorem is shown in Theorem 2. Moreover, $u_0$ on $D$ satisfies the asymptotic inequality

$$|u_0(r,\theta) - \Phi(\theta)| \le C^* h_{m_1}(r)$$

for all $(r,\theta)$, given $r > r_0$, where the constant $C^* > 0$ does not depend on $(r,\theta)$.

Next, consider the function $U = u_0 \cdot \phi$, where $\phi$ is a smooth function on $M_g$, such that $\phi = 0$ on the precompact set $B' \subset B$ and $\phi = 1$ outside $\overline{B}$. Then $U \in C^{2m_1}(M_g)$ and $\Delta U = \Delta(u_0 \cdot \phi) = f^*$, where $f^* \in C^{0,\alpha}(M_g)$, $f^*(x) = 0$ on $B'$, $f^*(x) = f(x)$ outside $\overline{B}$.

Let $\{B_k\}_{k=1}^{\infty}$ be an arbitrary smooth exhaustion of the manifold $M_g$, that is, a sequence of precompact open non-empty subsets of $B_k \subset M_g$, such that $\overline{B_k} \subset B_{k+1}$, $M_g = \bigcup_{k=1}^{\infty} B_k$, $\partial B_k$ are the smooth boundaries and $\overline{B} \subset B_k$ for every $k$.

We will construct a sequence of functions $\phi_k$, solution of the problem

$$\Delta \phi_k = f, \quad \text{in} \quad B_k \qquad \phi_k|_{\partial B_k} = u_0|_{\partial B_k}$$

and a sequence of functions $\psi_k = \phi_k - U$. For these functions, we have:

$$\Delta \psi_k = f - f^* \quad \text{in} \quad B_k, \qquad \psi_k|_{\partial B_k} = 0.$$

On every set $B_k$, there exists a Green's function, that is, a function $G_k(x,y)$ such that

$$\Delta_x G_k(x,y) = -\delta_y(x), \quad G_k|_{x \in \partial B_k} = 0$$

for every $y \in B_k$, where $\delta_y(x)$ is a Dirac $\delta$-function. Therefore, by Green's formula in $B_k$, we have

$$\psi_k(x) = -\int_{B_k} G_k(x,y)(f(y) - f^*(y))dy.$$

From the condition on convergence of the integral $h_{m_1}(r_0)$, the manifold $M_g$ is non-parabolic. As a non-trivial capacity potential of the precompact $B$ we can use the function $v(r) = \frac{q(r)}{q(r_0)}$. Non-parabolicity of the manifold $M_g$ yields the existance of the finite Green's function $G(x,y) = \lim_{k\to\infty} G_k(x,y)$ on all $M_g$.

The existence of Green's functions yields the existence of the limit of the sequence $\{\psi_k\}_{k=1}^{\infty}$. Let $\psi = \lim\limits_{k\to\infty} \psi_k$, then $\Delta\psi = f - f^*$ on $M_g$ (see also [9]). We will show that $\psi$ satisfies the asymptotic inequality

$$|\psi(r,\theta)| \le C_3 h_{m_1}(r)$$

for all $(r,\theta)$ given $r > r_0$, where the constant $C_3 > 0$ does not depend on $(r,\theta)$.

Due to the fact that the function $\psi(x)$ is continuous, there exists

$$U = \max_{\partial B} |\psi(x)|.$$

Obviously, the inequalities

$$-(U+1) \le \psi|_{\partial B} \le U+1 \qquad \text{and} \qquad -(U+1) \le \psi_k|_{\partial B} \le U+1$$

hold for every sufficiently large $k$.

Consider the functions $\underline{u} = -(U+1)\cdot \upsilon$ and $\overline{u} = (U+1)\cdot\upsilon$ on $M_g \backslash B$, where $\upsilon = \frac{q(r)}{q(r_0)}$ is a capacity potential of a precompact set $B$, moreover, $\upsilon(r) \le C_0 h_{m_1}(r)$ holds for all $r > r_0$, where $C_0 > 0$ is some constant. The functions $\underline{u}$ and $\overline{u}$ are the solutions of the equation $\Delta u = 0$ on $D$ and satisfy the conditions

$$\underline{u}|_{\partial B} = -(U+1), \quad -(U+1) \le \underline{u} \le 0, \quad |\underline{u}| \le C_1 h_{m_1}(r),$$

$$\overline{u}|_{\partial B} = (U+1), \quad 0 \le \overline{u} \le (U+1), \quad |\overline{u}| \le C_2 h_{m_1}(r).$$

Then $\underline{u} \le \overline{u}$ on $M\backslash B$ and, by the comparison principle for harmonic functions, we have

$$\underline{u} \le \psi_k \le \overline{u}$$

on $B_k\backslash B$ for every $k$. Turning to the limit for $k \to \infty$, we obtain $\underline{u} \le \psi \le \overline{u}$. Taking into account that the asymptotic inequalities hold for the functions $\underline{u}$ and $\overline{u}$, we obtain $|\psi| \le C_3 h_{m_1}(r)$.

From the existence of the function $\psi = \lim\limits_{k\to\infty} \psi_k$ follows the existence of the limit function

$$u = \lim_{k\to\infty} \phi_k = \lim_{k\to\infty} (\psi_k + U) = \psi + U.$$

Moreover, $\Delta u = \Delta\psi + \Delta U = f - f^* + f^* = f$ on $M_g$, and on $D$ the asymptotic inequality

$$|u(r,\theta) - \Phi(\theta)| = |\psi + U - \Phi| = |\psi + u_0 - \Phi| \le |\psi| + |u_0 - \Phi| \le$$

$$\le C_3 h_{m_1}(r) + C^* h_{m_1}(r) = C h_{m_1}(r)$$

holds for all $(r,\theta)$, given $r > r_0$, where $C > 0$ is some constant that does not depend on $(r,\theta)$. The theorem is proved. □

## References


[1] L.V. Ahlfors, *Sur le type d'une surface de Riemann*, C.R. Acad. Sci. Paris, **201** (1935), 30–32.

[2] R. Nevanlinna, *Ein Satz über offene Riemannsche Flächen*, Ann. Acad. Sci. Fenn., Ser. A., **54**:3 (1940), 1–18. Zbl 0024.22104

[3] A. Grigor'yan, *Analitic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Am. Math. Soc., New Ser., **36**:2 (1999), 135–249. Zbl 0927.58019

[4] S.Y. Cheng, S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math., **28**:3 (1975), 333–354. Zbl 0312.53031

[5] V. M. Kesel'man, *The concept and criteria of the capacitive type of the non-compact Riemannian manifold based on the Generalized capacity*, Mathematical Physics and Computer Modeling, **22**:2 (2019), 21–32.
[6] A.G. Losev, *On the hyperbolicity criterion for noncompact Riemannian manifolds of special type*, Math. Notes, **59**:4 (1996), 400–404. Zbl 0871.53037
[7] T. Lyons, D. Sullivan, *Function theory, random path and covering spaces*, J. Diff. Geom., **19** (1984), 299–323. Zbl 0554.58022
[8] A.A. Grigor'yan, *On Liouville theorems for harmonic functions with finite Dirichlet integral*, Math. USSR. Sb., **60**:2 (1988), 485–504. Zbl 0699.31018
[9] A.G. Losev, *Certain Liouville theorems for Riemannian manifolds of a special form*, Sov. Math., **35**:12 (1991), 15–23. Zbl 0764.58035
[10] P. Li, R. Schoen, *$L^p$ and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math., **153**:3-4 (1984), 279–300. Zbl 0556.31005
[11] M. Murata, *Positive harmonic functions on rotationary symmetric Riemannian manifolds*, Potential Theory, Proc. Int. Conf. Nagoya/Jap. 1990, 251–259, 1992. Zbl 0777.53039
[12] S.Y. Cheng, S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math., **28**:3 (1975), 333–354. Zbl 0312.53031
[13] A.G. Losev, E.A. Mazepa, *On the asymptotic behavior of solutions of certain elliptic type equations on noncompact Riemannian manifolds*, Russ. Math., **43**:6 (1999), 39–47. Zbl 0988.58008
[14] S.A. Korolkov, A.G. Losev, *Generalized harmonic functions of Riemannian manifolds with ends*, Math. Z., **272**:1-2 (2012), 459–472. Zbl 1259.53062
[15] C.-J. Sung, L.-F.Tam, J. Wang, *Spaces of harmonic functions*, J. Lond. Math. Soc., II. Ser., **61**:3 (2000), 789–806. Zbl 0963.31004
[16] M.T. Anderson, *The Dirichlet problem at innity for manifolds of negative curvature*, J. Diff. Geom., **18** (1983), 701–721. Zbl 0541.53036
[17] D. Sullivan, *The Dirichlet problem at infinity for a negatively curved manifold*, J. Diff. Geom., **18** (1983), 723–732. Zbl 0541.53037
[18] A. Losev, E. Mazepa, I. Romanova, *Eigenfunctions of the Laplace operator and harmonic functions on model Riemannian manifolds*, Lobachevskii J. Math., **41**:11 (2020), 2190–2197. Zbl 1455.35159
[19] A.G. Losev, *Solvability of the Dirichlet problem for the Poisson equation on some noncompact Riemannian manifolds*, Differ. Equ., **53**:12 (2017), 1595–1604. Zbl 1397.35075
[20] P. Mastrolia, D.D. Monticelli, F. Punzo, *Elliptic and parabolic equations with Dirichlet conditions at infinity on Riemannian manifolds*, Adv. Differ. Equ., **23**:1-2 (2018), 89–108. Zbl 1386.35072
[21] O. Munteanu, N. Sesum, *The Poisson equation on complete manifolds with positive spectrum and applications*, Adv. Math., **223**:1 (2010), 198–219. Zbl 1181.58014
[22] A.K. Gushchin, *A strengthening of the interior Hölder continuity property for solutions of the Dirichlet problem for a second-order elliptic equation*, Theor. Math. Phys., **157**:3 (2008), 1655–1670. Zbl 1155.35325
[23] E.M. Landis, *Second order equations of elliptic and parabolic types*, Nauka, Moscow, 1971. Zbl 0226.35001
[24] D. Grieser, *Uniform bounds for eigenfunctions of the Laplacian on manifolds with boundary*, Commun. Partial Diff. Equations, **27**:7-8 (2002), 1283–1299. Zbl 1034.35085

Losev Alexander Georgievoch
Volgograd State Univercity,
100, Universitetsky ave.,
Volgograd, 400062, Russia
*Email address*: alexander.losev@volsu.ru

Mazepa Elena Alexeevna
Volgograd State Univercity,
100, Universitetsky ave.,
Volgograd, 400062, Russia
*Email address*: elena.mazepa@volsu.ru