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THE RENEWAL EQUATION WITH UNBOUNDED INHOMOGENEOUS TERM

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Abstract. We consider the renewal equation whose kernel is a probability distribution with positive mean. The inhomogeneous term behaves like a submultiplicative function tending to infinity. Asymptotic properties of the solution are established depending on the asymptotics of the submultiplicative function.

Keywords: renewal equation, probability distribution, positive mean, unbounded inhomogeneous term, submultiplicative function, asymptotic behavior.

1. Introduction

Consider the renewal equation

$$z(x) = \int_{\mathbb{R}} z(x-y)\,F(dy) + g(x), \qquad x \in \mathbb{R}, \tag{1}$$

where z is the function sought, F is a given probability distribution on $\mathbb{R}$ and the inhomogeneous term g is a known complex function. A probability distribution F in $\mathbb{R}$ is *arithmetic* if it is concentrated on a set of points of the form $0, \pm\lambda, \pm 2\lambda, \dots$. The largest λ with this property is called the *span* of F (see [1, Chapter V, § 2, Definition 3]). A probability distribution G on $\mathbb{R}$ is called *nonarithmetic* if it is not concentrated on the set of points of the form $0, \pm\lambda, \pm 2\lambda, \dots$. Let $\mathbb{R}_+$ be the set of all nonnegative numbers and $\mathbb{R}_- := \mathbb{R} \setminus \mathbb{R}_+$ be the set of all negative numbers. A positive function $\varphi(x)$, $x \in \mathbb{R}(\mathbb{R}_+)$, is called *submultiplicative* if it is finite, Borel measurable and satisfies the conditions: $\varphi(0) = 1$, $\varphi(x+y) \le \varphi(x)\,\varphi(y)$,

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x, $y \in \mathbb{R}(\mathbb{R}_+)$. The following properties are valid for submultiplicative functions defined on the whole line [2, Theorem 7.6.2]:

$$(2) \qquad -\infty < r_-(\varphi) := \lim_{x\to-\infty} \frac{\log\varphi(x)}{x} = \sup_{x<0} \frac{\log\varphi(x)}{x} \le \inf_{x>0} \frac{\log\varphi(x)}{x} = \lim_{x\to\infty} \frac{\log\varphi(x)}{x} =: r_+(\varphi) < \infty.$$

Here are some examples of submultiplicative function on $\mathbb{R}_+$: (i) $\varphi(x) = (x+1)^r$, $r > 0$; (ii) $\varphi(x) = \exp(cx^\beta)$, where $c > 0$ and $0 < \beta < 1$; (iii) $\varphi(x) = \exp(\gamma x)$, where $\gamma \in \mathbb{R}$. In (i) and (ii), $r_+(\varphi) = 0$, while in (iii), $r_+(\varphi) = \gamma$. The product of a finite number of submultiplicative function is again a submultiplicative function.

Let ν and $\varkappa$ be finite measures on the σ-algebra $\mathscr{B}$ of Borel sets in $\mathbb{R}$. Their *convolution* is the measure

$$\nu * \varkappa(A) := \iint_{\{x+y\in A\}} \nu(dx)\,\varkappa(dy) = \int_{\mathbb{R}} \nu(A-x)\,\varkappa(dx), \qquad A \in \mathscr{B};$$

here $A - x := \{y \in \mathbb{R} : x + y \in A\}$. Define the Laplace transform of a measure $\varkappa$ as

$$\widehat{\varkappa}(s) := \int_{\mathbb{R}} \exp(sx)\,\varkappa(dx).$$

Let F be a nonarithmetic probability distribution with finite positive mean μ. Denote by F^{n*} the n-th convolution power of F:

$$F^{0*} := \delta_0, \quad F^{1*} := F, \quad F^{(n+1)*} := F^{n*} * F, \quad n \ge 1,$$

where δ_0 is the measure of unit mass concentrated at zero. Let U be the renewal measure generated by F: $U := \sum_{n=0}^{\infty} F^{n*}$. It is well known that if $g \in L_1(\mathbb{R})$ then $z(x) = U * g(x) := \int_{\mathbb{R}} g(x-y)\,U(dy)$, $x \in \mathbb{R}$, is the solution to equation (1) which coincides with the solution obtained by successive approximations. If g is directly Riemann integrable and $z(x) = U * g(x)$, then

$$z(x) \to \frac{1}{\mu} \int_{\mathbb{R}} g(y)\,dy \qquad \text{as } x \to \infty$$

and $z(x) \to 0$ as $x \to -\infty$ [3, Theorem 2.5.3]; see also [1, Section XI.1, Theorem 2 and Section XI.9, Theorem 1]. Suppose additionally that F^{n*} has an absolutely continuous component for some n and $g \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. Then $z(x)$ satisfies the same relations as above [3, Theorem 2.6.4]. For F on the whole line, the case $g \notin L_1(\mathbb{R})$ does not seem to have been considered in the literature so far.

For $c \in \mathbb{C}$, we assume that c/∞ is equal to zero. The relation $a(x) \sim cb(x)$ as $x \to \infty$ means that $a(x)/b(x) \to c$ as $x \to \infty$; if $c = 0$, then $a(x) = o(b(x))$.

In the present paper we investigate the asymptotic behavior of the solution $z(x)$ to equation (1) when the inhomogeneous term g is asymptotically equivalent (up to a constant factor) to an unbounded nondecreasing submultiplicative function φ: $g(x) \sim c\varphi(x)$ as $x \to \infty$. In the main theorems (Theorems 1 and 2), $\varphi(x)$, $x \in \mathbb{R}_+$, is a nondecreasing submultiplicative function such that there exists $\lim_{x\to\infty} \dfrac{\varphi(x+y)}{\varphi(x)}$ for each $y \in \mathbb{R}$. It can be proved that if such a limit exists, then it is equal to $\exp(r_+(\varphi)y)$. The value $r_+(\varphi)$ is well defined since the function $\varphi(x)$ can be extended to the whole line preserving the submultiplicativity property, e.g., $\varphi(x) \equiv$

1, $x \in \mathbb{R}_-$. The asymptotic behavior of the solution depends on whether $r_+(\varphi) = 0$ or $r_+(\varphi) > 0$.

Theorem 1. *Let F be a nonarithmetic probability distribution with finite positive mean μ and let $\varphi(x)$, $x \in \mathbb{R}_+$, be a nondecreasing continuous submultiplicative function tending to infinity as $x \to \infty$ such that $r_+(\varphi) = 0$ and $\lim_{x\to\infty} \varphi(x + y)/\varphi(x) = 1$ for each $y \in \mathbb{R}$. Suppose that the inhomogeneous term $g(x)$ is bounded on finite intervals, the function $\mathbf{1}_{\mathbb{R}_-} g$ is directly Riemann integrable and $g(x) \sim c\varphi(x)$ as $x \to \infty$, where $c \in \mathbb{C}$. Assume that*

$$I_F := \int_{-\infty}^{0} |x|\varphi(|x|)F((-\infty, x])\, dx < \infty. \tag{3}$$

Then the solution $z(x)$, $x \in \mathbb{R}$, to equation (1) satisfies the asymptotic relation

$$z(x) \sim \frac{c}{\mu} \int_0^x \varphi(y)\, dy \qquad \text{as } x \to \infty.$$

Theorem 2. *Let F be a nonarithmetic probability distribution with $\mu \in (0, +\infty]$ and let $\varphi(x)$, $x \in \mathbb{R}_+$, be a submultiplicative function such that $r_+(\varphi) > 0$,*

$$\lim_{x\to\infty} \frac{\varphi(x + y)}{\varphi(x)} = \exp(r_+(\varphi)y), \qquad y \in \mathbb{R},$$

and $\varphi(x)/\exp(r_+(\varphi)x)$ is nondecreasing on $\mathbb{R}_+$. Suppose that the inhomogeneous term $g(x)$ is bounded on finite intervals, the function $\mathbf{1}_{\mathbb{R}_-} g$ is directly Riemann integrable and $g(x) \sim c\varphi(x)$ as $x \to \infty$. Assume that $\widehat{F}(-r_+(\varphi)) < 1$. Then the solution $z(x)$ to equation (1) satisfies the asymptotic relation

$$z(x) \sim \frac{c}{1 - \widehat{F}(-r_+(\varphi))}\varphi(x) \qquad \text{as } x \to \infty.$$

The proofs of Theorems 1 and 2 are given in Section 4.

2. Preliminaries

Consider the collection $S(\varphi)$ of all complex-valued measures $\varkappa$ such that

$$\|\varkappa\|_\varphi := \int_{\mathbb{R}} \varphi(x)\, |\varkappa|(dx) < \infty;$$

here $|\varkappa|$ stands for the total variation of $\varkappa$. The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and $\varkappa$ of $S(\varphi)$ is defined as their convolution $\nu * \varkappa$ [2, Section 4.16]. The unit element of $S(\varphi)$ is the measure δ_0. It follows from (2) that the Laplace transform of any $\varkappa \in S(\varphi)$ converges absolutely with respect to $|\varkappa|$ for all $s \in \mathbb{C}$ such that $r_-(\varphi) \le \Re s \le r_+(\varphi)$.

Denote by $\mathbf{1}_{\mathbb{R}_-}$ the indicator of the subset $\mathbb{R}_-$ in $\mathbb{R}$: $\mathbf{1}_{\mathbb{R}_-}(x) = 1$ for $x \in \mathbb{R}_-$ and $\mathbf{1}_{\mathbb{R}_-}(x) = 0$ for $x \in \mathbb{R}_+$. A similar meaning has the notation $\mathbf{1}_{\mathbb{R}_+}$. Let ν be a measure defined on $\mathscr{B}$, and $a(x)$, $x \in \mathbb{R}$, a function. Define the convolution $\nu * a(x)$ as the function $\int_{\mathbb{R}} a(x - y)\, \nu(dy)$, $x \in \mathbb{R}$.

Let ν be a finite complex-valued measure. Denote by $T\nu$ the σ-finite measure with density $\nu((x, \infty))$ for $x \ge 0$ and $-\nu((-\infty, x])$ for $x < 0$. If $\int_{\mathbb{R}} |x|\, |\nu|(dx) < \infty$, then $T\nu$ is a finite measure whose Laplace transform is given by $\widehat{T\nu}(s) = (\widehat{\nu}(s) - \widehat{\nu(0)})/s$, $\Re s = 0$, the value $\widehat{T\nu}(0)$ being defined by continuity as $\int_{\mathbb{R}} x\, \nu(dx) \in \mathbb{C}$.

3. Lemmas

Lemma 1. *Let F be a nonarithmetic probability distribution with finite positive mean $\mu = \int_{\mathbb{R}} x\, F(dx)$ and let $\varphi(x)$, $x \in \mathbb{R}_+$, be a nondecreasing continuous submultiplicative function such that $r_+(\varphi) = 0$. Assume that condition* (3) *is fulfilled. Then*

$$\int_{-\infty}^{0} \varphi(|x|)\, U(dx) < \infty.$$

Proof. Extend the function φ onto the whole line by setting $\varphi(x) := \varphi(|x|)$, $x \in \mathbb{R}_-$. The extended function retains the submultiplicative property. Consider the auxiliary submultiplicative function

$$\psi(x) = \begin{cases} (1+|x|)\varphi(x) & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$

We have $r_-(\psi) = r_+(\psi) = 0$. Condition (3) together with finite μ implies $TF \in S(\psi)$. Suppose first that F has a nonzero absolutely continuous component. Let L be the restriction of Lebesgue measure to $\mathbb{R}_+$. By [4, Theorem 3.1], $U = U_1 + U_2$, where $U_2 \in S(\psi)$ and $U_1 = L/\mu + rTU_2$ for some $r > 0$. Hence

$$U\big|_{\mathbb{R}_-} = U_2\big|_{\mathbb{R}_-} + rTU_2\big|_{\mathbb{R}_-}.$$

Obviously, $U_2\big|_{\mathbb{R}_-} \in S(\psi)$. Therefore,

$$\big\|U_2\big|_{\mathbb{R}_-}\big\|_{\psi} = \int_{-\infty}^{0} (1+|x|)\varphi(x)\, |U_2|(dx) < \infty.$$

By [5, Theorem 3],

$$\int_{-\infty}^{0} \varphi(x)\, |TU_2|(dx) < \infty,$$

i.e., $TU_2\big|_{\mathbb{R}_-} \in S(\varphi)$, and hence $U\big|_{\mathbb{R}_-} \in S(\varphi)$. This proves the lemma under the additional assumption that F has a nonzero absolutely continuous component. Consider now the general case. Denote by $\mathscr{U}$ the uniform distribution in the interval $[-h, 0)$, $h > 0$. Let X and Y be independent random variables with distributions F and $\mathscr{U}$. Denote by G the distribution of the random variable $X + Y$. Let $\mathsf{F}(x)$ be the distribution function of X: $\mathsf{F}(x) := \mathsf{P}(X \leq x) = F((-\infty, x])$, and, similarly, set $\mathsf{G}(x) := \mathsf{P}(X + Y \leq x) = G((-\infty, x])$. Then $\mathsf{F}(x) \leq \mathsf{G}(x)$. Choosing h sufficiently small, we can achieve that

$$\mu_G := \mathsf{E}(X+Y) = \mu - h/2 > 0.$$

Let U_G be the renewal measure corresponding to the distribution G. By induction, $\mathsf{F}^{n*}(x) \leq \mathsf{G}^{n*}(x)$ for all $n \geq 1$ and hence

$$\mathsf{U}(x) := U((-\infty, x]) \leq U_G((-\infty, x]) = \sum_{n=0}^{\infty} \mathsf{G}^{n*}(x) =: \mathsf{U}_G(x).$$

The distribution G satisfies the hypotheses of the lemma. Indeed,

$$I_G = \int_{-\infty}^{0} |x|\varphi(x)\mathsf{G}(x)\, dx \leq \int_{-\infty}^{0} |x|\varphi(x)\mathsf{F}(x+h)\, dx = \int_{-\infty}^{h} |u-h|\varphi(u-h)\mathsf{F}(u)\, du.$$

Performing simple calculations, we get $I_G \le (1+h)I_F + h\varphi^2(h) + h\varphi(h) < \infty$. Moreover, the distribution G is absolutely continuous. Since the function $\varphi(x)$ is nonincreasing on $\mathbb{R}_-$, we have

$$\begin{aligned}\infty > \int_{-\infty}^{0} \varphi(x)\, U_G(dx) &= \varphi(x)\mathsf{U}_G(x)\big|_{-\infty}^{0} - \int_{-\infty}^{0} \mathsf{U}_G(x)\, d\varphi(x) \\ &= \mathsf{U}_G(0) - \int_{-\infty}^{0} \mathsf{U}_G(x)\, d\varphi(x) \ge \mathsf{U}(0) - \int_{-\infty}^{0} \mathsf{U}(x)\, d\varphi(x) = \int_{-\infty}^{0} \varphi(x)\, U(dx).\end{aligned}$$

The proof of the lemma is complete. □

Lemma 2. *Under the hypotheses of Lemma 1,*

$$\mathsf{U}(x) \sim \frac{x}{\mu} \qquad \text{as } x \to \infty.$$

Proof. We have $\mathsf{U}(x) = \mathsf{U}_1(x) + \mathsf{U}_2(x)$. Since $U_2 \in S(\varphi)$, U_2 is a finite measure and $\mathsf{U}_2(x) \to U_2(\mathbb{R}) \in \mathbb{R}$ as $x \to \infty$. Obviously, $\mathsf{U}_1(x) = x/\mu + rTU_2((-\infty, x])$, $x > 0$. It remains to show that $TU_2((-\infty, x]) = o(x)$ as $x \to \infty$. By the definition of T,

$$|TU_2([0,x])| \le \int_0^x |U_2|((y,\infty))\, dy = o(x) \qquad \text{as } x \to \infty.$$

since $|U_2|((y,\infty)) \downarrow 0$ as $y \uparrow \infty$. By Lemma 1, $rTU_2(\mathbb{R}_-) = U(\mathbb{R}_-) - U_2(\mathbb{R}_-)$ is finite. The proof of the lemma is complete. □

Lemma 3. *Let $a(x)$, $x \in \mathbb{R}_+$, be a monotone nondecreasing positive function such that $\lim_{x\to\infty} a(x+y)/a(x) = 1$ for each $y \in \mathbb{R}$. Then*

$$a(x) = o\left(\int_0^x a(y)\, dy\right) \qquad \text{as } x \to \infty.$$

Proof. Let $M > 0$ be arbitrary. We have

$$\int_0^x \frac{a(y)}{a(x)}\, dy \ge \int_{x-M}^{x} \frac{a(y)}{a(x)}\, dy \ge \int_{x-M}^{x} \frac{a(x-M)}{a(x)}\, dy = M\frac{a(x-M)}{a(x)}.$$

It follows that $\liminf_{x\to\infty} \int_0^x a(y)\, dy/a(x) = \infty$. The proof of the lemma is complete. □

Lemma 4. *Let F be a nonarithmetic probability distribution with $\mu \in (0,\infty]$ and let $\varphi(x)$, $x \in \mathbb{R}_+$, be a nondecreasing submultiplicative function such that $r_+(\varphi) > 0$, $\lim_{x\to\infty} \varphi(x+y)/\varphi(x) = \exp(r_+(\varphi)y)$ for each $y \in \mathbb{R}$ and $\varphi(x)/\exp(r_+(\varphi)x)$ is nondecreasing on $\mathbb{R}_+$. Suppose that $\int_{-\infty}^{0} \varphi(|x|)\, F(dx) < \infty$ and $\widehat{F}(-r_+(\varphi)) < 1$. Then*

$$\int_{-\infty}^{0} \varphi(x)\, U(dx) < \infty.$$

Proof. Put $\varphi(x) := \varphi(|x|)$ for $x \in \mathbb{R}_-$. Consider the auxiliary submultiplicative function

$$\psi(x) = \begin{cases} \varphi(x) & \text{for } x < 0, \\ \exp(-r_+(\varphi)x) & \text{for } x \ge 0. \end{cases}$$

We have $r_\pm(\psi) = -r_+(\varphi)$. It suffices to show that $U \in S(\psi)$. Obviously, $F \in S(\psi)$. Prove that the element $\nu := \delta_0 - F$ is invertible in $S(\psi)$. Let $\nu = \nu_{\mathfrak{c}} + \nu_{\mathfrak{d}} + \nu_{\mathfrak{s}}$ be the decomposition of ν into absolutely continuous, discrete and singular components.

Denote $\Pi := \{s \in \mathbb{C} : \Re s = -r_+(\varphi)\}$. By [6, Theorem 1], the element $\nu \in S(\psi)$ has an inverse if $\widehat{\nu}(s) \neq 0$ for all $s \in \Pi$ and if

$$\inf_{s\in\Pi} \left|\widehat{\nu_{\mathfrak{d}}}(s)\right| > \widehat{|\nu_{\mathfrak{s}}|}(-r_+(\varphi)). \tag{4}$$

Let $F = F^{\mathfrak{c}} + F^{\mathfrak{d}} + F^{\mathfrak{s}}$ be the decomposition of $F \in S(\psi)$ into absolutely continuous, discrete and singular components. Then $\nu^{\mathfrak{d}} = \delta_0 - F^{\mathfrak{d}}$ and $\nu^{\mathfrak{s}} = -F^{\mathfrak{s}}$. We have

$$\inf_{s\in\Pi} \left|\widehat{\nu_{\mathfrak{d}}}(s)\right| \geq 1 - \sup_{s\in\Pi}\left|\widehat{F^{\mathfrak{d}}}(s)\right| = 1 - \widehat{F^{\mathfrak{d}}}(-r_+(\varphi)).$$

On the other hand, $\widehat{|\nu_{\mathfrak{s}}|}(-r_+(\varphi)) = \widehat{F^{\mathfrak{s}}}(-r_+(\varphi))$. By assumption, $\widehat{F}(-r_+(\varphi)) < 1$. Hence

$$1 - \widehat{F^{\mathfrak{d}}}(-r_+(\varphi)) - \widehat{F^{\mathfrak{s}}}(-r_+(\varphi)) \geq 1 - \widehat{F}(-r_+(\varphi)) > 0,$$

and (4) follows. Therefore, by Theorem 1 in [6], the measure $\delta_0 - F$ is invertible in the Banach algebra $S(\psi)$ and $U = (\delta_0 - F)^{-1} \in S(\psi)$. The proof of the lemma is complete. □

4. Proofs

Proof of Theorem 1. Extend the function $\varphi(x)$ onto the whole line $\mathbb{R}$ by setting $\varphi(x) = \varphi(|x|)$ for $x \in \mathbb{R}_-$. The extended function retains the submultiplicative property and $r_\pm(\varphi) = 0$. First, let us prove the theorem in the specific case when $g(x) = \varphi(x)$ for $x \in \mathbb{R}_+$. Denote the corresponding solution to (1) by z_φ. Let $x \in \mathbb{R}_+$. We have

$$z_\varphi(x) = \int_{-\infty}^0 \varphi(x-y)\,U(dy) + \int_0^x \varphi(x-y)\,U(dy) + \int_x^\infty g(x-y)\,U(dy)$$
$$=: I_1(x) + I_2(x) + I_3(x).$$

Put $M(x) = \int_0^x \varphi(y)\,dy$. By Lemma 1 and Lemma 3 with $a(x) = \varphi(x)$,

$$I_1(x) \leq \varphi(x) \int_{-\infty}^0 \varphi(y)\,U(dy) = o(M(x)) \qquad \text{as } x \to \infty. \tag{5}$$

Next, let us establish that

$$I_2(x) \sim \frac{1}{\mu} M(x) \qquad \text{as } x \to \infty. \tag{6}$$

Integrating by parts (see [7, Chapter 6, Theorem 6.30]), we get

$$I_2(x) = U(x) - \varphi(x)U(0) - \int_0^x U(y)\,d_y\varphi(x-y).$$

The following three estimates hold:

$$\varphi(x),\ x,\ U(x) = o(M(x)) \qquad \text{as } x \to \infty. \tag{7}$$

The first estimate follows from Lemma 3. The second one follows from the assumption $\varphi(y) \to \infty$ as $y \to \infty$. The third estimate is a consequence of Lemma 2. Show that

$$-\int_0^x U(y)\,d_y\varphi(x-y) \sim -\frac{1}{\mu}\int_0^x y\,d_y\varphi(x-y) \sim \frac{1}{\mu}M(x) \qquad \text{as } x \to \infty. \tag{8}$$

The last equivalence follows from the second estimate in (7) and the equality

$$-\int_0^x y\,d_y\varphi(x-y) = -y\varphi(x-y)\big|_{y=0}^x + \int_0^x \varphi(x-y)\,dy = -x\varphi(0) + M(x).$$

Let $\varepsilon > 0$ be arbitrary. Use Lemma 2 and choose $y_0 = y_0(\varepsilon)$ such that

$$(1-\varepsilon)\frac{y}{\mu} \le U(y) \le (1+\varepsilon)\frac{y}{\mu}, \qquad y \ge y_0.$$

Write the left-hand side of (8) in the form

$$-\Big(\int_0^{y_0} + \int_{y_0}^x\Big) U(y)\,d_y\varphi(x-y) =: K_1(x) + K_2(x).$$

Put $N(x) := -\int_{y_0}^x y\,d_y\varphi(x-y)/\mu$. Obviously,

$$(1-\varepsilon)N(x) \le K_2(x) \le (1+\varepsilon)N(x). \tag{9}$$

Integrating by parts, we obtain

$$N(x) = -\frac{x}{\mu} + \frac{y_0}{\mu}\varphi(x-y_0) + \frac{1}{\mu}\int_0^{x-y_0} \varphi(y)\,dy.$$

It follows from (7) that

$$\int_{x-y_0}^x \varphi(y)\,dy \le \varphi(x)\varphi(y_0)y_0 = o(M(x)) \qquad \text{as } x \to \infty.$$

Hence $N(x) \sim M(x)/\mu$ as $x \to \infty$. Divide all parts of (9) by $N(x)$ and let x tend to ∞. We obtain

$$1 - \varepsilon \le \liminf_{x\to\infty} \frac{K_2(x)}{N(x)} \le \limsup_{x\to\infty} \frac{K_2(x)}{N(x)} \le 1 + \varepsilon.$$

It follows that $K_2(x) \sim M(x)/\mu$ as $x \to \infty$. Relation (8) is proven since

$$K_1(x) \le -U([0,y_0]) \int_0^{y_0} d_y\varphi(x-y) \le U([0,y_0])\varphi(x) = o(M(x)) \qquad \text{as } x \to \infty.$$

In view of (7), relation (6) has been established. By the key renewal theorem for directly Riemann integrable functions (see, e.g., [3, Theorem 2.5.3].),

$$I_3(x) \to \frac{1}{\mu}\int_{-\infty}^0 g(y)\,dy = o(M(x)) \qquad \text{as } x \to \infty. \tag{10}$$

Relations (5), (6) and (10) prove the theorem for z_φ.

Let g satisfy the hypotheses of the theorem. If, for some $C > 0$, $|g(x)| \le C\varphi(x)$, $x \in \mathbb{R}_+$, then

$$\limsup_{x\to\infty} \frac{|z(x)|}{M(x)} \le \frac{C}{\mu}.$$

If $c = 0$, then $z(x) = o(z_\varphi(x))$ as $x \to \infty$. To see this, choose a small $\varepsilon > 0$ and a number $n \in \mathbb{R}_+$ such that $|g(x)| \le \varepsilon\varphi(x)$, $x \ge n$. Denote by $\mathbf{1}_{[0,n]}$ the indicator of $[0,n]$. By hypothesis, there exists a constant $C > 0$ such that $|g(x)| \le C$ for $x \in [0,n]$. Write

$$g = \mathbf{1}_{[0,n]}g + (g - \mathbf{1}_{[0,n]}g) =: g_1 + g_2.$$

Let z_1 and z_2 be the solutions of (1) corresponding to g_1 and g_2, respectively. Then $z = z_1 + z_2$ and $|g_2(x)| \le \varepsilon\varphi(x)$, $x \in \mathbb{R}_+$. We have

$$z_1(x) = U * (\mathbf{1}_{[0,n]}g)(x) = \int_{x-n}^x g(x-y)\,U(dy).$$

By [1, Section XI.9, Theorem 1],

$$|z_1(x)| \le CU([x-n,x]) \sim Cn/\mu = o(M(x)) \qquad \text{as } x \to \infty.$$

Therefore,

$$\limsup_{x\to\infty} \frac{|z(x)|}{M(x)} \le \frac{\varepsilon}{\mu}.$$

Since $\varepsilon > 0$ is arbitrary, the assertion of the theorem is true for $c = 0$. Let $c \ne 0$. Write g in the form $g = c\mathbf{1}_{\mathbb{R}_+}\varphi + g_1$. Then $g_1(x) = o(\varphi(x))$ and $z_1(x) = o(M(x))$ as $x \to \infty$, whence $z(x) \sim cM(x)/\mu$ as 1 $x \to \infty$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. We begin with the solution z_φ corresponding to the case $g(x) = \varphi(x)$ on $\mathbb{R}_+$ and prove that

$$\text{(11)} \qquad \frac{z_\varphi(x)}{\varphi(x)} \to \int_{\mathbb{R}} \exp(-r_+(\varphi)y)\, U(dy) = \frac{1}{1-\widehat{F}(-r_+(\varphi))} \qquad \text{as} \quad x \to \infty.$$

We use the notation of the preceding proof. By Lebesgue's bounded convergence theorem [8, § 26, Theorem D],

$$\text{(12)} \qquad \frac{I_1(x)}{\varphi(x)} = \int_{-\infty}^{0} \frac{\varphi(x-y)}{\varphi(x)}\, U(dy) \to \int_{-\infty}^{0} e^{-r_+(\varphi)y}\, U(dy) \qquad \text{as } x \to \infty$$

since the integrand tends to $e^{-r_+(\varphi)y}$ as $x \to \infty$ by assumption and, according to Lemma 1, it is majorized by the U-integrable function $\varphi(y)$, $y \in \mathbb{R}_-$. As before, $I_3(x)$ tends to a finite limit and, therefore,

$$\text{(13)} \qquad I_3(x) = o(\varphi(x)) \qquad \text{as } x \to \infty.$$

Let us prove that

$$\text{(14)} \quad \frac{I_2(x)}{\varphi(x)} = \int_0^\infty \mathbf{1}_{[0.x]}(y) \frac{\varphi(x-y)}{\varphi(x)}\, U(dy) \to \int_0^\infty e^{-r_+(\varphi)y}\, U(dy) \qquad \text{as } x \to \infty.$$

The integrand $\mathbf{1}_{[0.x]}(y)\varphi(x-y)/\varphi(x)$ tends to $e^{-r_+(\varphi)y}$ as $x \to \infty$. Choose a majorant for the integrand in the form $Me^{\beta y}$ with $\beta \in (-r_+(\varphi), 0)$. Then, by Lebesgue's theorem, we may pass to the limit under the integral sign in the left-side integral in (14) as $x \to \infty$ and thus prove relation (14). To this end, put $f(x) = \log \varphi(x) - r_+(\varphi)x$. By hypothesis, we have

$$\text{(15)} \qquad f(x-y) - f(x) = \log\varphi(x-y) - \log\varphi(x) + r_+(\varphi)x \to 0 \qquad \text{as } x \to \infty$$

for each $y \in \mathbb{R}$. According to Lemma 1.1 in [9], relation (15) is fulfilled uniformly in $y \in [0,1]$. Hence

$$\frac{\varphi(x-y)\exp(r_+(\varphi)y)}{\varphi(x)} \to 1 \qquad \text{as } x \to \infty$$

uniformly in $y \in [0,1]$. Choose a small $\varepsilon > 0$ such that $\beta := \log(1+\varepsilon) - r_+(\varphi) < 0$. Let $N > 0$ be an integer such that

$$\frac{\varphi(x-y)\exp(r_+(\varphi)y)}{\varphi(x)} \le 1 + \varepsilon, \qquad x \ge N, \quad y \in [0,1].$$

Denote by $[x]$ the integral part of a real number x, i.e., $[x]$ is the maximal integer not exceeding x: $x = [x] + \vartheta$, $\vartheta \in [0, 1)$. For $y \in [l, l+1]$, $l = 0, \dots, [x] - N - 1$, we have

$$\frac{\varphi(x-y)}{\varphi(x)} = \frac{\varphi(x-l-(y-l))}{\varphi(x-l)} \frac{\varphi(x-l)}{\varphi(x)},$$

$$\frac{\varphi(x-l-(y-l))}{\varphi(x-l)} \le (1+\varepsilon) \exp(-r_+(\varphi)(y-l)),$$

$$\frac{\varphi(x-l)}{\varphi(x)} = \frac{\varphi(x-l)}{\varphi(x-l+1)} \frac{\varphi(x-l+1)}{\varphi(x-l+2)} \cdots \frac{\varphi(x-1)}{\varphi(x)} \le (1+\varepsilon)^l \exp(-lr_+(\varphi)).$$

Hence

$$\frac{\varphi(x-y)}{\varphi(x)} \le (1+\varepsilon)^{l+1} \exp(-r_+(\varphi)(y-l)) \exp(-lr_+(\varphi))$$
$$= (1+\varepsilon)^{l+1} \exp(-r_+(\varphi) y) \le (1+\varepsilon) \exp(\beta y).$$

Let $y \in ([x] - N - 1, x]$. We have

$$\frac{\varphi(x-y)}{\varphi(x)} \le \frac{\varphi(N+2)}{\varphi(x)} \le \frac{\varphi(N+2)}{\exp(r_+(\varphi)x)} \le \frac{\varphi(N+2)}{\exp(r_+(\varphi)y)} \le \varphi(N+2) \exp(\beta y).$$

Thus, the U-integrable majorant sought, which does not depend on x, is of the form

$$\max\{(1+\varepsilon), \varphi(N+2)\} \exp(\beta y), \qquad y \in \mathbb{R}_+,$$

and this completes the proof of (14). Relation (11) now follows from (12)–(14). The last equality in (11) is a consequence of $\widehat{F}(-r_+(\varphi)) < 1$.

In the general case it suffices to repeat the concluding reasoning of the previous proof using the estimate

$$\limsup_{x \to \infty} \frac{|z(x)|}{\varphi(x)} \le \frac{C}{1 - \widehat{F}(-r_+(\varphi))}$$

for $|g(x)| \le C\varphi(x)$, $x \in \mathbb{R}_+$, and, considering the case $c = 0$, take into account the relation $z_1(x) = o(\varphi(x))$ as $x \to \infty$. The proof of Theorem 2 is complete. □

Remark 1. Under the hypotheses of Theorems 1 and 2, $\lim_{x \to -\infty} z(x)/\varphi(x) = 0$. Indeed,

$$z(x) = \left(\int_{-\infty}^{x} + \int_{x}^{\infty} \right) g(x-y)\, U(dy) =: J_1(x) + J_2(x).$$

There exists $C > 0$ with $|g(x)| \le C\varphi(x)$, $x \in \mathbb{R}_+$. By Lemma 1,

$$|J_1(x)| \le C \int_{-\infty}^{x} \varphi(x-y)\, U(dy) \le C\varphi(x) \int_{-\infty}^{x} \varphi(y)\, U(dy) = o(\varphi(x)) \quad \text{as } x \to -\infty.$$

Since $\mathbf{1}_{\mathbb{R}_-} g$ is directly Riemann integrable,

$$J_2(x) = U * (\mathbf{1}_{\mathbb{R}_-} g)(x) \to 0 \qquad \text{as} \quad x \to -\infty;$$

see, e.g., [3, Theorem 2.5.3].

Remark 2. Theorem 1 remains true even if $\mu = +\infty$:

$$z(x) = o\left(\int_0^x \varphi(y)\, dy \right) \qquad \text{as } x \to \infty.$$

Remark 3. The case of arithmetic F has been considered in [10].

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