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ON THE MODELING OF STATIONARY SEQUENCES USING THE INVERSE DISTRIBUTION FUNCTION

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ABSTRACT. We study a method for modeling stationary sequences, which is implemented generally speaking by a nonlinear transformation of Gaussian noise. The paper establishes limit theorems in the metric space $D[0,1]$ for normalized processes of partial sums of sequences obtained as a result of the mentioned Gaussian noise transformation. Application of this method for simulating function words in fiction is investigated.

Keywords: modeling of stationary processes, long-range dependence, limit theorems, function words in fiction.

1. Introduction.

Under broad assumptions, a stationary sequence can be represented by a moving average formed on the basis of a non-random square-summable sequence and white noise, which provides a method for modeling such sequences. More precisely, let $\{X_j,\ j \in \mathbb{Z}\}$ be a stationary sequence of random variables that has a spectral density; then, the representation

$$X_j = \sum_{k=-\infty}^{\infty} a_{j-k}\xi_k, \tag{1}$$

holds, in which $\{\xi_k,\ k \in \mathbb{Z}\}$ is a sequence of uncorrelated random variables with zero mean and unit variance (white noise) and $\{a_k,\ k \in \mathbb{Z}\}$ is a non-random square-summable sequence of real numbers. Note that representation (1) provides a method for modeling stationary sequences.

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However, if a sample implementing a stationary time series is known, we have the problem of modeling a stationary sequence that inherits the properties of the original time series. A method for solving this problem is the method of inverse distribution function, which makes it possible to model such sequences while preserving the covariance function and the marginal distribution of the original series. The basic properties and a theoretical justification of this method were given in [1, 2]. This paper continues these studies. In particular, the application of this method for modeling stationary sequences with strong dependence is studied. At the same time, for the covariance sequence of the original time series, the case of divergence in absolute value and the case of absolute convergence are distinguished. In the former case (if the covariance function has a regular behavior at infinity), the C-convergence to the fractional Brownian motion is proved (see Theorem 2). In the latter case, a one-dimensional limit theorem for the corresponding normalized processes of the partial sums of the sequence modeling the original stationary sequence is obtained (see Theorem 3). Note that the proven convergence statements are based on the limit theorems from [3, 4].

To illustrate the results obtained in this paper, we examine the stationarity of the time series consisting of indicators of function words in Dostoevsky's novel *Crime and Punishment* [5] and construct a model of this series using the method of inverse distribution function (see Section 4).

Throughout the paper, the indices j, k can range over all integer values. By a sequence, say $\{a_j\}$, we mean the set $\{a_j,\ j \in \mathbb{Z}\}$, where any letter can be used instead of a.

2. Transformation of Gaussian Noise.

2.1. **Preliminaries.** Let $\{X_k\}$ be a stationary (in a broad sense) sequence of identically distributed random variables with the covariance function $\gamma(k)$. Denote by F the cumulative distribution function (cdf) of the random variable X_1 (further we use the convention that F is left-continuous). We assume that F specifies a non-degenerate distribution with a finite second moment and zero mean.

Let $\{z_k\}$ be a stationary sequence of *standard normal random variables* (Gaussian noise) with the covariance function $\rho = \rho(k)$. Along with $\{X_k\}$, we consider the sequence

$$Y_k := F^{-1}(\Phi_{0,1}(z_k)), \tag{2}$$

where $\Phi_{0,1}$ is the cdf of the standard normal law and F^{-1} is the *quantile transform* of the function F defined by $F^{-1}(t) = \inf\{x : F(x) \geq t\}$.

Below, we will need the following definition. A sequence of real numbers $\{\beta(j)\}$ is said to be *positive definite* if $\beta(j) = \beta(-j)$ for all $j = 0, 1, \dots$ and, for each $n = 1, 2, \dots$, the matrix $A_n = (a_{ij})_{n\times n}$, where $a_{ij} = \beta(i-j)$, is positive definite (e.g., see [6]).

We will say that $\{X_k\}$ is modeled by the method of inverse distribution function if there exists a Gaussian sequence $\{z_k\}$ such that

$$\mathbf{E}(Y_0 Y_k) = \gamma(k) \tag{3}$$

for all $k \in \mathbb{Z}$ (recall that $\gamma(k) = \mathbf{E}(X_0 X_k)$).

The existence of $\{z_k\}$ can be analyzed using the following algorithm:

1) equation (3) is solved for $\rho(k)$;
2) the positive definiteness of $\{\rho(k)\}$ is checked;

3) if the check at the preceding step is successful, then $\{z_k\}$ is reconstructed given the matrices $A_n = (a_{ij})_{n\times n}$, where $a_{ij} = \rho(i-j)$ (e.g., see [7, Chapter, §13]).

2.2. **Investigation of the algorithm of the inverse distribution function method.** Define the random variables Z_1 and Z_2 as follows:

$$Z_1 := F^{-1}(\Phi_{0,1}(z)), \quad Z_2 := F^{-1}(\Phi_{0,1}(w)), \tag{4}$$

where (z, w) is a Gaussian vector with standard normal components and correlation coefficient r. Consider the function

$$R(r) := \mathbf{E}(Z_1 Z_2). \tag{5}$$

It is well known that the random variables $\nu_1 = \Phi_{0,1}(z)$ and $\nu_2 = \Phi_{0,1}(w)$ are uniformly distributed on $[0,1]$, so the variables $F^{-1}(\nu_1)$ and $F^{-1}(\nu_2)$ have the common cdf F, and therefore have the same distribution as X_1. These facts imply, in particular, the following relations

$$R(1) = \int_{-\infty}^{+\infty} (F^{-1}(\Phi_{0,1}(x))^2 \varphi_{0,1}(x)\, dx < +\infty \tag{6}$$

and

$$R(-1) = \int_{-\infty}^{+\infty} F^{-1}(\Phi_{0,1}(x)) F^{-1}(1-\Phi_{0,1}(x)) \varphi_{0,1}(x)\, dx, \tag{7}$$

where $\varphi_{0,1}(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$. These equalities immediately imply that

$$R(1) = \int_0^1 (F^{-1}(u))^2\, du$$

and $R(-1) = \int_0^1 F^{-1}(u) F^{-1}(1-u)\, du$.

Theorem 1. *Let the cdf F specify a distribution with zero mean and unit variance. Then, the function R have the following properties*:
1) *R is continuous on $[-1,1]$ and analytic on $(-1,1)$.*
2) *R is strictly increasing on $[-1,1]$.*
3) *It holds that $c := \int_{-\infty}^{\infty} x F^{-1}(\Phi_{0,1}(x)) \varphi_{0,1}(x)\, dx > 0$ and $R(r) \sim c^2 r$ as $r \to 0$.*
4) *For all $r \in [-1,1]$, the inequality $|R(r)| \le |r|$ holds. Moreover, if the distribution specified by F differs from the standard normal distribution, then for all $r \in (-1,0) \cup (0,1)$ satisfy $|R(r)| < |r|$.*

The monograph [1] considers the situation when the function R is formed by two, generally speaking, different distribution functions, in this paper we consider the case of one distribution function (see (4) and (5)). In this case items 1 and 2 of Theorem 1 improve items 2—4 of Lemma 3.3 from [1]. In particular, we establish the strict monotonicity of the function R (in item 4 of Lemma 3.3 of [1], the non-strict monotonicity of R is proved) and remove the restriction on the continuity of F.

In [1, Section 3.2], the following inequality is presented without proof: $|R(r)| \le |r|$ for all $r \in [-1,1]$, which corresponds to the first part of item 4 of Theorem 1. We note that the second part of item 4 is of primary value for the present work; nevertheless, the first part of this item, for the sake of completeness, is proved in Section 5.2.

The properties of R formulated in Theorem 1 will play a central role in what follows.

The next remark shows the solvability of the equation (3).

Remark 1. The monotonicity of R implies that the following inequalities hold for all $r \in [-1, 1]$

$$\int_0^1 F^{-1}(u)F^{-1}(1-u)\,du \le R(r) \le 1.$$

Now let T_1 and T_2 be arbitrary identically distributed random variables given on the same probability space, with $T_1 \sim F$. It holds that (e.g., see [8, Theorem 2.5], [1, Lemma 3.1])

$$\int_0^1 F^{-1}(u)F^{-1}(1-u)\,du \le \mathbf{E}(T_1T_2) \le 1.$$

This immediately implies that the quantity $R^{-1}(t_0)$, where $t_0 = \mathbf{E}(T_1T_2)$, is defined.

Remark 2. Let t_0 satisfy the relations: $\int_0^1 F^{-1}(u)F^{-1}(1-u)\,du \le t_0 \le 1$. Then from items 2 and 4 of Theorem 1 one can obtain the following inequalities: if $t_0 \ge 0$, then $t_0 \le R^{-1}(t_0) \le 1$; if $t_0 < 0$, then $-1 \le R^{-1}(t_0) \le t_0$. These inequalities can be used for the numerical search for $R^{-1}(t_0)$.

In the next proposition, in particular, we obtain an explicit representation of the function R in the case when the distribution of the stationary time series to be modeled is discrete. Note that, if the distribution of the sample realizing the stationary time series is not known, then a discrete approximation of this distribution may be used for the modeling by the method of inverse distribution function and the obtained representation may also be used (e.g., see [9]).

Proposition 1. *Let the cdf F be such that there exist finite a and b satisfying $F(b) - F(a) = 1$ (recall that F has zero mean). Then, the function R can be represented in the form*

$$R(r) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi_r(x,y)\,dF^{-1}(\Phi_{0,1}(x))\,dF^{-1}(\Phi_{0,1}(y)) - (F^{-1}(1))^2,$$

where $\Phi_r(x,y)$ is the joint cdf of the two-dimensional Gaussian vector with standard normal components and the correlation coefficient r. In particular, if F specifies a discrete distribution with atoms at points $a_1 < a_2 < \cdots < a_n$, $n \ge 2$. Then, it holds that

$$R(r) = 2\sum_{1\le i<j\le n-1}(a_{i+1}-a_i)(a_{j+1}-a_j)f_{ij}(r) + \sum_{i=1}^{n-1} f_{ii}(r)(a_{i+1}-a_i)^2 - a_n^2,$$

where $q_1 := F(a_1+0), \ldots, q_{n-1} := F(a_{n-1}+0)$, $f_{ij}(r) := \Phi_r(\Phi_{0,1}^{-1}(q_i), \Phi_{0,1}^{-1}(q_j))$.

Remark 3. Note that the function R is especially simple for the two-point distribution in which $q_1 = 1/2$ because in this case we have $f_{11}(r) = \frac{1}{4} + \frac{1}{2\pi}\arcsin(r)$ (e.g., see [10]). Therefore, the function R can be represented as $R(r) = \frac{2a_1^2}{\pi}\arcsin(r)$.

3. Limit Theorems.

3.1. **Preliminaries.** Denote by $B_H(t)$ the *fractional Brownian motion*, i.e., the centered Gaussian process with the covariance function (see [11])

$$R(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

where $H \in (0,1)$ (the case $H = 1/2$ corresponds to the ordinary Brownian motion). In particular, note that the equality $\mathbf{E}B_H^2(t) = t^{2H}$ holds. If $H < 1/2$ ($H > 1/2$), then in this case we will speak about subdiffusion (superdiffusion) transport regime. Accordingly, in the case of $H = 1/2$, we will speak about a diffusion transport regime.

For a real number x, let $[x]$ denote the largest integer not exceeding x.

Without loss of generality, in what follows we will assume that F specifies a distribution with *zero mean and unit variance.*

3.2. **The case of absolute divergence of the covariance sequence.** If $\{X_j\}$ can be modeled using the method of inverse distribution function, then the following proposition explains the asymptotic properties of the corresponding Gaussian sequence. Furthermore, this proposition formulates asymptotic properties of $\{Y_j\}$, see (2).

Theorem 2. *Let the covariance function of $\{X_j\}$ satisfy the relation $\gamma(j) = L(j)j^{2H-2}$ as $j \to +\infty$, where L is a slowly varying function on $+\infty$ and $H \in (1/2, 1)$. Moreover, let $\{\rho(j)\}$, where $\rho(j) = R^{-1}(\gamma(j))$, be positive definite. Then*

i) *There exists a Gaussian sequence $\{z_j\}$ with the covariance function $\rho = \rho(j)$ satisfying the relation*:

$$\mathbf{D}(\sum_{j=1}^{n} z_j) \sim \frac{L(n)}{c^2} H^{-1}(2H-1)^{-1} n^{2H}, \ n \to \infty, \tag{8}$$

where $c = \int_{-\infty}^{\infty} xF^{-1}(\Phi_{0,1}(x))\varphi_{0,1}(x)\,dx$ (see Theorem 1). As $n \to \infty$, the processes $\frac{\sum_{j=1}^{[nt]} z_j}{\sqrt{L(n)c^{-2}H^{-1}(2H-1)^{-1}n^H}}$ C-converge in $D[0,1]$ to $B_H(t)$.

ii) *The sequence $\{Y_j\}$ satisfies the relation*:

$$\mathbf{D}(\sum_{j=1}^{n} Y_j) \sim L(n)H^{-1}(2H-1)^{-1} n^{2H}, \ n \to \infty \tag{9}$$

and, in addition, the processes $\frac{\sum_{j=1}^{[nt]} Y_j}{\sqrt{L(n)H^{-1}(2H-1)^{-1}n^H}}$ C-converge in $D[0,1]$ to $B_H(t)$ as $n \to \infty$.

Recall that by C-convergence in $D[0,1]$ we mean the weak convergence of distributions of measurable (in Skorokhod's topology) functionals on $D[0,1]$ that are continuous in the uniform topology at the points of the space $C[0,1]$ (e.g., see [12]).

Remark 4. The sequence $\{Y_j\}$ provides a model of the sequence $\{X_j\}$, and the corresponding normalized processes of its partial sums C-converge to B_H in $D[0,1]$. Note that the original sequence $\{X_j\}$ does not generally have this property. However, if $\{X_j\}$ can be represented by a moving average formed on the basis of i.i.d. random variables and if the covariance function of this sequence tends to 0 and has regular behavior at infinity, then the C-convergence mentioned above also holds for $\{X_j\}$. Indeed, let $\{X_j\}$ can be represented by $X_j = \sum_{k=-\infty}^{\infty} a_{j-k}\xi_k$, where $\{\xi_k\}$ is a sequence of i.i.d. random variables with zero mean and unit variance, $\{a_k\}$ be a non-random square-summable sequence of real numbers, and $\gamma(j) \sim bj^{2H-2}$, $j \to +\infty$, where $b > 0$ and $H \in (1/2, 1)$. Then, for $n \to \infty$, the processes $\frac{\sum_{j=1}^{[nt]} X_j}{\sqrt{bH^{-1}(2H-1)^{-1}n^H}}$ C-converge to $B_H(t)$ in $D[0,1]$ (see [13, Corollary 1]).

Theorem 2 illustrates the stability property of noise with the superdiffusion transport regime ($H > 1/2$) under the transform $F^{-1} \circ \Phi_{0,1}$; i.e., the Gaussian noise $\{z_j\}$ with the superdiffusion transport regime is transformed into a generally speaking non-Gaussian sequence $\{Y_j\}$ with the same transport regime. Below, we show that the subdiffusion noise ($H < 1/2$) does not possess such a property (see Proposition 2).

3.3. **The case of absolute summability of a covariance sequence.** Next, we consider a sequence $\{X_j\}$ the covariance function of which satisfies the condition $\sum_{j=-\infty}^{+\infty} |\gamma(j)| < +\infty$. Note that in this case the spectral density $f(\lambda)$ satisfies $f(\lambda) = \frac{1}{2\pi}\sum_{j=-\infty}^{+\infty} e^{-i\lambda j}\gamma(j)$; therefore it holds that $\sum_{j=-\infty}^{+\infty}\gamma(j) \geq 0$.

Theorem 3 gives asymptotic properties of the sequence $\{Y_j\}$, which provides a model of the original sequence $\{X_j\}$.

Theorem 3. *Let $\{\rho(j)\}$, where $\rho(j) = R^{-1}(\gamma(j))$, be positive definite. Then, the convergence in distribution*
$\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \xrightarrow{d} \sqrt{g}\mathcal{N}(0,1)$, *where $g := \sum_{j=-\infty}^{+\infty}\gamma(j)$, holds.*
In addition,
i) *if $g > 0$, then $\mathbf{D}(\sum_{j=1}^n Y_j) \sim gn,\ n \to \infty$;*
ii) *if $g = 0$ and $\gamma(j) \sim aj^{2H-2},\ j \to +\infty$ for certain $a < 0$ and $H \in (0, 1/2)$, then $\mathbf{D}(\sum_{j=1}^n Y_j) \sim aH^{-1}(2H-1)^{-1}n^{2H},\ n \to \infty$.*

Remark 5. Let $\{X_j\}$ satisfy the condition of Theorem 3, as well as its item ii). Consider the degenerate case when this sequence is Gaussian (in this case, $\{X_j\}$ coincides with $\{Y_j\}$). First, note that $\{X_j\}$ has a spectral density; therefore, this sequence can be obtained using moving average (e.g., see [14]). Whence, there exist a square-summable sequence $\{a_j\}$ and a sequence $\{\varepsilon_j\}$ that is a white noise such that $X_j = \sum_{k=-\infty}^{+\infty} a_{j-k}\varepsilon_k$. Moreover, since $\{X_j\}$ is a Gaussian sequence, then $\{\varepsilon_j\}$ may be assumed to be a sequence of independent standard normal random variables. Then, [15, Proposition 1] implies that the processes $\frac{\sum_{j=1}^{[nt]} X_j}{\sqrt{aH^{-1}(2H-1)^{-1}n^H}}$ C-converge to $B_H(t)$ in $D[0,1]$ as $n \to \infty$.

The centered Gaussian sequence is called a *fractional noise* with the parameter $H \in (0,1)$ and variance δ^2 (e.g., see [16]) if its covariance function $\rho = \rho(j)$ has the form

$$\rho(j) = \frac{\delta^2}{2}(|j+1|^{2H} + |j-1|^{2H} - 2|j|^{2H}). \tag{10}$$

Consider properties of $\{Y_j\}$ formed from the fractional noise with the parameter $H < 1/2$.

Proposition 2. *Let $\{z_j\}$ be a fractional noise with the parameter $H < 1/2$, unit variance, and the covariance function $\rho(j)$. Let, in addition, the distribution specified by F be different from the standard normal one. Then, the covariance function $R(\rho(j))$ of the sequence $\{Y_j\}$ is absolutely summable and*

$$\sum_{j=-\infty}^{+\infty} R(\rho(j)) > 0.$$

Thus, if $\{Y_j\}$ is formed from the fractional noise with the parameter $H < 1/2$, then Theorem 3 implies that the noise with the subdiffusion transport regime ($H < 1/2$) deforms (with respect to $F^{-1} \circ \Phi_{0,1}$) into a noise with the diffusion regime ($H = 1/2$).

4. Estimate of Stationarity and Simulation Example.

Let $\{X_j\}$ be a stationary sequence of identically distributed random variables such that X_1 has the Bernoulli distribution with the parameter p. As a numerical implementation of such a sequence, we consider the time series of indicators of function words in Dostoevsky's novel *Crime and Punishment*—the value 1 is chosen for function words, and 0 is chosen otherwise. Let us explain such a choice of numerical implementation. Function words are words that have little lexical meaning and express grammatical relationships among other words within a sentence or parts of a sentence. It is noted in [17] that the frequency of occurrence of function words in the works by a certain author is stable and characterizes the style of this author. Therefore, we may assume that the occurrence of function words in a text is a stationary process. Next we give a stationarity test.

Let $\{X_j^*\}$ be an arbitrary stationary sequence of random variables with zero mean and finite variance. Define $S_n := \sum_{i=1}^n X_j^*$. We assume that $\mathbf{D}S_n \sim \sigma^2 n^{2H}$, $n \to \infty$. Under broad assumptions, finite-dimensional distributions of the processes $\frac{S_{[nt]}}{\sigma n^H}$, $t \in [0,1]$ converge to the corresponding finite-dimensional distributions $B_H(t)$, $t \in [0,1]$ as $n \to \infty$ (see [13]). Next, we propose a test for verifying the *main hypothesis of stationarity and power-like behavior of variance* based on the closeness of the joint distribution $(\frac{S_\tau}{\sigma\tau^H}, \frac{S_{2\tau}-S_\tau}{\sigma\tau^H}, \ldots, \frac{S_{k\tau}-S_{(k-1)\tau}}{\sigma\tau^H})$ to the joint distribution $(B_H(1), B_H(2) - B_H(1), \ldots, B_H(k) - B_H(k-1))$ as $\tau \to \infty$.

Consider a finite sample $(X_j^*,\ j = 1, \ldots, n)$ realizing $\{X_j^*\}$. Form the sample $(X_j^{(\tau)},\ j = 1, \ldots, [n/\tau])$, where $X_j^{(\tau)} = \sum_{i=(j-1)\tau+1}^{j\tau} X_i^*$. Note that the value of τ must be sufficiently large ($\tau >> 1$) to "gather normality". We will test the hypothesis that this sample is a fractional noise with variance $\sigma^2\tau^{2H}$. Estimates for the parameters σ and H can be obtained using known methods, e.g., the method of variances (see [18]).

Let R be the covariance matrix of the fractional noise. There exists an orthogonal matrix C and a diagonal matrix D such that $C^T RC = D$. Define $B = \sqrt{D}$. Note that the product $(B^{-1}C^T)(X_j^{(\tau)})^T$ gives the sample that can be tested for standard normality (e.g., see [7]); for this purpose, it is natural to use the parametric χ^2 test. To ensure the validity of using the χ^2 test, $[n/\tau]$ must be sufficiently large, i.e., $\tau << n$. The actually achieved significance level of this test gives an estimate of the closeness of the sample $(X_j^{(\tau)})$ to the fractional noise and shows the significance level at which the *main hypothesis* may be accepted.

Now, we give more detailed descriptions of the computational procedures mentioned above.

Method:
1. Given the sample $(X_j,\ j = 1, \ldots, n)$, find the estimates σ_n and H_n of the parameters σ and H, respectively (the method of variances may be used).
2. Using $a_n = \frac{1}{n}\sum_{i=1}^n X_i$, center the sample $(X_k,\ k = 1, \ldots, n)$ to obtain $X_k^* = X_k - a_n,\ k = 1, \ldots, n$.
3. Form the sample $X_j^{(\tau)} = \sum_{i=(j-1)\tau+1}^{j\tau} X_i^*,\ j = 1, \ldots, [n/\tau]$.
4. Define the covariance matrix R of the fractional noise (see (10)), where $\delta_n = \sigma_n \tau^{H_n}$.
5. Using the matrix R, find the matrices B and C. Multiply $B^{-1}C^T$ by the vector

$(X_j^{(\tau)},\ j=1,\ldots,[n/\tau])^T$ to obtain the sample $(\eta_i,\ i=1,\ldots,[n/\tau])$.
6. On the sample realization $(\eta_i,\ i=1,\ldots,[n/\tau])$, find the actually achieved significance level of the χ^2 test under the basic hypothesis that the sample has the standard normal distribution.
6.1. Let $m=[n/\tau]$. Divide the number line into $k=[\sqrt{m}]+1$ non-overlapping intervals $\Delta_1,\Delta_2,...,\Delta_k$ so that $m\Phi_{0,1}(\Delta_1)=m\Phi_{0,1}(\Delta_2)=...=m\Phi_{0,1}(\Delta_k)=m/k$. On the sample realization $(\eta_i,\ i=1,\ldots,[n/\tau])$, find the value of the test statistic $X^2=\sum_{i=1}^k(\frac{\nu_i-m/k}{m/k})^2$, where ν_i is the number of sample elements (η_i) in Δ_i, $i=1,\ldots,k$.
6.2. Keeping in mind the three parameters a, σ, and H to be estimated, find the actually achieved significance level of the χ^2 test, namely, $\varepsilon(\tau)=1-\chi^2_{k-4}(X^2)$.

The disadvantages of this test are as follows: 1) stationarity is checked for the smoothed sample $(X_j^{(\tau)},\ j=1,\ldots,[n/\tau])$, which reduces the power of the test; 2) there is the problem of ambiguity of choosing the parameter τ.

Thus, we have the sample of indicators of function words $(X_j,\ j=1,\ldots,n)$ of size $n=172586$. The estimates of the parameters σ, H, and a (a coincides with p) are, respectively, $\sigma_n=0.32$, $H_n=0.60$, and $p_n=0.25$. We have already mentioned that τ must satisfy the relations $1<<\tau<<n$. Taking into account these relations, we consider two variants for the values of τ: 1) $\tau_1=[\sqrt{n}]$; 2) $\tau_2=2[\sqrt{n}]$ ($\tau_1=415$, $\tau_2=830$). The actually achieved level of significance of the stationarity test in the first case is $\varepsilon(\tau_1)=0.002$ and, accordingly, in the second case is $\varepsilon(\tau_2)=0.16$. Therefore, we obtain fairly high levels of agreement between the main hypothesis of stationarity, as well as the power-like behavior of the variance, and real data.

For the further simulation using the method of inverse distribution function, we center and normalize the sample $(X_j,\ j=1,\ldots,n)$ using p_n and $\sqrt{p_n(1-p_n)}$. Then we obtain $(X_j^*,\ j=1,\ldots,n)$ (where $X_j^*=(X_j-p_n)/\sqrt{p_n(1-p_n)}$). Next, we calculate the estimate of the covariance function $\lambda_n(l)=\frac{1}{n-l}\sum_{k=1}^{n-l}X_k^*X_{k+l}^*$, where $l=0,\ldots,m$. In this paper, $m=3000$ was used.

At the next step, the sequence $\gamma_n(l)=R^{-1}(\lambda_n(l))$, $l=0,\ldots,m$, is numerically found, for which purpose the representation of the function R obtained in Proposition 1 is used (where, as the function F, we apply the empirical distribution function F_n of the sample $(X_j^*,\ j=1,\ldots,n)$). Next, positive definiteness of this sequence is verified (the matrix $G_n=(\gamma_n(|i-j|))_{i,j=1,\ldots,m+1}$ is considered). Note that in this case the sequence is strictly positive definite, which allows us to simulate the Gaussian sequence $g_n=(g_n(k),\ k=1,\ldots,m+1)$ for which G_n is the covariance matrix. At the final step, we obtain $Y_k=\sqrt{p_n(1-p_n)}F_n^{-1}(\Phi_{0,1}(g_n(k)))+p_n$, $k=1,\ldots,m+1$ (for details see [9, Method 3]).

5. Proofs.

5.1. **Preliminaries on Chebyshev-Hermite polynomials.** The Chebyshev-Hermit polynomial of degree k is defined by $H_k(x)=\frac{(-1)^k}{\varphi_{0,1}(x)}\frac{d^k\varphi_{0,1}(x)}{dx^k}$. The space of functions G such that

$$\int_{-\infty}^{+\infty}G^2(x)\varphi_{0,1}(x)\,dx<+\infty \tag{11}$$

is a Hilbert space with the scalar product $(G, H) = \int_{-\infty}^{+\infty} G(x)H(x)\varphi_{0,1}(x)\,dx$. The Chebyshev-Hermit polynomials form an orthogonal basis in this space.

Note that (e.g., see [19, Theorem 3.1])

$$\mathbf{E}H_k(z)H_l(w) = \delta_{kl} r^k k!, \tag{12}$$

where δ_{kl} is the Kronecker symbol, z and w are jointly Gaussian random variables such that $z, w \sim \mathcal{N}(0,1)$, $\mathbf{E}(zw) = r$. In particular, $\mathbf{E}H_k(z)H_l(z) = \delta_{kl}k!$. Therefore, according to (11), we have the expansion

$$G(x) = \sum_{k=0}^{+\infty} c_k H_k(x), \tag{13}$$

and the relation $\sum_{k=0}^{+\infty} k! c_k^2 < +\infty$ holds. Due to (6), the function $F^{-1}(\Phi_{0,1}(\cdot))$ satisfies (13).

Note that $m := \min\{k : c_k \neq 0\}$ is called the *Hermitian rank* of the function G.

5.2. **Proof of Theorem 1.** 1) Relations (12) and (13) imply that, for all $r \in [-1, 1]$, the function R can be represented by the power series

$$R(r) = \sum_{k=0}^{+\infty} k! c_k^2 r^k, \tag{14}$$

where $c_k = \frac{(-1)^k \int_{-\infty}^{+\infty} F^{-1}(\Phi_{0,1}(x))\varphi_{0,1}^{(k)}(x)\,dx}{k!}$. This expansion in the power series and its convergence for $r = 1$ imply the continuity of $R(r)$ on the entire interval $[-1, 1]$ and, in addition, the analyticity on $(-1, 1)$.

2) Relation (14) implies

$$c_1 = \int_{-\infty}^{+\infty} xF^{-1}(\Phi_{0,1}(x))\varphi_{0,1}(x)\,dx. \tag{15}$$

Let us show that $c_1 > 0$. It is clear that $c_1 = \mathbf{E}zF^{-1}(\Phi_{0,1}(z))$, where $z \sim \mathcal{N}(0, 1)$. Denote by μ the median of the distribution specified by F (this means that $F(\mu + 0) \geq 1/2$ and $F(\mu) \leq 1/2$). Note that the distribution specified by $F_0(x) := F(x + \mu)$, has a zero median, and it holds that $F^{-1}(\Phi_{0,1}(z)) = \mu + F_0^{-1}(\Phi_{0,1}(z))$ almost surely (a.s.); therefore, $\mathbf{E}zF^{-1}(\Phi_{0,1}(z)) = \mathbf{E}zF_0^{-1}(\Phi_{0,1}(z))$.

Let $z \geq 0$, then $\Phi_{0,1}(z) \geq 1/2$ and, therefore, due to the fact that 0 is the median for F_0, we obtain the inequality $F_0^{-1}(\Phi_{0,1}(z)) \geq 0$. If now $z < 0$, then $\Phi_{0,1}(z) < 1/2$, which immediately implies that $F_0^{-1}(\Phi_{0,1}(z)) \leq 0$. Combining these two cases, we conclude that $zF_0^{-1}(\Phi_{0,1}(z)) \geq 0$ a.s. Assume that $\mathbf{E}zF_0^{-1}(\Phi_{0,1}(z)) = 0$; then, taking into account the preceding inequality, we obtain that $zF_0^{-1}(\Phi_{0,1}(z)) = 0$ a.s.; however, this implies that $F_0^{-1}(\Phi_{0,1}(z)) = 0$ a.s., which contradicts the non-degeneracy of the distribution defined by F. Finally, we conclude that $c_1 > 0$, which implies that $R'(r) = \sum_{k=1}^{+\infty} k! c_k^2 k r^{k-1} \geq c_1^2 > 0$ on the interval $[0, 1)$.

Continue R' to the set of complex numbers $\mathbb{C}$ by defining

$$R'(r) = \sum_{k=1}^{+\infty} k! c_k^2 k r^{k-1}, \quad r \in \mathbb{C}.$$

Note that the complex function $R'(r)$ is analytic on $D = \{r \in \mathbb{C} : |r| < 1\}$. Furthermore, $R'(r)$ is not identically equal to zero in this domain (see above); therefore, the zeros of this function are isolated points (e.g., see [20, Theorem 3.2.8]).

It was proved in [1, Lemma 3.3] that R is a not strictly increasing function on the interval $[-1,1]$. Assume that there exists an interval $[a,b] \subset (-1,1)$ on which R is constant. Then, $R'(r)=0$ on $[a,b]$, which contradicts the fact that the zeros of R' are isolated. Finally, we conclude that R is strictly increasing on $[-1,1]$.

3) Since $c_0=0$, (14) immediately implies that $R(r) \sim c_1^2 r,\ r \to 0$. Above, we have proved that $c_1>0$; therefore, it remains to set $c := c_1$.

4) Let us prove the first part of this item. Consider the function $\tau(r) := \frac{R(r)}{r}$ on $[0,1]$ (the value of $\tau(\cdot)$ at zero is set to c^2). From (14) we get that $\tau'(r)$ is non-negative on $(0,1)$. This implies that the function τ is increasing on $[0,1]$. Note that $\tau(1)=1$, so $R(r) \le r$ for all $r \in [0,1]$. Obviously, for all $r \in [-1,1]$, the inequality $|R(r)| \le R(|r|)$ is satisfied, so $|R(r)| \le |r|$ for all $r \in [-1,1]$.

Now let the distribution specified by F differ from the standard normal one. Let us show that there is a $k \ge 2$ in expansion (14) such that $c_k \ne 0$. Assume that this is not the case. Then, $R(r) = c_1^2 r$. Therefore, taking into account that $R(1)=1$ and $c_1>0$, we conclude that $c_1=1$. Hence, $c_1 = \mathbf{E} z F^{-1}(\Phi_{0,1}(z)) = 1$, where $z \sim \mathcal{N}(0,1)$. On the other hand, we have the obvious inequality $\frac{z^2+(F^{-1}(\Phi_{0,1}(z)))^2}{2} - zF^{-1}\Phi_{0,1}(z) \ge 0$ a.s. However, since $\mathbf{E}(\frac{z^2+(F^{-1}(\Phi_{0,1}(z)))^2}{2} - zF^{-1}(\Phi_{0,1}(z))) = 0$, we have $z = F^{-1}(\Phi_{0,1}(z))$ a.s. Therefore, the distribution specified by F coincides with the standard normal distribution. This is a contradiction; therefore, there exists a $k \ge 2$ such that $c_k \ne 0$.

The fact that $c_k \ne 0$ for a certain $k \ge 2$ implies that $\tau'(r) > 0$ on the interval $(0,1)$, i.e., the function τ strictly increases on $[0,1]$. Note that $\tau(1)=1$; therefore, for all $r \in (0,1)$, it holds that $R(r) < r$. Since $|R(r)| \le R(|r|)$, we immediately obtain that $|R(r)| < |r|$ for all $r \in (-1,0)$. □

In what follows, to prove Theorem 2 and 3, we will need the following corollary of Theorem 1.

Corollary 1. *The Hermitian rank of the function* $F^{-1}(\Phi_{0,1}(\cdot))$ *is equal to* 1.

Proof. First of all, note that $c_0=0$ since $\mathbf{E}Z_1=0$ (see (4)), which implies that the Hermitian rank of the function $F^{-1}(\Phi_{0,1}(\cdot))$ at least 1. Since the first Hermite polynomial has the form $H_1(x)=x$, therefore, it is sufficient to establish that $\int_{-\infty}^{+\infty} xF^{-1}(\Phi_{0,1}(x))\varphi_{0,1}(x)\,dx \ne 0$. But above in item 3 of Theorem 1 it was found that the mentioned integral is greater than 0 (see (15)). □

5.3. **Proof of Proposition 1.** In what follows, we will need the following equality due to Hoeffding (see [21]; see, e.g., also [22, Lemma 2]).

Lemma 1 (Hoeffding)**.** *Let the random vector* (A,B) *have the joint cdf* H *with marginals* J *and* G*. Then*

$$\mathbf{E}AB - \mathbf{E}A\mathbf{E}B = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (H(a,b) - J(a)G(b))\,da\,db$$

provided the expectations on the left hand side exist.

Let us prove the first part of the proposition. Using Lemma 1, we find that the function $R(r)$ can be represented as

$$R(r) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (H(a,b) - F(a)F(b))\,da\,db, \tag{16}$$

where H is the joint cdf for $(F^{-1}(\Phi_{0,1}(z)), F^{-1}(\Phi_{0,1}(w)))$ (recall that z and w are jointly Gaussian random variables such that $z, w \sim \mathcal{N}(0,1)$ and $\mathbf{E}(zw) = r$). Making the substitutions: $a = F^{-1}(\Phi_{0,1}(x))$ and $b = F^{-1}(\Phi_{0,1}(y))$ in (16), we deduce

$$R(r) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (\Phi_r(x,y) - \Phi_{0,1}(x)\Phi_{0,1}(y))\, dF^{-1}(\Phi_{0,1}(x))\, dF^{-1}(\Phi_{0,1}(y)).$$

From where we immediately get

(17)

$$R(r) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi_r(x,y)\, dF^{-1}(\Phi_{0,1}(x))\, dF^{-1}(\Phi_{0,1}(y)) - \left(\int_0^1 x\, dF^{-1}(x)\right)^2.$$

Next, it is obvious that

$$\int_0^1 x\, dF^{-1}(x) = F^{-1}(1) - \int_0^1 F^{-1}(x)\, dx.$$

Substituting this equality into (17) (taking into account that $\int_0^1 F^{-1}(x)\, dx = 0$), we at once obtain the assertion of the first part of the proposition.

We turn to the proof of the second part. It is clear that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi_r(x,y)\, dF^{-1}(\Phi_{0,1}(x))\, dF^{-1}(\Phi_{0,1}(y))$$

$$= 2\sum_{1\le i<j\le n-1} (a_{i+1} - a_i)(a_{j+1} - a_j) f_{ij}(r) + \sum_{i=1}^{n-1} f_{ii}(r)(a_{i+1} - a_i)^2.$$

In addition, we have $F^{-1}(1) = a_n$. These two relations immediately imply the assertion of the proposition. □

5.4. **Proof of Theorem** 2. The next lemma plays the key role in the proof of Theorem 2. This lemma is a consequence of Theorems 3.1 and 5.1 from [3].

Lemma 2. *Let $\{z_j\}$ be a stationary sequence of standard normal random variables the covariance function of which satisfies the relations*
$\rho(j) \to 0,\ j \to \infty,$
$\sum_{i=1}^n \sum_{j=1}^n \rho(i-j) \sim L(n)n^{2H},\ n \to \infty,$
$\sum_{i=1}^n \sum_{j=1}^n |\rho(i-j)| = O(L(n)n^{2H}),\ n \to \infty,$
where L is a slowly varying function at $+\infty$ and $H \in (1/2, 1)$. Let, in addition, the function G satisfy the conditions: $\mathbf{E}G(z_1) = 0$, $\mathbf{E}G^2(z_1) < +\infty$, *and let the Hermitian rank of this function be* 1. *Then*

(i) $\mathbf{D}(\sum_{i=1}^n G(z_i)) \sim J^2(1)n^{2H}L(n)$ *as* $n \to \infty$, *where* $J(1) = \mathbf{E}z_1G(z_1)$;

(ii) *the processes*

$$Z_n(t) = \frac{\sum_{i=1}^{[nt]} G(z_i)}{\sqrt{L(n)}n^H}$$

C-converge in $D[0,1]$ to $J(1)B_H(t)$ as $n \to \infty$.

Proof of Theorem 2. i) For each $\gamma(j)$, $j \in \mathbb{Z}$, there exists a preimage $R^{-1}(\gamma(j))$ (see Remark 1), which is denoted by $\rho(j)$. In addition, item 3 of Theorem 1 implies that

$$R^{-1}(\gamma(j)) \sim \frac{1}{c^2}\gamma(j),\ j \to \infty.$$

Therefore,

$$\rho(j) \sim \frac{L(j)}{c^2} j^{2H-2}, \ j \to \infty. \tag{18}$$

Next, (18) and Lemma 3.1 from [3] entail

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \rho(i-j) \sim \frac{L(n)}{c^2} H^{-1}(2H-1)^{-1} n^{2H}, \ n \to \infty \tag{19}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |\rho(i-j)| = O(L(n) n^{2H}), \ n \to \infty. \tag{20}$$

Since item i) deals with a Gaussian sequence, in this case we consider the identity transformation $G(x) = x$ of this sequence. It is clear that the Hermitian rank of such a function G is 1. Relations (19) and (20) satisfy the condition of Lemma 2. Therefore, equivalence (8) holds (in this case, $J(1)$ equals 1), and the processes $\frac{\sum_{j=1}^{[nt]} z_j}{\sqrt{L(n)c^{-2}H^{-1}(2H-1)^{-1}n^H}}$ C-converge to $B_H(t)$.

ii) First, we note that the Hermitian rank of the function $F^{-1}(\Phi_{0,1}(\cdot))$ is 1 (see Corollary 1). Furthermore, (19) and (20) satisfy the condition of Lemma 2. Therefore, equivalence (9) holds (note that $J(1)$ coincides with the constant c (see the condition of the theorem)). Again, according to Lemma 2, we conclude that the processes $\frac{\sum_{j=1}^{[nt]} Y_j}{\sqrt{L(n)c^{-2}H^{-1}(2H-1)^{-1}n^H}}$ C-converge to $cB_H(t)$. □

5.5. **Proof of Theorem 3 and Proposition 2.** As before, we denote by $\{z_k\}$ the stationary sequence of standard normal variables. We will consider the function G such that $\mathbf{E}G(z_1) = 0$ and $\mathbf{E}G^2(z_1) < +\infty$. In this case, we have expansion (13):

$$G(x) = \sum_{k=0}^{+\infty} c_k H_k(x).$$

We formulate a one-dimensional version of Theorem 1 from [4] for functions with the unit Hermitian rank.

Lemma 3. *Let the Hermitian rank of the function G be 1 and the covariance function $\rho = \rho(j)$ of the sequence $\{z_j\}$ satisfy the condition*

$$\sum_{j=-\infty}^{+\infty} |\rho(j)| < +\infty.$$

Then, the limits

$$\lim_{n\to\infty} \frac{\mathbf{E}(\sum_{i=1}^{n} H_l(z_i))^2}{n} = \lim_{n\to\infty} n^{-1} l! \sum_{i=1}^{n}\sum_{j=1}^{n} \rho^l(i-j) = \sigma_l^2 l! \tag{21}$$

exist for all $l \geq 1$, and the following sum is finite:

$$\sigma^2 = \sum_{l=1}^{+\infty} c_l^2 \sigma_l^2 l!. \tag{22}$$

Moreover, as $n \to +\infty$, we have the convergence in distribution $\frac{\sum_{i=1}^{n} G(z_i)}{\sqrt{n}}$ to $\sigma\mathcal{N}(0,1)$.

Lemma 4. *Let the conditions of Lemma 3 be satisfied. Then, for σ_l^2, it holds that*

$$\sigma_l^2 = \sum_{j=-\infty}^{+\infty} \rho^l(j).$$

Proof. It is clear that

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\rho^l(i-j)}{n} = \frac{n\rho^l(0) + 2\sum_{j=1}^{n-1}\sum_{i=1}^{j}\rho^l(i)}{n}. \tag{23}$$

Consider the right-hand side of (23). It follows from [23, Chapter 9, Lemma 9] (it suffices to put $b_k = 1$ and $x_k = \sum_{i=1}^{k}\rho^l(i)$) that $\frac{\sum_{k=1}^{n-1}\sum_{i=1}^{k}\rho^l(i)}{n} \to \sum_{i=1}^{+\infty}\rho^l(i)$ as $n \to \infty$. This immediately implies the assertion of the lemma. □

Proof of Theorem 3. According to item 3 of Theorem 1, the sequence $\{\rho(j)\}$ is absolutely summable. Then, taking into account the Corollary 1, the condition of the Lemma 3 is satisfied. Consider (22). Using Lemma 4, we conclude that $\sigma^2 = \sum_{j=-\infty}^{+\infty}\sum_{l=1}^{+\infty} l!c_l^2\rho^l(j)$. Recall that $R(\rho(j)) = \sum_{l=1}^{+\infty} l!c_l^2\rho^l(j)$ (see (14)). Therefore, it holds that $\sigma^2 = \sum_{j=-\infty}^{+\infty} R(\rho(j))$; hence $\sigma^2 = g$. This implies the convergence in distribution of the sequence $\{\frac{\sum_{i=1}^{n} Y_i}{\sqrt{n}}\}$ to $\sqrt{g}\mathcal{N}(0,1)$ as $n \to \infty$.

i) Let $g > 0$. Since $\{Y_n\}$ is a stationary sequence with spectral density, it can be obtained using moving average (e.g., see [14]). Thus, there exists a square-summable sequence $\{a_j\}$ and $\{\varepsilon_j\}$ that is a white noise such that $Y_n = \sum_{j=-\infty}^{+\infty} a_{n-j}\varepsilon_j$, $n \in \mathbb{Z}$. Along with $\{Y_n\}$, consider the Gaussian analog of this sequence $G_n = \sum_{j=-\infty}^{+\infty} a_{n-j}\zeta_j$, where $\{\zeta_j\}$ is the sequence of independent standard normal random variables. It is clear that $D_n := \mathbf{D}(\sum_{j=1}^{n} Y_j) = \mathbf{D}(\sum_{j=1}^{n} G_j)$ for all $n \geq 1$. Furthermore, $R(\rho(j)) = \mathbf{E}(G_0 G_j)$ for all $j \in \mathbb{Z}$. Note that $\{\frac{G_n}{\sqrt{g}\sqrt{n}}\}$ weakly converges to the standard normal law (see item 2 of Theorem 7.2.11 in [16]). The random variable $G_n/\sqrt{D_n}$ has the standard normal distribution for all n. Hence, $D_n \sim gn$ as $n \to \infty$.

ii) The proof of this assertion is similar to the proof of the preceding assertion. Next, we note the differences. The sequence $\{\frac{G_n}{\sqrt{aH^{-1}(2H-1)^{-1}n^H}}\}$ weakly converges to the standard normal law (see item 3 of Theorem 7.2.11 in [16]). In this case, the random variable $G_n/\sqrt{D_n}$ has the standard normal distribution for all n. Finally, we conclude that $D_n \sim aH^{-1}(2H-1)^{-1}n^{2H}$ as $n \to \infty$.

□

Proof of Proposition 2. According to Theorem 7.2.10 from [3], we conclude that $\{\rho(j)\}$ is absolutely summable, and the corollary to this theorem implies that $\sum_{j=-\infty}^{+\infty}\rho(j) = 0$. Using Theorem 1, we obtain the absolute summability of $\{R(\rho(j)\}$. According to items 2 and 4 of Theorem 1, we obtain the inequalities $|R(\rho(j))| < |\rho(j)|$ and $R(\rho(j)) < 0$ for all $j \neq 0$; also note that $\rho(0) = R(\rho(0)) = 1$. Therefore, $\sum_{j=-\infty}^{+\infty} R(\rho(j)) > \sum_{j=-\infty}^{+\infty}\rho(j) = 0$.

□

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