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ON THE COLLECTION PROCESS FOR POSITIVE WORDS

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Abstract. We present an approach to studying the divisibility of the exponents of the commutators that arise in collection formulas obtained for positive words of a free group. It deals with logical formulas that establish a connection between the exponents of the commutators and the structure of the positive word to which the collection process is applied. Using our approach, we obtain several generalizations of known collection formulas with some divisibility properties of the exponents.

Keywords: commutator, collection formula, free group, divisibility.

1. Introduction

In [3], P. Hall introduced the concept of a *collection process*, which can be briefly described as follows. Let W be a positive word on the generators $a_1, \dots, a_n$, $n \geqslant 2$, of the free group $F = F(a_1, \dots, a_n)$, i.e. the word W does not contain inverses of the generators. By rearranging step by step consecutive occurrences of elements in W with use of commutators: $QR = RQ[Q, R]$, $Q, R \in F$, the collection process transforms W into the following form:

$$W = q_1^{e_1} \dots q_j^{e_j} T_j, \quad j \geqslant 1, \tag{1}$$

where $q_1, \dots, q_j$ are commutators in $a_1, \dots, a_n$ arranged in order of increasing weights, T_j consists of commutators of weights not less than $w(q_j)$ (the weight of q_j), the exponents $e_1, \dots, e_j$ are positive integers.

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P. Hall applied the collection process to the word $(a_1a_2)^m$, $m \geqslant 1$, and obtained the following collection formula in [3, Theorems 3.1 and 3.2]:

$$(a_1a_2)^m = q_1^{e_1} \dots q_{j(s)}^{e_{j(s)}} \pmod{\Gamma_s(F)}, \quad s \geqslant 2, \tag{2}$$

where $q_1, \dots, q_{j(s)}$ are commutators in a_1, a_2 of weights less than s and $\Gamma_s(F)$ is the s-th term of the lower central series of F, which is defined as follows: $\Gamma_1(F) = F$, $\Gamma_k(F) = [\Gamma_{k-1}(F), F]$, $k \geqslant 2$. It was proved that the exponents of the commutators can be expressed as integer-valued polynomials in m vanishing for $m = 0$:

$$e_i(m) = \sum_{k=1}^{w(q_i)} c_k \binom{m}{k}, \tag{3}$$

where non-negative integers c_k do not depend on m. This result is significant for the theory of p-groups, since the expression $e_i(p^\alpha)$ is divisible by the prime power p^α if $w(q_i) < p$.

Research has been carried out in different directions. On the one hand, an explicit form for some series of the exponents $e_i(m)$ has been found [5], [8], [4]. On the other hand, collection formulas for different words of the free group with the same divisibility property of exponents have been obtained.

In [2, Theorem 12.3.1], M. Hall applied the collection process to the word $(a_1 \dots a_n)^m$, $n \geqslant 1$, and using P. Hall's approach, obtained exactly the same result for the exponents.

In [7, Theorems 5.13A and 5.13B], an arbitrary word W (not necessarily positive) of the group F was considered. By making use of Lie algebras, it was proved that W^m, $m \geqslant 1$, can be expressed as follows:

$$W^m = q_1^{e_1} \dots q_{j(s)}^{e_{j(s)}} \pmod{\Gamma_s(F)}, \quad s \geqslant 2, \tag{4}$$

where $q_1, \dots, q_{j(s)}$ are commutators in $a_1, \dots, a_n$ of weights less than s and e_i is divisible by m if m is a prime power number p^α and $w(q_i) < p$.

R. R. Struik dealt with the word $a_1^{m_1}a_2^{m_2}$, where $m_1, m_2 \geqslant 1$, in [9, Lemma 4]. Using a modification of P. Hall's approach, she proved the following collection formula:

$$a_1^{m_1}a_2^{m_2} = q_1^{e_1} q_2^{e_2} q_3^{e_3} \dots q_{j(s)}^{e_{j(s)}} \pmod{\Gamma_s(F)}, \quad s \geqslant 2, \tag{5}$$

where $q_1 = a_2$, $q_2 = a_1$ and

$$e_i = \sum_{k=1}^{w_{a_1}(q_i)} \sum_{t=1}^{w_{a_2}(q_i)} c_{k,t} \binom{m_1}{k} \binom{m_2}{t}, \quad c_{k,t} \in \mathbb{N}_0.$$

Here $w_{a_l}(q_i)$ is the weight of q_i in a_l, $l = 1, 2$. If m_l, $l = 1, 2$, is a prime power number p^α and $1 \leqslant w_{a_l}(q_i) < p$, then e_i is divisible by m_l.

H. W. Waldinger obtained formula (5) in [10] by methods of [7] for $m_1 = 1$, $m_2 = p$, $s = p + 2$, where p is prime. A. M. Gaglione proved (5) by means of the Magnus Algebra [7] for $m_1 = 1$, $m_2 = p^\alpha$, $\alpha \geqslant 1$, $s = p^2 + 1$ in [1].

In this paper we present an approach to studying the divisibility of the exponents of the commutators that arise in collection formulas obtained for positive words of a free group. Our approach is based on P. Hall's idea that he used in the proof of formula (2). Let us very briefly describe three basic steps of that proof.

Labeling process. All occurrences of a_1 and a_2 in $(a_1a_2)^m$ are assigned labels as follows: $a_1(1)a_2(1)a_1(2)a_2(2)\dots a_1(m)a_2(m)$. During the collection process, new occurrences of commutators are assigned labels by the following rule:

$$Q(\lambda_1,\dots,\lambda_i)R(\mu_1,\dots,\mu_j) = R(\mu_1,\dots,\mu_j)Q(\lambda_1,\dots,\lambda_i)[Q,R](\lambda_1,\dots,\lambda_i,\mu_1,\dots,\mu_j).$$

Thus, the exponent e_j is equal to the number of sequences $(\lambda_1,\dots,\lambda_{w(q_j)}) \in \{1,\dots,m\}^{w(q_j)}$ that are labels of q_j's occurrences.

Existence and precedence conditions. Let E_{q_j} be a condition that is satisfied by the sequence $(\lambda_1,\dots,\lambda_{w(q_j)}) \in \{1,\dots,m\}^{w(q_j)}$ iff the occurrence $q_j(\lambda_1,\dots,\lambda_{w(q_j)})$ has arisen during the collection process. Then E_{q_j} can be obtained by disjunction and conjunction of such conditions as $\lambda_i = \lambda_j$, $\lambda_i < \lambda_j$.

Combinatorial result. Suppose C is an arbitrary condition obtained by disjunction and conjunction of such conditions as $\lambda_i = \lambda_j$, $\lambda_i < \lambda_j$. Then the number of elements $(\lambda_1,\dots,\lambda_r) \in \{1,\dots,m\}^r$, $r,m \in \mathbb{N}$, that satisfy the condition C can be expressed as $c_1\binom{m}{1} + \dots + c_r\binom{m}{r}$, where non-negative integers c_k do not depend on m.

We will generalize this idea to obtain an approach applicable to various positive words of the free group.

In Section 2 we introduce in a formal manner basic notation of the collection process and give some necessary properties. Note that we define the collection process without restrictions on the arranging of the commutators. So, the commutators can be collected not only in order of increasing weights.

In Section 3 existence and precedence conditions are defined. Using a combinatorial interpretation of the collection process in terms of binary representations of integers (Theorems 4.1, 4.4), we provide a system of recurrence relations for the existence and precedence conditions (Theorem 4.6) in Section 4. That system establishes a connection between the exponent of a commutator and the structure of the initial positive word.

In Section 5 we give a generalization of the P. Hall's combinatorial result, which makes it possible to deal with numerous positive words of the free group.

Finally, in Section 6 we discuss the details of our approach and use it to obtain several series of collection formulas with some divisibility properties of the exponents. Theorem 6.9 extends R.R. Struik's result to the product of several letters. Theorems 6.1 and 6.6 deal with the words as in the M. Hall's formula and in the formula from [7] (when W is a positive word), respectively, where some occurrences of the letters may be deleted. Moreover, the commutators can be collected in an arbitrary oreder as we have mentioned above.

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2. Basic notation

Definition 2.1. *For the letters $a_1, \dots, a_n$, $n \geqslant 2$, we define the set of formal commutators $\Gamma(a_1, \dots, a_n)$ and the weight w by induction:*

(1) $\{a_1, \dots, a_n\} \subset \Gamma(a_1, \dots, a_n)$, $w(a_1) = \dots = w(a_n) = 1$;
(2) *If $c_1, c_2 \in \Gamma(a_1, \dots, a_n)$, then $[c_1, c_2] \in \Gamma(a_1, \dots, a_n)$ and $w([c_1, c_2]) = w(c_1) + w(c_2)$.*

We will usually suppress the adjective 'formal' and so call a formal commutator simply a commutator.

Definition 2.2. *A finite sequence of commutators from $\Gamma(a_1, \dots, a_n)$ is called a commutator word. We say that two commutator words $c_1 \dots c_m$ and $d_1 \dots d_k$ are equal and write $c_1 \dots c_m \equiv d_1 \dots d_k$ iff they have the same length $m = k$ and $c_i = d_i$, for $i \in \overline{1, m}$.*

Example 2.3. *The commutator words of length 1,2, and 3, respectively:*

$$[a_1, a_2], \qquad a_3[a_1, a_2], \qquad [a_3, [a_1, a_2]]a_3[a_1, a_2].$$

When we work with collection process, it is convenient to deal with commutator words (sequences of formal symbols) instead of products of group elements. After the collection process is finished, we consider commutator words as elements of the free group $F(a_1, \dots, a_n)$ by putting $[y, x] = y^{-1}x^{-1}yx$ for any $x, y \in F(a_1, \dots, a_n)$, i.e. we go from formal commutators to group-theoretical ones.

If a commutator word contains several occurrences of the same commutator, then we should clearly distinguish these occurrences from each other. For this purpose, it would be natural to use the following tool.

Definition 2.4. *Suppose that a commutator word X of length $m \geqslant 1$ contains an occurrence of a commutator R. A certain finite sequence of integers assigned to that occurrence is called the label of the occurrence of R. If any two occurrences of the same commutator in X have pairwise different labels of the same length, then the map that takes $i \in \overline{1, m}$ to the label of the occurrence of a commutator at the i-th position is called the* labeling *of X.*

If an occurrence of a commutator R has the label Λ and we need to point out this fact, then we denote that occurrence by $R(\Lambda)$. Obviously, any commutator word $t_1 \dots t_m$ can be labeled, for example: $t_1(1) \dots t_m(m)$.

Further on, we use multiplicative notation for the operation of concatenation of finite integer sequences. For example, if $\Lambda_1 = (1, 1, 2)$, $\Lambda_2 = (1, 2)$, then $\Lambda_1\Lambda_2 = (1, 1, 2, 1, 2)$.

Definition 2.5. *Suppose $X \equiv t_1(\Lambda_1) \dots t_m(\Lambda_m)$ is a commutator word with some labeling, and there exist $e \geqslant 1$ occurrences of the commutator q in X. A stage of the collection process applied to X is a transformation of X to the commutator word Y by the following algorithm. Let $t_{i_1}(\Lambda_{i_1}), \dots, t_{i_e}(\Lambda_{i_e})$ be all occurrences of q in X, and $1 \leqslant i_1 < \dots < i_e \leqslant n$. We move $t_{i_1}(\Lambda_{i_1})$ to the beginning of the word rearranging step by step consecutive occurrences of commutators by the rule*

$$y(\Lambda_u)x(\Lambda_v) = x(\Lambda_v)y(\Lambda_u)[y, x](\Lambda_u\Lambda_v).$$

Then we move $t_{i_2}(\Lambda_{i_2})$ *immediately to the right of* $t_{i_1}(\Lambda_{i_1})$ *by the same way. Continuing this line of reasoning, we get the commutator word*

$$Y \equiv t_{i_1}(\Lambda_{i_1}) \ldots t_{i_e}(\Lambda_{i_e}) \prod_{i=1}^{m'} t'_i \equiv q^e T,$$

where T *consists of all commutators that have occurrences in* X *except* q*, as well as of commutators arising from the collection of occurrences of* q*. All occurrences of commutators in* T *are labeled.*

Definition 2.6. *Let* W_0 *be a labeled commutator word consisting only of commutators of weight 1. The collection process applied to* W_0 *is a construction of the sequence of commutator words*

$$\{W_j \equiv q_1^{e_1} \ldots q_j^{e_j} T_j\}_{j \geqslant 0} \tag{6}$$

by the following rule. The word W_j*,* $j \geqslant 1$*, obtained by the* j*-th stage of the collection process applied to the commutator word* T_{j-1}*. At that stage an arbitrary commutator* q_j *in* T_{j-1} *has been collected. The words* $q_1^{e_1} \ldots q_j^{e_j}$ *and* T_j *are called, respectively,* collected part *and* uncollected part *of* W_j*. The initial word* $W_0 \equiv T_0$ *with empty collected part is considered as the result of zero stage of the collection process.*

We will use the following notation for commutators:

$$[y, {}_0x] = y; \qquad [y, {}_ix] = [[y, {}_{i-1}x], x], \quad i \geqslant 1; \qquad [y, x, z] = [[y, x], z].$$

Example 2.7. *Let us consider the commutator word* $W_0 \equiv a_1(1)a_2(1)a_1(2)a_2(2)$*. Here is a possible variant of the collection process applied to* W_0 *(for clarity, the uncollected parts of the words are separated by a dot):*

$$\begin{aligned}
&W_0 \equiv a_1(1)a_2(1)a_1(2)a_2(2),\\
&W_1 \equiv a_1(1)a_1(2) \cdot a_2(1)[a_2, a_1](1,2)a_2(2),\\
&W_2 \equiv a_1(1)a_1(2)a_2(1)a_2(2) \cdot [a_2, a_1](1,2)[a_2, a_1, a_2](1,2,2),\\
&W_3 \equiv a_1(1)a_1(2)a_2(1)a_2(2)[a_2, a_1](1,2) \cdot [a_2, a_1, a_2](1,2,2),\\
&W_4 \equiv a_1(1)a_1(2)a_2(1)a_2(2)[a_2, a_1](1,2)[a_2, a_1, a_2](1,2,2).
\end{aligned}$$

By collecting the commutators in the following order: a_1*,* a_2*,* $[a_2, a_1]$*,* $[a_2, a_1, a_2]$*, we get the commutator word* $W_4 \equiv a_1^2 a_2^2 [a_2, a_1][a_2, a_1, a_2]$ *with empty uncollected part.*

Note that the sequence (6) can be finite in two cases: we decided to stop the collection process at some stage, or the collection process terminated at some word W_j, $j \geqslant 1$, the uncollected part of which turned out to be empty (see the previous example).

Any commutator from $\Gamma(a_1, \ldots, a_n)$ can be written in various ways, for example,

$$\Big[[[a_2, a_1], a_1], a_1\Big] = [a_2, {}_3a_1] = \big[[a_2, a_1], {}_2a_1\big] = \big[[a_2, {}_3a_1], {}_0a_3\big].$$

From now on we will use the following important rule.

Remark 2.8. *For any commutators* $Q, R \in \Gamma(a_1, \ldots, a_n)$*, we can write* $[Q, {}_uR]$ *only with maximum possible parameter* u*, i.e. for any* $S \in \Gamma(a_1, \ldots, a_n)$ *we have* $[Q, {}_uR] \neq [S, {}_{u+1}R]$*.*

Let us note some obvious facts about the collection process.

Proposition 2.9. *Consider an arbitrary word W_j, $j \geqslant 1$, from the sequence (6). The following statements hold:*

(1) *In the free group $F(a_1, \ldots, a_n)$ we have the equality $W_0 = W_j$. In other words, the collection process does not change the element $W_0 \in F(a_1, \ldots, a_n)$.*
(2) *If the uncollected part T_j is non-empty, then any commutator in T_j has the form $[R, {}_p q_j]$ for some $R \in \Gamma(a_1, \ldots, a_n)$ and $p \geqslant 0$.*
(3) *Any two occurrences of the same commutator in T_j have different labels of the same length. In other words, the labels in T_j form the labeling of T_j. Moreover, labels do not change during the collection process.*
(4) *If the occurrence $R(\Lambda_u)$ precedes (is to the left of) $Q(\Lambda_v)$ in T_{j-1} and $R \neq q_j$, $Q \neq q_j$, then $R(\Lambda_u)$ precedes $Q(\Lambda_v)$ in T_j. In other words, the j-th stage of the collection process preserves the relative arrangement of occurrences of commutators different from q_j in T_{j-1}.*

Since the initial word W_0 contains only the commutators $a_1, \ldots, a_n$, the following statement holds.

Proposition 2.10. *For any commutator word W_j, $j \geqslant 1$, from the sequence (6), the uncollected part T_j does not contain occurrences of the commutators $q_1, \ldots, q_j$. As a consequence, the commutators $q_1, \ldots, q_j$ are pairwise different.*

Proof. We prove the statement by induction on the number of the stage of the collection process (on the number of the word in the sequence). At the first stage the occurrences of q_1 were collected and only occurrences of the commutators $[R, {}_p q_j]$, $p \geqslant 1$, arose. Therefore, the word W_1 contains occurrences of q_1 only in its collected part.

Assume that the statement is true for all words in the sequence up to $W_j = q_1^{e_1} \ldots q_j^{e_j} T_j$, $j \geqslant 1$, and the uncollected part T_j is nonempty. Let us prove the statement for the word

$$W_{j+1} = q_1^{e_1} \ldots q_k^{e_K} \ldots q_j^{e_j} q_{j+1}^{e_{j+1}} T_{j+1}$$

Suppose q_k has an occurrence in T_{j+1} for some $k \in \overline{1, j+1}$. Note that $k \neq j+1$ since at the stage $j+1$ all occurrences of the commutator q_{j+1} in the word T_j were collected. Further, at the stage k all occurrences of q_k were collected. By the inductive assumption, new occurrences of q_k could arise only at the stage $j+1$. Therefore, $q_k = [R, q_{j+1}]$ for some commutator R. Recall that the initial word W_0 contains only the elements $a_1, \ldots, a_n$, so the commutator $q_k = [R, q_{j+1}]$ arose during the collection process at some stage h, where $h < k < j+1$, i.e. $q_k = [Q, q_h]$ for some Q. From the equality $[R, q_{j+1}] = [Q, q_h]$ of formal commutators it follows that $q_h = q_{j+1}$. This contradicts the inductive assumption: q_h has occurrences both in the collected part $q_1 \ldots q_j$ and in the uncollected part T_j. □

3. Existence and precedence conditions

Suppose W_0 is a labeled commutator word consisting only of commutators of weight 1: $a_1, \dots, a_n$, $n \in \mathbb{N}$. Consider an arbitrary variant of the collection process

$$\{W_j \equiv q_1^{e_1} \dots q_j^{e_j} T_j\}_{j \geqslant 0}. \tag{7}$$

Denote by $D(a_k)$, $k \in \overline{1,n}$, an arbitrary fixed set of integer sequences of the same length that contains all labels of the occurrences of a_k in W_0.

Assume that the uncollected part T_m, $m \geqslant 0$ contains some commutator R and the parenthesis-free notation of R is $(a_{i_1}, \dots, a_{i_{w(R)}})$.

Example 3.1. *The parenthesis-free notation of* $\Big[[a_2, [a_1, a_2]], a_4\Big]$ *is* (a_2, a_1, a_2, a_4).

From Definition 2.5 it follows that any occurrence of R in T_m has a label of the form

$$\Lambda_1 \dots \Lambda_{w(R)}, \quad \text{where } \Lambda_1 \in D(a_{i_1}),\ \dots,\ \Lambda_{w(R)} \in D(a_{i_{w(R)}}).$$

Since different occurrences of R have different labels, the number of all occurrences of R in T_m is equal to the number of elements from the Cartesian product $D(a_{i_1}) \times \dots \times D(a_{i_{w(R)}})$ that are the labels of occurrences of R. We denote that Cartesian product by $D(R)$.

Definition 3.2. *Suppose that some uncollected part in (7) contains an occurrence of the commutator R. The existence condition of the commutator R is the predicate E_R^{Λ}, $\Lambda \in D(R)$, that is equal to 1 iff there exists a commutator word in (7) such that its uncollected part contains the occurrence $R(\Lambda)$.*

Definition 3.3. *Suppose that some uncollected part in (7) contains occurrences of the commutators R and Q. The precedence condition for the commutators R and Q is the predicate $P_{Q,R}^{\Lambda_1\Lambda_2}$, $\Lambda_1\Lambda_2 \in D(Q) \times D(R)$, that is equal to 1 iff there exists a commutator word in (7) such that, in its uncollected part, $Q(\Lambda_1)$ precedes (is to the left of) $R(\Lambda_2)$.*

Note that we can determine all values of the predicate E_R by considering an arbitrary uncollected part T_m that contains occurrences of R. In other words, any uncollected part either does not contain occurrences of R, or it contains all occurrences of R that have ever arisen during the collection process. Indeed, the occurrences of R arose at some stage $m_1 \leqslant m$. From statement 3 of Proposition 2.9 it follows that the labels of the occurrences do not change at the next stages of the collection process. By Proposition 2.10, new occurrences of R will never arise in uncollected parts. Finally, the occurrences can "disappear" only at some stage $m_2 > m \geqslant m_1$, when all of them, without exception, will be collected.

From the discussion above and statement 4 of Proposition 2.9 it follows that all values of the predicate $P_{Q,R}$ can be determined by considering an arbitrary uncollected part containing occurrences of R and Q.

Proposition 3.4. *For the sequence (7) we have*

$$e_j = |\{\Lambda \in D(q_j) \mid E_{q_j}^{\Lambda} = 1\}|.$$

This statement establishes the connection between the exponent e_j of the commutator and its existence condition E_{q_j}. So, by investigating the properties of the predicate E_{q_j}, we get information about e_j.

Definition 3.5. *Let R_1, R_2 be formal commutators. The predicate $R_1 \prec R_2$ is equal to 1 iff there exist commutators q_i, q_j in (7) such that $q_i = R_1$, $q_j = R_2$, $i < j$, i.e., the occurrences of R_1 were collected at an earlier stage than the occurrences of R_2 in the variant of the collection process (7).*

Example 3.6. *Let us consider the variant of the collection process from Example 2.7. We have the following relations:*

$$D(a_1) = D(a_2) = \{1, 2\};$$
$$D([a_2, a_1]) = D(a_2) \times D(a_1) = \{(1, 1), (1, 2), (2, 1), (2, 2)\};$$
$$D([a_2, a_1, a_2]) = D(a_2) \times D(a_1) \times D(a_2) = \{1, 2\}^3;$$
$$E_{a_1}^{\lambda_1} = E_{a_2}^{\lambda_2} = 1,$$
$$P_{a_1,a_2}^{(\lambda_1,\lambda_2)} = (\lambda_1 < \lambda_2) \vee (\lambda_1 = \lambda_2), \quad \textit{where} \;\; \lambda_1 \in D(a_1),\; \lambda_2 \in D(a_2);$$
$$a_1 \prec a_2; \quad [a_2, a_1] \prec [a_2, a_1, a_2];$$
$$e_3 = \left| \left\{ (\lambda_1, \lambda_2) \in D([a_2, a_1]) \mid E_{[a_2,a_1]}^{(\lambda_1,\lambda_2)} = 1 \right\} \right| = |\{(1, 2)\}| = 1.$$

4. Recurrence relations for the existence and precedence conditions

Theorem 4.1. *Let G be a group, $m \in \mathbb{N}$, and $y, x_1, x_2, \ldots, x_m \in G$. Then the following identity holds:*

$$(8) \qquad [y, x_1x_2 \ldots x_m] = \prod_{i=1}^{2^m-1} [y, {}_{i_1}x_1, {}_{i_2}x_2, \ldots, {}_{i_m}x_m],$$

where $\overline{i_1i_2 \ldots i_m}$ is the m-bit binary representation of the index i.

Proof. We prove the equivalent identity by induction on m:

$$(9) \qquad yx_1x_2 \ldots x_m = x_1x_2 \ldots x_m y \prod_{i=1}^{2^m-1} [y, {}_{i_1}x_1, {}_{i_2}x_2, \ldots, {}_{i_m}x_m].$$

For $m = 1$ the identity has the form $yx_1 = x_1y[y, x_1]$. Consider the product $yx_1x_2 \ldots x_mx_{m+1}$. By the inductive assumption, we get

$$(yx_1 \ldots x_m)x_{m+1} = x_1 \ldots x_m y \left(\prod_{i=1}^{2^m-1} [y, {}_{i_1}x_1, \ldots, {}_{i_m}x_m] \right) x_{m+1}$$
$$= x_1 \ldots x_{m+1} \prod_{i=0}^{2^m-1} [y, {}_{i_1}x_1, \ldots, {}_{i_m}x_m, {}_0x_{m+1}][y, {}_{i_1}x_1, \ldots, {}_{i_m}x_m, {}_1x_{m+1}],$$

where $\overline{i_1 \ldots i_m0}$ and $\overline{i_1 \ldots i_m1}$ are the binary representations of the numbers $2i$ and $2i + 1$, respectively. Thus, we have the equality

$$(yx_1 \ldots x_m)x_{m+1} = x_1 \ldots x_{m+1} \prod_{i=0}^{2^{m+1}-1} [y, {}_{i_1}x_1, \ldots, {}_{i_m}x_m, {}_{i_{m+1}}x_{m+1}],$$

where $\overline{i_1 \ldots i_{m+1}}$ is the binary representation of the index i. □

Example 4.2. *For $m = 2$ identity (8) transforms into the well-known formula:*

$$[y, x_1x_2] = [y, {}_0x_1, {}_1x_2][y, {}_1x_1, {}_0x_2][y, {}_1x_1, {}_1x_2] = [y, x_2][y, x_1][y, x_1, x_2].$$

Using the theorem, we can express the fact of precedence for two occurrences of commutators in terms of precedence for two binary numbers.

We denote by $\vee$ and $\wedge$ the disjunction and conjunction of two propositions, respectively. The sign $\wedge$ will be usually omitted to save space. For example, we write $AB \vee C$ instead of $(A \wedge B) \vee C$.

Denote by $\omega(i)$ the number of units in the binary representation of the non-negative integer i.

Proposition 4.3. *Let $N, M \in \overline{1, 2^m - 1}$ and $\omega(N) = u$, $\omega(M) = v$, where $u, v, m \geqslant 1$. Suppose $(j_1, \dots, j_u)$ and $(k_1, \dots, k_v)$ are two tuples such that j_s, $s \in \overline{1, u}$, and k_l, $l \in \overline{1, v}$ are the positions of the s-th unit and the l-th unit in the binary representations of N and M, respectively (we count units and digit positions from left to right starting from one). Then*

$$N < M \;\Leftrightarrow\; (u < v) \bigwedge_{t=1}^{u} (j_t = k_t) \vee \bigvee_{t=1}^{\min\{u,v\}} (k_t < j_t) \bigwedge_{h=1}^{t-1} (j_h = k_h).$$

Theorem 4.4. *Suppose $F(Q, R)$ is a free group, and the commutator word*

$$W_0 \equiv Q(\Lambda_0) R(\Lambda_1) \dots R(\Lambda_m), \quad m \in \mathbb{N}, \tag{10}$$

has an arbitrary labeling with some sets $D(Q)$ and $D(R)$. If we collect all occurrences of R at the first stage of the collection process, then we get the commutator word

$$W_1 \equiv R(\Lambda_1) \dots R(\Lambda_m) \cdot Q(\Lambda_0) \prod_{i=1}^{2^m - 1} [Q, {}_{\omega(i)} R](\phi(i)), \tag{11}$$

where $\phi(i) = \Lambda_0 \Lambda_{j_1} \dots \Lambda_{j_{\omega(i)}}$ and j_s, $s \in \overline{1, \omega(i)}$, is the position of the s-th unit in the binary representation of the index i (we count units and digit positions from left to right starting from one). Moreover, for any $u, v \in \mathbb{N}$, we have the following equalities:

$$E_{[Q, {}_u R]}^{\Lambda_0^1 \Lambda_1^1 \dots \Lambda_u^1} = E_Q^{\Lambda_0^1} \bigwedge_{k=1}^{u-1} P_{R,R}^{\Lambda_k^1 \Lambda_{k+1}^1},$$

$$P_{[Q, {}_u R], [Q, {}_v R]}^{\Lambda_0^1 \dots \Lambda_u^1 \Lambda_0^2 \dots \Lambda_v^2}$$
$$= E_{[Q, {}_u R]}^{\Lambda_0^1 \dots \Lambda_u^1} E_{[Q, {}_v R]}^{\Lambda_0^2 \dots \Lambda_v^2} \left((u < v) \bigwedge_{k=1}^{u} (\Lambda_k^2 = \Lambda_k^1) \vee \bigvee_{k=1}^{\min\{u,v\}} P_{R,R}^{\Lambda_k^2 \Lambda_k^1} \bigwedge_{h=1}^{k-1} (\Lambda_h^2 = \Lambda_h^1) \right),$$

where $\Lambda_0^1, \Lambda_0^2 \in D(Q)$, $\Lambda_1^1, \dots, \Lambda_u^1, \Lambda_1^2, \dots, \Lambda_v^2 \in D(R)$.

Proof. The first statement of the theorem immediately follows from identity (9).

Consider the product over i in (11). Any m-bit binary number is uniquely determined by the positions of units in its binary representation. Therefore, ϕ is a bijection map of $\{1, \dots, 2^m - 1\}$ to

$$\{\Lambda_0 \Lambda_{j_1} \dots \Lambda_{j_s} \mid s \geqslant 1,\; 1 \leqslant j_1 < \dots < j_s \leqslant m\}.$$

Any two integer-valued sequences

$$\Lambda_0 \Lambda_{j_1} \dots \Lambda_{j_u} \in D([Q, {}_u R]) = D(Q) \times D(R)^u,$$
$$\Lambda_0 \Lambda_{k_1} \dots \Lambda_{k_v} \in D([Q, {}_v R]) = D(Q) \times D(R)^v,$$

are labels of some occurrences of $[Q, {}_uR]$ and $[Q, {}_vR]$ in (11), respectively, iff $\Lambda_0\Lambda_{j_1}\dots\Lambda_{j_u} = \phi(i_1)$ and $\Lambda_0\Lambda_{k_1}\dots\Lambda_{k_v} = \phi(i_2)$ for some i_1, i_2. In other words, the following conditions must be satisfied:

$$\bigwedge_{t=1}^{u-1}(j_t < j_{t+1}), \quad \bigwedge_{t=1}^{v-1}(k_t < k_{t+1}).$$

If in addition the occurrence of $[Q, {}_uR]$ must precede the occurrence of $[Q, {}_vR]$, then we get the condition $i_1 < i_2$. By Proposition 4.3, it is equivalent to

$$\bigwedge_{t=1}^{u-1}(j_t < j_{t+1}) \bigwedge_{t=1}^{v-1}(k_t < k_{t+1}) \\ \wedge \left((u < v)\bigwedge_{t=1}^{u}(j_t = k_t) \vee \bigvee_{t=1}^{\min\{u,v\}}(k_t < j_t)\bigwedge_{h=1}^{t-1}(j_h = k_h)\right).$$

Note that

$$j_t < j_{t+1} \Leftrightarrow P_{R,R}^{\Lambda_{j_t}\Lambda_{j_{t+1}}}, \quad k_t < k_{t+1} \Leftrightarrow P_{R,R}^{\Lambda_{k_t}\Lambda_{k_{t+1}}},$$
$$j_t = k_t \Leftrightarrow \Lambda_{j_t} = \Lambda_{k_t}, \quad k_t < j_t \Leftrightarrow P_{R,R}^{\Lambda_{k_t}\Lambda_{j_t}}.$$

Thus, the existence and precedence conditions for commutators in the product over i with variables $\Lambda_0^1, \Lambda_0^2 \in D(Q)$, $\Lambda_1^1, \dots, \Lambda_u^1, \Lambda_1^2, \dots, \Lambda_v^2 \in D(R)$ can be written as follows:

$$E_{[Q,{}_uR]}^{\Lambda_0^1\Lambda_1^1\dots\Lambda_u^1} = E_Q^{\Lambda_0^1}\bigwedge_{k=1}^{u}E_R^{\Lambda_k^1}\bigwedge_{k=1}^{u-1}P_{R,R}^{\Lambda_k^1\Lambda_{k+1}^1} = E_Q^{\Lambda_0^1}\bigwedge_{k=1}^{u-1}P_{R,R}^{\Lambda_k^1\Lambda_{k+1}^1},$$

$$P_{[Q,{}_uR],[Q,{}_vR]}^{\Lambda_0^1\dots\Lambda_u^1\Lambda_0^2\dots\Lambda_v^2} \\ = E_{[Q,{}_uR]}^{\Lambda_0^1\dots\Lambda_u^1}E_{[Q,{}_vR]}^{\Lambda_0^2\dots\Lambda_v^2}\left((u < v)\bigwedge_{k=1}^{u}(\Lambda_k^2 = \Lambda_k^1) \vee \bigvee_{k=1}^{\min\{u,v\}}P_{R,R}^{\Lambda_k^2\Lambda_k^1}\bigwedge_{h=1}^{k-1}(\Lambda_h^2 = \Lambda_h^1)\right).$$

□

Corollary 4.5. *Suppose there are $e \geqslant 1$ occurrences of a commutator R in the following commutator word with arbitrary labeling:*

$$X \equiv \prod_{k=1}^{m} t_k(\Lambda_k).$$

After one stage of the collection process we get the word

$$Y \equiv R^e \prod_{\substack{k=1 \\ t_k \neq R}}^{m} t_k(\Lambda_k)\Omega_k, \quad \textit{where} \quad \Omega_k \equiv \prod_{i=1}^{2^{m_k}-1}[t_k, {}_{\omega(i)}R](\phi_k(i)),$$

m_k is the number of occurrences of R that are to the right of $t_k(\Lambda_k)$ in X, the label $\phi_k(i)$ is defined according to the previous theorem.

Further we will call Ω_k the *ω-product with initial element $t_k(\Lambda_k)$*.

Theorem 4.6. *Suppose* $\{W_j \equiv q_1^{e_1} \dots q_j^{e_j} T_j\}_{j\geqslant 0}$ *is an arbitrary variant of the collection process. Then the following recurrence relations hold (if the left-hand side of a relation is defined for* $\{W_j\}_{j\geqslant 0}$*):*

$$E^{\Lambda_0^1\dots\Lambda_u^1}_{[Q_1,\,_uR_1]} = P^{\Lambda_0^1\Lambda_1^1}_{Q_1,R_1} \bigwedge_{k=1}^{u-1} P^{\Lambda_k^1\Lambda_{k+1}^1}_{R_1,R_1}, \quad u \geqslant 1; \tag{12}$$

$P^{\Lambda_0^1\dots\Lambda_u^1\Lambda_0^2\dots\Lambda_v^2}_{[Q_1,\,_uR_1],[Q_2,\,_vR_2]}$ *is equal to*

(13a) $E^{\Lambda_0^1\dots\Lambda_u^1}_{[Q_1,\,_uR_1]} E^{\Lambda_0^2\dots\Lambda_v^2}_{[Q_2,\,_vR_2]} F,$ *if* $u+v \geqslant 1,\ R_1 = R_2,\ Q_1 = Q_2;$

(13b) $E^{\Lambda_0^1\dots\Lambda_u^1}_{[Q_1,\,_uR_1]} E^{\Lambda_0^2\dots\Lambda_v^2}_{[Q_2,\,_vR_2]} P^{\Lambda_0^1\Lambda_0^2}_{Q_1,Q_2},$ *if* $u+v \geqslant 1,\ R_1 = R_2,\ Q_1 \neq Q_2,$ $u = 0 \Rightarrow w(Q_1) = 1,\ v = 0 \Rightarrow w(Q_2) = 1;$

(13c) $E^{\Lambda_0^1\dots\Lambda_u^1}_{[Q_1,\,_uR_1]} E^{\Lambda_0^2\dots\Lambda_v^2}_{[Q_2,\,_vR_2]} P^{\Lambda_0^1\dots\Lambda_u^1\Lambda_0^2}_{[Q_1,\,_uR_1],Q_2},$ *if* $u, v \geqslant 1, \quad R_1 \prec R_2;$

(13d) $E^{\Lambda_0^1\dots\Lambda_u^1}_{[Q_1,\,_uR_1]} E^{\Lambda_0^2\dots\Lambda_v^2}_{[Q_2,\,_vR_2]} P^{\Lambda_0^1\Lambda_0^2\dots\Lambda_v^2}_{Q_1,[Q_2,\,_vR_2]},$ *if* $u, v \geqslant 1, \quad R_2 \prec R_1;$

where $[Q_1,\,_uR_1] \neq Q_2$ *for* $u \geqslant 1$ *and* $[Q_2,\,_vR_2] \neq Q_1$ *for* $v \geqslant 1$,

$$\Lambda_0^1 \in D(Q_1),\ \Lambda_0^2 \in D(Q_2),\ \Lambda_1^1, \dots, \Lambda_u^1 \in D(R_1),\ \Lambda_1^2, \dots, \Lambda_v^2 \in D(R_2),$$

$$F = P^{\Lambda_0^1\Lambda_0^2}_{Q_1,Q_2} \vee (\Lambda_0^1 = \Lambda_0^2)\Big((u < v) \bigwedge_{k=1}^{u} (\Lambda_k^2 = \Lambda_k^1) \vee \bigvee_{k=1}^{\min\{u,v\}} P^{\Lambda_k^2\Lambda_k^1}_{R_1,R_2} \bigwedge_{h=1}^{k-1} (\Lambda_h^2 = \Lambda_h^1)\Big).$$

Proof. Relations (12) and (13a). Let $u, v \geqslant 1$. Since the existence condition $E_{[Q_1,\,_uR_1]}$ is defined for $\{W_j\}_{j\geqslant 0}$, it follows that there are occurrences of the commutator $[Q_1,\,_uR_1]$ in some uncollected part T_m. Therefore, they arose during the l-th stage of the collection process, $l \in \overline{1,m}$, when occurrences of the commutator R_1 were being collected. Let us consider a schematic representation of the commutator word T_{l-1}, where we single out two occurrences of Q_1 (at least one such occurrence exists):

$$T_{l-1} = \dots Q_1(\Lambda_1) \dots Q_1(\Lambda_2) \dots.$$

From Corollary 4.5 it follows that T_l has the following form:

$$T_l = \dots Q_1(\Lambda_1)\Omega_{Q_1(\Lambda_1)} \dots Q_1(\Lambda_2)\Omega_{Q_1(\Lambda_2)} \dots,$$

where $\Omega_{Q_1(\Lambda_1)}$, $\Omega_{Q_1(\Lambda_2)}$ are ω-products with corresponding initial elements.

Let $\Lambda_0^1 \in D(Q_1)$, $\Lambda_1^1, \dots, \Lambda_u^1 \in D(R_1)$ and $P^{\Lambda_0^1\Lambda_1^1}_{Q_1,R_1} = 1$. Then by Theorem 4.4 the occurrence $[Q_1,\,_uR_1](\Lambda_0^1 \dots \Lambda_u^1)$ exists in ω-product with initial element $Q_1(\Lambda_0^1)$ iff

$$E^{\Lambda_0^1}_{Q_1} \bigwedge_{k=1}^{u-1} P^{\Lambda_k^1\Lambda_{k+1}^1}_{R_1,R_1} = 1.$$

Thus, we have

$$E^{\Lambda_0^1\dots\Lambda_u^1}_{[Q_1,\,_uR_1]} = P^{\Lambda_0^1\Lambda_1^1}_{Q_1,R_1} E^{\Lambda_0^1}_{Q_1} \bigwedge_{k=1}^{u-1} P^{\Lambda_k^1\Lambda_{k+1}^1}_{R_1,R_1} = P^{\Lambda_0^1\Lambda_1^1}_{Q_1,R_1} \bigwedge_{k=1}^{u-1} P^{\Lambda_k^1\Lambda_{k+1}^1}_{R_1,R_1}.$$

Further, commutators of the form $[Q_1,\,_tR_1]$, $t \geqslant 1$, have occurrences only in the ω-products. Moreover, the label of any occurrence in an ω-product begins with the sequence Λ_1 if the initial element is equal to $Q_1(\Lambda_1)$, with Λ_2 if the initial element is equal to $Q_1(\Lambda_2)$, and so on.

Hence, for arbitrary occurrences $[Q_1, {}_uR_1](\Lambda_0^1 \dots \Lambda_u^1)$ and $[Q_1, {}_vR_1](\Lambda_0^2 \dots \Lambda_v^2)$, there are two alternatives: they are in the same ω-product (i.e. $\Lambda_0^1 = \Lambda_0^2$), or in different ones. In the second case the precedence condition $P^{\Lambda_0^1 \dots \Lambda_u^1 \Lambda_0^2 \dots \Lambda_v^2}_{[Q_1, {}_uR_1],[Q_1, {}_vR_1]}$ is equivalent to precedence of corresponding ω-products, i.e. of initial elements $Q_1(\Lambda_0^1)$ and $Q_1(\Lambda_0^2)$. In the first case we use the recurrence relation from Theorem 4.4. Thus, we get

$$P^{\Lambda_0^1 \dots \Lambda_u^1 \Lambda_0^2 \dots \Lambda_v^2}_{[Q_1, {}_uR_1],[Q_1, {}_vR_1]} = E^{\Lambda_0^1 \dots \Lambda_u^1}_{[Q_1, {}_uR_1]} E^{\Lambda_0^2 \dots \Lambda_v^2}_{[Q_1, {}_vR_1]} F.$$

It is easy to prove that the obtained relation is true when either $u = 0$ or $v = 0$. Indeed, since there are no occurrences of Q_1 in ω-products, for arbitrary occurrences $[Q_1, {}_uR_1](\Lambda_0^1 \dots \Lambda_u^1)$ and $[Q_1, {}_vR_1](\Lambda_0^2 \dots \Lambda_v^2)$ we have

$$P^{\Lambda_0^1 \dots \Lambda_u^1 \Lambda_0^2 \dots \Lambda_v^2}_{[Q_1, {}_uR_1],[Q_1, {}_vR_1]} \Leftrightarrow \begin{cases} P^{\Lambda_0^1 \Lambda_0^2}_{Q_1, Q_1} \vee (\Lambda_0^1 = \Lambda_0^2), & \text{if } u = 0, v \neq 0; \\ P^{\Lambda_0^1 \Lambda_0^2}_{Q_1, Q_1}, & \text{if } u \neq 0, v = 0. \end{cases}$$

Relation (13b). Let $u, v \geqslant 1$. As in the previous case the occurrences of the commutators $[Q_1, {}_uR_1]$ and $[Q_2, {}_vR_1]$ arose during the same stage of the collection process. However, the occurrences of $[Q_1, {}_uR_1]$ are in the ω-product with initial element $Q_1(\Lambda)$, $\Lambda \in D(Q_1)$, and the occurrences of $[Q_2, {}_vR_1]$ are in the ω-product with initial element $Q_2(\Lambda)$, $\Lambda \in D(Q_2)$. As we mentioned above precedence of ω-products is equivalent to precedence of corresponding initial elements. Thus, we get

$$P^{\Lambda_0^1 \dots \Lambda_u^1 \Lambda_0^2 \dots \Lambda_v^2}_{[Q_1, {}_uR_1],[Q_2, {}_vR_1]} = E^{\Lambda_0^1 \dots \Lambda_u^1}_{[Q_1, {}_uR_1]} E^{\Lambda_0^2 \dots \Lambda_v^2}_{[Q_2, {}_vR_1]} P^{\Lambda_0^1 \Lambda_0^2}_{Q_1, Q_2}.$$

Now let $u = 0$ and $w(Q_1) = 1$. The commutator Q_1 has occurrences in the initial commutator word $W_0 \equiv T_0$. Therefore Q_1 and Q_2 have occurrences in the same uncollected part at the stage when occurrences of R_1 were being collected and occurrences of the commutator $[Q_2, {}_vR_1]$ arose. Here we deal with precedence of an initial element and an ω-product, which is also equivalent to precedence of corresponding initial elements. Since $Q_1 \neq Q_2$, we have

$$P^{\Lambda_0^1 \Lambda_0^2 \dots \Lambda_v^2}_{Q_1,[Q_2, {}_vR_1]} = E^{\Lambda_0^1}_{Q_1} E^{\Lambda_0^2 \dots \Lambda_v^2}_{[Q_2, {}_vR_1]} P^{\Lambda_0^1 \Lambda_0^2}_{Q_1, Q_2}.$$

For the case $v = 0$, $w(Q_2) = 1$ the reasoning is analogous.

Relations (13c) and (13d). Assume that occurrences of R_1 were collected earlier than occurrences of R_2. Since there must be an uncollected part containing both occurrences of $[Q_1, {}_uR_1]$ and $[Q_2, {}_vR_2]$, at the stage when occurrences of R_2 are collected the corresponding uncollected part contains both occurrences of $[Q_1, {}_uR_1]$ and Q_2. Since $[Q_1, {}_uR_1] \neq Q_2$, we use the same argument as above to get the relation

$$P^{\Lambda_0^1 \dots \Lambda_u^1 \Lambda_0^2 \dots \Lambda_v^2}_{[Q_1, {}_uR_1],[Q_2, {}_vR_2]} = E^{\Lambda_0^1 \dots \Lambda_u^1}_{[Q_1, {}_uR_1]} E^{\Lambda_0^2 \dots \Lambda_v^2}_{[Q_2, {}_vR_2]} P^{\Lambda_0^1 \dots \Lambda_u^1 \Lambda_0^2}_{[Q_1, {}_uR_1], Q_2}.$$

If occurrences of R_2 were collected earlier than occurrences of R_1, the reasoning is analogous. □

Example 4.7. *Let us consider the variant of the collection process from Example 2.7 and express the existence condition*

$$E^{(\lambda_1, \lambda_2, \lambda_3)}_{[[a_2, a_1], a_2]}, \quad (\lambda_1, \lambda_2) \in D([a_2, a_1]) = D(a_2) \times D(a_1),\ \lambda_3 \in D(a_2),$$

in terms of $E_{a_1}, E_{a_2}, P_{a_1,a_2}, P_{a_2,a_1}, P_{a_1,a_1}, P_{a_2,a_2}$.

$$
\begin{aligned}
E^{(\lambda_1,\lambda_2,\lambda_3)}_{[[a_2,a_1],a_2]} &= P^{(\lambda_1,\lambda_2,\lambda_3)}_{[a_2,a_1],a_2} && \text{use (12) for } u=1,\ Q_1=[a_2,a_1],\ R_1=a_2\\
&= E^{(\lambda_1,\lambda_2)}_{[a_2,a_1]} E^{(\lambda_3)}_{a_2} P^{(\lambda_1\lambda_3)}_{a_2,a_2} && \text{use (13a) for } u=1,\ v=0,\ Q_1=Q_2=a_2,\ R_1=a_1\\
&= P^{(\lambda_1,\lambda_2)}_{a_2,a_1} E^{(\lambda_3)}_{a_2} P^{(\lambda_1\lambda_3)}_{a_2,a_2} && \text{use (12) for } u=1,\ Q_1=a_2,\ R_1=a_1.
\end{aligned}
$$

Using the information from Example 3.6, we get

$$E^{(\lambda_1,\lambda_2,\lambda_3)}_{[[a_2,a_1],a_2]} = (\lambda_1<\lambda_2)(\lambda_1<\lambda_3), \quad (\lambda_1,\lambda_2,\lambda_3)\in\{1,2\}^3.$$

The only case when $E^{(\lambda_1,\lambda_2,\lambda_3)}_{[[a_2,a_1],a_2]} = 1$ *is* $(\lambda_1,\lambda_2,\lambda_3)=(1,2,2)$.

Corollary 4.8. *Suppose a commutator* $R\in\Gamma(a_1,\dots,a_n)$ *arose in some variant of the collection process* $\{W_j\}_{j\geqslant 0}$. *Then the existence condition* E_R *is expressed by a formula containing at most the operations* $\vee$ *and* $\wedge$, *the predicates* E_{a_i}, P_{a_i,a_j}, $i,j\in\overline{1,n}$, *and the equality relation* $=$ *on* $\mathbb{Z}$.

Proof. First of all, we show that for any predicates

$$E_Q, \quad w(Q)\geqslant 2; \qquad P_{Q,R}, \quad w(Q)+w(R)\geqslant 3;$$

there are appropriate recurrence relations from the previous theorem. Any formal commutator Q, $w(Q)\geqslant 2$, can be expressed as $[Q_1, {}_uR_1]$, $u\geqslant 1$. Therefore, we use relation (12) for the existence condition E_Q.

Consider the precedence condition $P_{Q,R}$. Assume that $w(R)=1$ and write Q as $[Q_1, {}_uR_1]$, $u\geqslant 1$. If $Q_1\neq R$, then we use relation (13b). If $Q_1=R$, then we write $[Q_1, {}_uR_1]$, R as $[Q_1, {}_uR_1]$, $[Q_1, {}_0R_1]$, respectively, and use relation (13a). For the case $w(Q)=1$ the reasoning is analogous.

Let $w(Q), w(R)\geqslant 2$. Then Q and R can be written as $[Q_1, {}_uR_1]$, $u\geqslant 1$, and $[Q_2, {}_vR_2]$, $v\geqslant 1$, respectively. If $Q_1\neq[Q_2, {}_vR_2]$ and $Q_2\neq[Q_1, {}_uR_1]$, then we apply one of the relations (13) to $P_{Q,R}$. If $Q_1=[Q_2, {}_vR_2]$, then we represent $[Q_1, {}_uR_1]$ and $[Q_2, {}_vR_2]$ in the form $[Q_1, {}_uR_1]$ and $[Q_1, {}_0R_1]$, respectively, and use relation (13a). The reasoning is the same if $Q_2=[Q_1, {}_uR_1]$.

Thus, using the recurrence relations from the previous theorem a finite number of times, we express E_R by a formula containing at most the operations $\vee$ and $\wedge$, the predicates E_{a_i}, P_{a_i,a_j}, $i,j\in\overline{1,n}$, and the equality relation $\Lambda_u=\Lambda_v$. It remains to note that for any integer-valued sequences of the same length $\Lambda_u=(\lambda_{i_1},\dots,\lambda_{i_s})$, $\Lambda_v=(\lambda_{j_1},\dots,\lambda_{j_s})$ the relation $\Lambda_u=\Lambda_v$ can be written in terms of the equality relation on $\mathbb{Z}$:

$$\bigwedge_{k=1}^{s}(\lambda_{i_k}=\lambda_{j_k}).$$

□

5. L-conditions

In [6], the concept of an L-condition was introduced.

Definition 5.1. *Let C be a formula containing only the following logical operations: disjunction, conjunction, negation and the predicate symbols: $[\lambda_i < \lambda_j]$, $[\lambda_i = \lambda_j]$, where $i, j \in \mathbb{N}$. The formula C is called an* L-condition *of rank r, $r \in \mathbb{N}$, if C contains at least one of the following predicate symbols: $[\lambda_i < \lambda_r]$, $[\lambda_r < \lambda_i]$, $[\lambda_i = \lambda_r]$, $[\lambda_r = \lambda_i]$, $i \in \mathbb{N}$, and does not contain the predicate symbols $[\lambda_i < \lambda_j]$ and $[\lambda_i = \lambda_j]$, where $i > r$ or $j > r$. Logical constants 0 are 1 are called L-conditions of rank zero.*

Suppose M is a nonempty totally ordered set. Any L-condition of rank at most r can be interpreted as an r-place predicate $C(\lambda_1, \dots, \lambda_r)$ on an arbitrary subset N of $M^r = M \times \dots \times M$ if we consider $(\lambda_1, \dots, \lambda_r)$ as a tuple of variables from N and put $[\lambda_i = \lambda_j] = 1$ iff λ_i is equal to λ_j in M, and $[\lambda_i < \lambda_j] = 1$ iff λ_i is less than λ_j in M.

We say that a tuple of elements $(\lambda_1^*, \dots \lambda_r^*) \in N$ satisfies an L-condition C of rank at most r if $C(\lambda_1^*, \dots \lambda_r^*)$ is equal to 1 as a predicate on N.

Example 5.2. *Consider the following L-conditions of ranks 1, 2, and 3, respectively:*

$$C_1 = [\lambda_1 < \lambda_1], \quad C_2 = [\lambda_1 < \lambda_2] \vee [\lambda_1 = \lambda_2], \quad C_3 = [\lambda_2 = \lambda_3][\lambda_3 = \lambda_2].$$

Let $M = \mathbb{Z}$, $N = \{(0,0,0), (1,1,2), (1,2,2)\}$. Then there are 0, 3, 2 elements from N that satisfy C_1, C_2, C_3, respectively.

From Corollary 4.8, we get immediately the following lemma.

Lemma 5.3. *Suppose a commutator $R \in \Gamma(a_1, \dots, a_n)$ arose in some variant of the collection process $\{W_j\}_{j \geqslant 0}$. If the existence and precedence conditions E_{a_i}, P_{a_i,a_j} in the formula from Corollary 4.8 are expressed by L-conditions on the sets $D(a_i)$, $D(a_i) \times D(a_j)$, respectively, then E_R is expressed by some L-condition on $D(R)$.*

The following statement was proved in [6, Theorem 2]. Suppose $M_1, \dots, M_r$ are nonempty finite subsets of any totally ordered set, C is an L-condition of rank at most r. If the following conditions hold:

$$M = M_1 \cap \dots \cap M_r \neq \varnothing; \qquad [\min M, \max M] \cap (M_i \backslash M) = \varnothing, \quad i \in \overline{1, r};$$

then the number of elements from the Cartesian product

$$M_1 \times \dots \times M_r = \{(\lambda_1, \dots, \lambda_r) \mid \lambda_i \in M_i, i \in \overline{1, r}\}$$

that satisfy C are expressed as follows:

$$\sum_{t=1}^{r} a_t \binom{|M|}{t} + \sum_{t=0}^{r-1} b_t \binom{|M|}{t}, \quad a_s, b_s \in \mathbb{N}_0,$$

where b_s depend on $M_1, \dots, M_r$ and C, a_s depend only on C. Moreover, if $M \in \{M_i\}_{i=1}^r$, then $b_0 = 0$; if $M_1 = \dots = M_r$, then all b_s are equal to zero.

Using this statement, we obtain the following lemma.

Lemma 5.4. *Suppose finite sets $M_1, \dots, M_r \subset \mathbb{Z}$ satisfy the following conditions:*

$$M = M_1 \cap \dots \cap M_r \in \{M_j\}_{j=1}^r; \qquad [\min M, \max M] \cap (M_i \backslash M) = \varnothing, \quad i \in \overline{1, r}.$$

Then the number of elements from $M_1 \times \cdots \times M_r$ that satisfy an L-condition C of rank at most r is expressed as follows:

$$\sum_{t=1}^{r} c_t \binom{|M|}{t}, \quad c_t \in \mathbb{N}_0, \tag{14}$$

where c_t do not depend on $M_1, \dots, M_r$ if $M_1 = \cdots = M_r$.

We do not require here the sets $M_1, \dots, M_r$ to be nonempty. Indeed, if some $M_j = \varnothing$, then $M_1 \times \cdots \times M_r = \varnothing$, $M = \varnothing$, and all terms in sum (14) are equal to zero.

If $M_1 = \ldots = M_r = [1, n]$, then we get the P. Hall's combinatorial result (see Introduction).

Example 5.5. *Any family of intervals of integers with fixed left or right endpoint satisfies the conditions of Lemma 5.4. For example,*

$$M_1 = \{1, 2, 3\}, \quad M_2 = \{1, 2, 3, 4, 5\}, \quad M_3 = \{1, 2, 3, 4, 5, 6, 7\}.$$

6. Collection formulas

Suppose W is a positive word of the free group $F(a_1, \dots, a_n)$, $n \in \mathbb{N}$. According to our approach, if we want to apply the collection process to W and study the divisibility of the exponent of some commutator R with parenthesis-free notation $(a_{i_1}, \dots, a_{i_{w(R)}})$, then our main purpose is to find an appropriate labeling for the word W. 'Appropriate labeling' means that 1) the existence and precedence conditions $E_{a_{i_j}}$, $P_{a_{i_j}, a_{i_k}}$ for all $j, k \in \overline{1, w(R)}$ are expressed by L-conditions on the sets $D(a_{i_j})$, $D(a_{i_j}) \times D(a_{i_k})$, respectively, and 2) the sets $D(a_{i_j})$, $j \in \overline{1, w(R)}$, satisfy the conditions of Lemma 5.4. We will follow this "standard" way in Theorem 6.1.

However, in Theorem 6.6 we will consider restrictions of the predicate E_R by fixing special variables to get more powerful divisibility property than we could get by following the "standard" way. Finally, in Theorem 6.9 we use labeling for which the sets $D(a_{i_j})$ even do not satisfy the conditions of Lemma 5.4 until we use the same trick as in the previous case.

Theorem 6.1. *Suppose $\{W_j \equiv q_1^{e_1} \dots q_j^{e_j} T_j\}_{j \geqslant 0}$ is an arbitrary variant of the collection process in the free group $F(a_1, \dots, a_n)$, $n \in \mathbb{N}$, with the initial word*

$$W_0 \equiv \prod_{i=1}^{N} \left(a_1^{\rho(i,1)} \dots a_n^{\rho(i,n)} \right), \quad N \in \mathbb{N},\ \rho(i,k) : \{1, \dots, N\} \times \{1, \dots, n\} \to \{0, 1\}.$$

Suppose $M_k = \{i \mid \rho(i,k) = 1\}$, $k \in \overline{1, n}$, and $(a_{k_1}, \dots, a_{k_{w(q_j)}})$ is the parenthesis-free notation of q_j. If the set $M = M_{k_1} \cap \dots \cap M_{k_{w(q_j)}}$ satisfies the following conditions:

$$M \in \{M_{k_s}\}_{s=1}^{w(q_j)}; \qquad [\min M, \max M] \cap (M_{k_s} \backslash M) = \varnothing, \quad s \in \overline{1, w(q_j)};$$

then

$$e_j = \sum_{t=1}^{w(q_j)} c_t \binom{|M|}{t}, \quad c_t \in \mathbb{N}_0.$$

Proof. Consider the positive word $W \equiv (a_1 \dots a_n)^N$ with the following labeling:

$$W \equiv \prod_{i=1}^{N} (a_1(i) \dots a_n(i)); \qquad D(a_k) = \{1, \dots, N\}, \quad k \in \overline{1,n}.$$

We see that for any $\lambda_1 \in D(a_{k_1})$, $\lambda_2 \in D(a_{k_2})$, $k_1, k_2 \in \overline{1,n}$, the following equalities hold:

$$E^{(\lambda_1)}_{a_{k_1}} = 1, \quad P^{(\lambda_1,\lambda_2)}_{a_{k_1},a_{k_2}} = \begin{cases} [\lambda_1 < \lambda_2], & \text{if } k_1 = k_2; \\ [\lambda_1 < \lambda_2] \vee [\lambda_1 = \lambda_2], & \text{if } k_1 < k_2; \\ [\lambda_1 < \lambda_2], & \text{if } k_1 > k_2. \end{cases}$$

Now for each $k \in \overline{1,n}$ we delete all occurrences of the letter a_k in the word W whose labels belong to the set $D(a_k) \backslash M_k$. Thus, we get the labeled word W_0 for which $D(a_k) = M_k$, $k \in \overline{1,n}$. Moreover, the equalities for the existence and precedence conditions mentioned above remain true for W_0. Indeed, since $D(a_k)$ contains only the labels of occurrences of a_k from W_0, we have $E^{(\lambda_1)}_{a_k} = 1$ for any $\lambda_1 \in D(a_k)$. Further, deleting occurrences of letters in the word W does not change the relative arrangement of the remaining letters. In other words, an occurrence $a_{k_1}(\lambda_1)$ precedes an occurrence $a_{k_2}(\lambda_2)$ in W_0 iff $a_{k_1}(\lambda_1)$ precedes $a_{k_2}(\lambda_2)$ in W. Thus, the equality for $P^{(\lambda_1,\lambda_2)}_{a_{k_1},a_{k_2}}$ stays the same as we go from W to W_0.

Suppose $(a_{i_1}, \dots, a_{i_{w(q_j)}})$ is the parenthesis-free notation of the commutator q_j. From Lemma 5.3 it follows that the existence condition $E^{\Lambda}_{q_j}$ is expressed by an L-condition on the set $D(q_j) = D(a_{k_1}) \times \dots \times D(a_{k_{w(q_j)}}) = M_{k_1} \times \dots \times M_{k_{w(q_j)}}$. If the set $M = M_{k_1} \cap \dots \cap M_{k_{w(q_j)}}$ satisfies the following conditions:

$$M \in \{M_{k_s}\}_{s=1}^{w(q_j)}; \qquad [\min M, \max M] \cap (M_{k_s} \backslash M) = \varnothing, \quad s \in \overline{1, w(q_j)},$$

then by Proposition 3.4 and Lemma 5.4 we have

$$e_j = |\{\Lambda \in D(q_j) \mid E^{\Lambda}_{q_j} = 1\}| = \sum_{t=1}^{w(q_j)} c_t \binom{|M|}{t}, \quad c_t \in \mathbb{N}_0.$$

□

The theorem actually shows that for any family of sets that satisfies the conditions of Lemma 5.4 we can construct the corresponding initial word W_0. This provides numerous examples of collection formulas with nontrivial divisibility properties of the exponents of commutators. We consider only a few examples.

Example 6.2 (The M. Hall's formula (see Introduction))**.** *Let $M_k = \{1, \dots, m\}$, $k \in \overline{1,n}$, where $m \in \mathbb{N}$. Then we have the initial word $W_0 \equiv (a_1 \dots a_n)^m$ in the variant of the collection process $\{W_j \equiv q_1^{e_1} \dots q_j^{e_j} T_j\}_{j \geqslant 0}$. If we collect the commutators in order of increasing weights, then the following collection formula holds in the free group $F = F(a_1, \dots, a_n)$:*

$$(a_1 \dots a_n)^m = q_1^{e_1} \dots q_{j(s)}^{e_{j(s)}} \pmod{\Gamma_s(F)}, \quad s \in \mathbb{N}.$$

For any commutator q_j we have $M = \{1, \dots, m\}$, $|M| = m$. Therefore, e_j is divisible by m if m is a prime power number p^α and $w(q_j) < p$.

Example 6.3. *Let $M_k = \{1, \dots, m\}$ for $k \neq s$, and $M_s = \{1, \dots, m + r\}$, where $s \in \overline{1,n}$, $m, r \in \mathbb{N}$. Then we have the initial word $W_0 \equiv (a_1 \dots a_n)^m a_s^r$ in the variant* *of the collection process* $\{W_j \equiv q_1^{e_1} \dots q_j^{e_j} T_j\}_{j\geqslant 0}$. *Thus, the following collection formula holds in the free group* $F(a_1, \dots, a_n)$*:*

$$(a_1 \dots a_n)^m a_s^r = q_1^{e_1} \dots q_j^{e_j} T_j.$$

Assume that $w(q_j) \geqslant 2$. *Then the parenthesis-free notation of* q_j *contains a letter different from* a_s. *Therefore,* $M = \{1, \dots, m\}$, $|M| = m$, *and* e_j *is divisible by* m *if* m *is a prime power number* p^α *and* $1 < w(q_j) < p$.

Definition 6.4. *For the elements of* $\Gamma(a_1, \dots, a_n)$ *we define the weights* w_{a_i}, $i \in \overline{1,n}$ *by induction:*

1. $w_{a_i}(a_k) = 0$ *for* $k \neq i$, $w(a_i) = 1$*;*
2. *If* $[c_1, c_2] \in \Gamma(a_1, \dots, a_n)$, *then* $w_{a_i}([c_1, c_2]) = w_{a_i}(c_1) + w_{a_i}(c_2)$.

If $(a_{i_1}, \dots, a_{i_{w(R)}})$ is the parenthesis-free notation of $R \in \Gamma(a_1, \dots, a_n)$, then $w_{a_i}(R)$ is equal to the number of occurrences of a_i in the sequence $(a_{i_1}, \dots, a_{i_{w(R)}})$.

Example 6.5. *Let* M_1, M_2, M_3 *be as in Example 5.5. Then we get the word* $W_0 \equiv (a_1 a_2 a_3)^3 (a_2 a_3)^2 a_3^2$. *If* $w_{a_1}(q_j) \geqslant 1$ *and* $3 > w(q_j) \geqslant 1$, *then* e_j *is divisible by 3. If* $w_{a_1}(q_j) = 0$, $w_{a_2}(q_j) \geqslant 1$, *and* $5 > w(q_j) \geqslant 1$, *then* e_j *is divisible by 5.*

Theorem 6.6. *Suppose* $\{W_j \equiv q_1^{e_1} \dots q_j^{e_j} T_j\}_{j\geqslant 0}$ *is an arbitrary variant of the collection process in the free group* $F(a_1, \dots, a_n)$, $n \in \mathbb{N}$, *with the initial word*

$$W_0 \equiv a_1^{m_1} \dots a_n^{m_n}, \quad m_1, \dots, m_n \in \mathbb{N}.$$

If $w_{a_{s_1}}(q_j) \geqslant 1, \dots, w_{a_{s_r}}(q_j) \geqslant 1$, *then the exponent* e_j *is expressed as*

$$e_j = \sum_{t_1=1}^{w_{a_{s_1}}(q_j)} \cdots \sum_{t_r=1}^{w_{a_{s_r}}(q_j)} c(t_1, \dots, t_r) \binom{m_{s_1}}{t_1} \cdots \binom{m_{s_r}}{t_r},$$

where integers $c(t_1, \dots, t_r)$ *do not depend on* $m_{s_1}, \dots, m_{s_r}$.

Proof. Consider the positive word W_0 with the following labeling:

$$W_0 \equiv \prod_{i=1}^{n} a_i(1) \dots a_i(m_i); \qquad D(a_i) = \{1, \dots, m_i\}, \quad i \in \overline{1,n}.$$

We see that for any $\lambda_1 \in D(a_{i_1})$, $\lambda_2 \in D(a_{i_2})$, $i_1, i_2 \in \overline{1,n}$, the following equalities hold:

$$E_{a_{i_1}}^{(\lambda_1)} = 1, \quad P_{a_{i_1}, a_{i_2}}^{(\lambda_1, \lambda_2)} = \begin{cases} [\lambda_1 < \lambda_2], & i_1 = i_2; \\ 1, & i_1 < i_2; \\ 0, & i_2 < i_1. \end{cases}$$

Suppose $(a_{i_1}, \dots, a_{i_{w(q_j)}})$ is the parenthesis-free notation of the commutator q_j. From Lemma 5.3 it follows that the existence condition $E_{q_j}^{\Lambda}$, $\Lambda = (\lambda_1, \dots, \lambda_{w(q_j)})$, is expressed by an L-condition on the set $D(q_j) = D(a_{i_1}) \times \cdots \times D(a_{i_{w(q_j)}})$.

Let $w_{a_{s_1}}(q_j) \geqslant 1$. We arbitrarily fix those variables $\lambda_1, \dots, \lambda_{w(q_j)}$ in $E_{q_j}^{\Lambda}$ that correspond to labels of the letters a_k, $k \neq s_1$. The obtained predicate we denote by $\widetilde{E}_{q_j}^{M}$ where M is the tuple of the remaining $w_{a_{s_1}}(q_j)$ variables.

Note that the predicate $\widetilde{E}_{q_j}^{M}$ on the Cartesian power $D(a_{s_1})^{w_{a_{s_1}}(q_j)}$ is still an L-condition. Indeed, from the equalities for E_{a_i} and P_{a_i, a_j} above it follows that for any predicate $[\lambda_u < \lambda_v]$ in the L-condition $E_{q_j}^{\Lambda}$ the variables λ_u and λ_v correspond to labels of the same letter. This is also true for predicates $[\lambda_u = \lambda_v]$, which follows from the recurrence relations in Theorem 4.6. Therefore, $\widetilde{E}_{q_j}^M$ contains at most the logical constants 0, 1 and predicates of the form $[\lambda_u < \lambda_v]$, $[\lambda_u = \lambda_v]$, where λ_u, λ_v are variables.

Thus, from Lemma 5.4 it follows that

$$|\{M \in D(a_s)^{w_{a_{s_1}}(q_j)} \mid \widetilde{E}_{q_j}^M = 1\}| = \sum_{t=1}^{w_{a_{s_1}}(q_j)} b_t \binom{m_{s_1}}{t},$$

where non-negative integers b_t do not depend on m_{s_1}.

Considering all possible values of the variables in $E_{q_j}^\Lambda$ that correspond to labels of the letters a_k, $k \neq s_1$, and summing obtained expressions we finally get

$$e_j = |\{\Lambda \in D(q_j) \mid E_{q_j}^\Lambda = 1\}| = \sum_{t=1}^{w_{a_{s_1}}(q_j)} c_t \binom{m_{s_1}}{t}, \quad c_t \in \mathbb{N}_0,$$

where the coefficients c_t do not depend on m_{s_1} but may depend on $m_{s_2}, \ldots, m_{s_r}$.

Further proof is by induction on r. Let $w_{a_{s_1}}(q_j) \geqslant 1, \ldots, w_{a_{s_{r-1}}}(q_j) \geqslant 1$ for $r \geqslant 2$. Assume that

$$e_j = \sum_{t_1=1}^{w_{a_{s_1}}(q_j)} \cdots \sum_{t_{r-1}=1}^{w_{a_{s_{r-1}}}(q_j)} c(t_1, \ldots, t_{r-1}) \binom{m_{s_1}}{t_1} \cdots \binom{m_{s_{r-1}}}{t_{r-1}},$$

where the integers $c(t_1, \ldots, t_{r-1})$ do not depend on $m_{s_1}, \cdots m_{s_{r-1}}$. If $w_{a_{s_r}}(q_j) \geqslant 1$, then

$$e_j = \sum_{t=1}^{w_{a_{s_r}}(q_j)} a_t \binom{m_{s_r}}{t} = \sum_{t_1=1}^{w_{a_{s_1}}(q_j)} \cdots \sum_{t_{r-1}=1}^{w_{a_{s_{r-1}}}(q_j)} c(t_1, \ldots, t_{r-1}) \binom{m_{s_1}}{t_1} \cdots \binom{m_{s_{r-1}}}{t_{r-1}},$$

where a_t do not depend on m_{s_r}. Therefore, a_t depend on $m_{s_1}, \ldots, m_{s_{r-1}}$, and $c(t_1, \ldots, t_{r-1})$ depend on m_{s_r}. If $m_{s_r} = 1$, then we get

$$a_1 = \sum_{t_1=1}^{w_{a_{s_1}}(q_j)} \cdots \sum_{t_{r-1}=1}^{w_{a_{s_{r-1}}}(q_j)} c(t_1, \ldots, t_{r-1}; 1) \binom{m_{s_1}}{t_1} \cdots \binom{m_{s_{r-1}}}{t_{r-1}},$$

Further, if $m_{s_r} = 2$, then

$$2a_1 + a_2 = \sum_{t_1=1}^{w_{a_{s_1}}(q_j)} \cdots \sum_{t_{r-1}=1}^{w_{a_{s_{r-1}}}(q_j)} c(t_1, \ldots, t_{r-1}; 2) \binom{m_{s_1}}{t_1} \cdots \binom{m_{s_{r-1}}}{t_{r-1}},$$

$$a_2 = \sum_{t_1=1}^{w_{a_{s_1}}(q_j)} \cdots \sum_{t_{r-1}=1}^{w_{a_{s_{r-1}}}(q_j)} (c(t_1, \ldots, t_{r-1}; 2) - 2c(t_1, \ldots, t_{r-1}; 1)) \binom{m_{s_1}}{t_1} \cdots \binom{m_{s_{r-1}}}{t_{r-1}}.$$

Continuing this line of reasoning, we see that for any $t \in \overline{1, w_{a_{s_r}}(q_j)}$ there exist integers $h_t(t_1, \ldots, t_{r-1})$ such that

$$a_t = \sum_{t_1=1}^{w_{a_{s_1}}(q_j)} \cdots \sum_{t_{r-1}=1}^{w_{a_{s_{r-1}}}(q_j)} h_t(t_1, \ldots, t_{r-1}) \binom{m_{s_1}}{t_1} \cdots \binom{m_{s_{r-1}}}{t_{r-1}}.$$

Therefore,

$$e_j = \sum_{t=1}^{w_{a_{s_r}}(q_j)} \left(\sum_{t_1=1}^{w_{a_{s_1}}(q_j)} \cdots \sum_{t_{r-1}=1}^{w_{a_{s_{r-1}}}(q_j)} h_t(t_1,\ldots,t_{r-1}) \binom{m_{s_1}}{t_1} \cdots \binom{m_{s_{r-1}}}{t_{r-1}} \right) \binom{m_{s_r}}{t}$$
$$= \sum_{t_1=1}^{w_{a_{s_1}}(q_j)} \cdots \sum_{t_r=1}^{w_{a_{s_r}}(q_j)} c(t_1,\ldots,t_r) \binom{m_{s_1}}{t_1} \cdots \binom{m_{s_r}}{t_r},$$

where integers $c(t_1,\ldots,t_r)$ do not depend on $m_{s_1},\cdots,m_{s_r}$ □

Example 6.7. *Suppose we collect commutators in order of increasing weights, and the commutators of weight 1 are collected in the following order: a_{s+1}, a_{s+2}, $\ldots$, a_n, a_1, a_2, $\ldots$, a_s for some $s \in \overline{1, n-1}$. Then our variant of the collection process $\{W_j \equiv q_1^{e_1} \ldots q_j^{e_j} T_j\}_{j\geqslant 0}$ leads to the following collection formula in the free group $F = F(a_1,\ldots,a_n)$:*

$$[a_1^{m_1}\ldots a_s^{m_s}, a_{s+1}^{m_{s+1}}\ldots a_n^{m_n}] = q_{n+1}^{e_{n+1}}\ldots q_{j(k)}^{e_{j(k)}} \pmod{\Gamma_k(F)}, \quad k \geqslant 2,$$

where $q_{n+1},\ldots,q_{j(k)}$ are commutators of weights less than k, and e_j is divisible by m_i if m_i is a prime power number p^α and $1 \leqslant w_{a_i}(q_j) < p$.

Example 6.8 (The R. R. Struik's formula (see Introduction))**.** *If we put $n = 2$, $s = 1$ in the previous example, then we get formula (5).*

Theorem 6.9. *Suppose $\{W_j \equiv q_1^{e_1}\ldots q_j^{e_j} T_j\}_{j\geqslant 0}$ is an arbitrary variant of the collection process in the free group $F(a_1,\ldots,a_n)$, $n \in \mathbb{N}$, with the initial word*

$$W_0 \equiv \prod_{i=1}^{N} \left(a_{k_1}^{\rho(i,1)} \ldots a_{k_s}^{\rho(i,s)} \right), \quad N, s \in \mathbb{N},\ \rho(i,j) : \{1,\ldots,N\} \times \{1,\ldots,s\} \to \{0,1\},$$

where $k_1,\ldots,k_s \in \overline{1,n}$. Suppose

$$M_k = \{(i,j) \mid k_j = k,\ \rho(i,j) = 1\}, \qquad M_k(\mu) = \{i \mid (i,\mu) \in M_k\}, \quad k \in \overline{1,n},$$

and $(a_{i_1},\ldots,a_{i_{w(q_j)}})$ is the parenthesis-free notation of q_j. If for any $\mu_1,\ldots,\mu_{w(q_j)}$ the set $M = M(\mu_1,\ldots,\mu_{w(q_j)}) = M_{i_1}(\mu_1) \cap \cdots \cap M_{i_{w(q_j)}}(\mu_{w(q_j)})$ satisfies the following conditions:

$$M \in \{M_{i_k}(\mu_k)\}_{k=1}^{w(q_j)}; \qquad [\min M, \max M] \cap (M_{i_k}(\mu_k) \backslash M) = \varnothing, \quad k \in \overline{1, w(q_j)};$$

then

$$e_j = \sum_{t=1}^{w(q_j)} \sum_{\mu_1,\ldots,\mu_{w(q_j)}} c_t(\mu_1,\ldots,\mu_{w(q_j)}) \binom{|M(\mu_1,\ldots,\mu_{w(q_j)})|}{t},$$

where $c_t(\mu_1,\ldots,\mu_{w(q_j)}) \in \mathbb{N}_0$.

Proof. Consider the positive word $W \equiv (a_{k_1}\ldots a_{k_s})^N$ with the following labeling:

$$W \equiv \prod_{i=1}^{N} (a_{k_1}(i,1)\ldots a_{k_s}(i,s));$$
$$D(a_k) = \{(i,j) \mid 1 \leqslant i \leqslant N, k_j = k\} = \{1,\ldots,N\} \times \{j \mid k_j = k\}, \quad k \in \overline{1,n}.$$

We see that for any $(\lambda_1,\lambda_2)\in D(a_i)$, $(\lambda_3,\lambda_4)\in D(a_j)$, $i,j\in\overline{1,n}$, the following equalities hold:

$$E_{a_i}^{(\lambda_1,\lambda_2)}=1,\quad P_{a_i,a_j}^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}=[\lambda_1<\lambda_3]\vee[\lambda_1=\lambda_3][\lambda_2<\lambda_4].$$

Now for each $k=\overline{1,n}$ we delete all occurrences of the letter a_k in the word W whose labels belong to the set $D(a_k)\backslash M_k$. Thus, we get the labeled word W_0 for which $D(a_k)=M_k$, $k\in\overline{1,n}$. Using the same argument as in the proof of Theorem 6.1 we claim that the equalities for $E_{a_i}^{(\lambda_1,\lambda_2)}$ and $P_{a_i,a_j}^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}$ stay the same as we go from W to W_0.

Suppose $(a_{i_1},\dots,a_{i_{w(q_j)}})$ is the parenthesis-free notation of the commutator q_j. From Lemma 5.3 it follows that the existence condition $E_{q_j}^{\Lambda}$ is expressed by an L-condition on the set $D(q_j)=D(a_{i_1})\times\cdots\times D(a_{i_{w(q_j)}})=M_{i_1}\times\cdots\times M_{i_{w(q_j)}}$, where the tuple $\Lambda=(\lambda_1,\dots,\lambda_{2w(q_j)})$ is such that

$$(\lambda_1,\lambda_2)\in M_{i_1},\ (\lambda_2,\lambda_3)\in M_{i_2},\ \dots,\ (\lambda_{2w(q_j)-1},\lambda_{2w(q_j)})\in M_{i_{w(q_j)}}.$$

Introduce the following notation:

$$\Lambda_1=(\lambda_1,\lambda_3\dots,\lambda_{2w(q_j)-1}),\qquad \Lambda_2=(\lambda_2,\lambda_4\dots,\lambda_{2w(q_j)}).$$

Let us arbitrarily fix $\lambda_2,\lambda_4,\dots,\lambda_{2w(q_j)}$ in $E_{q_j}^{\Lambda}$. Denote the obtained predicate by

$$P_{\Lambda_2}^{\Lambda_1},\quad \text{where } \Lambda_1\in M_{i_1}(\lambda_2)\times\cdots\times M_{i_{2w(q_j)-1}}(\lambda_{2w(q_j)}).$$

From the equalities for E_{a_i} and P_{a_i,a_j} it follows that $P_{\Lambda_2}^{\Lambda_1}$ contains at most logical constants 0, 1 and predicates of the form $[\lambda_u<\lambda_v]$, $[\lambda_u=\lambda_v]$, where λ_u,λ_v are variables. In other words, for any fixed Λ_2 the predicate $P_{\Lambda_2}^{\Lambda_1}$ is expressed by an L-condition on the set $M_{i_1}(\lambda_2)\times\cdots\times M_{i_{2w(q_j)-1}}(\lambda_{2w(q_j)})$.

Thus, using Proposition 3.4 and Lemma 5.4, we obtain

$$\begin{aligned} e_j&=|\{\Lambda\in D(q_j)\mid E_{q_j}^{\Lambda}=1\}|\\ &=\sum_{\Lambda_2}\left|\{\Lambda_1\in M_{i_1}(\lambda_2)\times\cdots\times M_{i_{2w(q_j)-1}}(\lambda_{2w(q_j)})\mid P_{\Lambda_2}^{\Lambda_1}=1\}\right|\\ &=\sum_{\Lambda_2}\sum_{t=1}^{w(q_j)}c_t(\Lambda_2)\binom{|M(\Lambda_2)|}{t}=\sum_{t=1}^{w(q_j)}\sum_{\Lambda_2}c_t(\Lambda_2)\binom{|M(\Lambda_2)|}{t}. \end{aligned}$$

□

Example 6.10 (The formula from [7] (see Introduction))**.** *Let $W_0\equiv(a_{k_1}\dots a_{k_s})^m$, where $m\in\mathbb{N}$. Then we put*

$$M_k=\{1,\dots,m\}\times\{j\mid k_j=k\},\quad k\in\overline{1,n}.$$

Therefore, for any μ the set $M_k(\mu)$ is equal to $\{1,\dots,m\}$ or $\emptyset$. Then for any $\mu_1,\dots,\mu_{w(q_j)}$ the intersection $M=M(\mu_1,\dots,\mu_{w(q_j)})$ is equal to $\{1,\dots,m\}$ or $\emptyset$, so $|M|$ is equal to m or 0. Thus, if we collect commutators in order of increasing weights, then we get the following collection formula in the free group $F=F(a_1,\dots,a_n)$:

$$(a_{k_1}\dots a_{k_s})^m=q_1^{e_1}\dots q_{j(c)}^{e_{j(c)}}\pmod{\Gamma_c(F)},\quad c\in\mathbb{N},$$

where e_j is divisible by m if m is a prime power number p^α and $w(q_j)<p$.

Example 6.11. *Let* $W_0 \equiv a_l^r \left(a_{k_1} \dots a_{k_s}\right)^m$, *where* $l \in \overline{1,n}$, $m, r \in \mathbb{N}$. *Then we put*

$$M_k = \{r+1, \dots, m+r\} \times \{j \mid k_j = k\}, \quad k \neq l;$$

$$M_l = \Big(\{1, \dots, m+r\} \times \{j'\}\Big) \cup \Big(\{r+1, \dots, m+r\} \times \{j \mid k_j = l\}\Big);$$

where j' *is such that* $k_{j'} = l$. *Therefore, for any* μ *the set* $M_k(\mu)$, $k \neq l$, *is equal to* $\{r+1, \dots, m+r\}$ *or* $\emptyset$. *Assume that* $w(q_j) \geqslant 2$, *hence* $w_{a_k}(q_j) \geqslant 1$ *for some* $k \neq l$. *Then for any* $\mu_1, \dots, \mu_{w(q_j)}$ *the intersection* $M = M(\mu_1, \dots, \mu_{w(q_j)})$ *is equal to* $\{r+1, \dots, m+r\}$ *or* $\emptyset$, *so* $|M|$ *is equal to* m *or 0. Thus, we get the following collection formula in the free group* $F(a_1, \dots, a_n)$:

$$a_l^r \left(a_{k_1} \dots a_{k_s}\right)^m = q_1^{e_1} \dots q_j^{e_j} T_j,$$

where e_j *is divisible by* m *if* m *is a prime power number* p^α *and* $1 < w(q_j) < p$.

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