S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

*Том 19, №2, стр. 562–577 (2022)* УДК 510.64
DOI 10.33048/semi.2022.19.047 MSC 03B20, 03B45

# SOME REMARKS ON DOŠEN'S LOGIC N AND ITS EXTENSIONS

S. O. SPERANSKI

Abstract. This paper collects some observations about Došen's logic N, where negation is treated as a modal operator, and its extensions. We shall see what happens when we add the contraposition axiom to several important extensions of N, show that certain extensions of N are canonical, and also revisit the method of filtration.

**Keywords:** modal negation, intuitionistic modal logic, Heyting–Ockham logic, Hype, Routley star.

## 1. Introduction

Došen's logic N, proposed in [3], enriches the positive fragment of intuitionistic logic by adding a negative modality, which is weaker than the negation of Johansson's minimal logic. (For information about quantified versions of logics containing N, the reader may consult [11].) Among the interesting extensions of N are the logics $\mathsf{N}^*$ and Hype. The former was introduced in [2] in the course of developing a framework for the study of logic programs with negation. The latter has been advocated in [6] as a system suitable for dealing with 'hyperintensional' contexts, but was first described in [7]; the reader may consult [8] for further discussion. Following [11], we shall write $\mathsf{N}^\bullet$ instead of Hype. Note that $\mathsf{N}^\bullet$ extends $\mathsf{N}^*$.

While the system for N employs the contraposition rule, the corresponding scheme

$$(\phi \to \psi) \to (\neg\psi \to \neg\phi)$$

cannot be derived even in $\mathsf{N}^\bullet$. In Section 3 we shall see what happens when we

---

Speranski, S.O., Some remarks on Došen's logic N and its extensions.

This work was supported by the Russian Science Foundation under grant no. 21-11-00318.
*Received March, 21, 2022, published August, 29, 2022.*

add the above scheme to some important extensions of $\mathsf{N}$.[1] In Section 4 we shall prove that certain extensions of $\mathsf{N}$ — which are obtained by adding various schemes involved in the definitions of $\mathsf{N}^*$ and $\mathsf{N}^\bullet$ — are canonical. In Section 5 we shall revisit the method of filtration, which was used in [4] to establish the decidability of $\mathsf{N}$ and $\mathsf{N}^*$. This will lead to further decidability results.

It should be noted that these remarks are inspired by the work of K. Došen and that of S. Odintsov, and are intended to complement [3], [9], [4] and [8]. The technique used in the paper is quite simple, but the results may be of interest to those working in non-classical logics.

## 2. PRELIMINARIES

Fix once and for all a countable set Prop of *propositional variables*. The syntax of $\mathsf{N}$ is exactly the same as that of intuitionistic logic; so the *connective symbols* are $\to$, $\wedge$, $\vee$ and $\neg$. However, one should bear in mind that

> *in the semantics of* $\mathsf{N}$, $\neg$ *will be interpreted as a negative modal operator, and thus many intuitionistic principles involving* $\neg$ *will not be valid.*

Denote by Form the collection of all formulas — i.e. the set of all expressions that can be built up from Prop using the connective symbols. We treat $\leftrightarrow$ as defined in the obvious way, viz.

$$\phi \leftrightarrow \psi \ := \ (\phi \to \psi) \wedge (\phi \to \psi).$$

For convenience, when concerned only with non-empty sets of formulas, we shall abbreviate the condition '$\Delta \neq \varnothing$ and $\Delta \subseteq$ Form' as $\Delta \sqsubseteq$ Form.

### 2.1. The logics N, N°, N* and N•.

The Hilbert-type system for $\mathsf{N}$ was described in [3]. It employs the following axiom schemes:

I1. $\phi \to (\psi \to \phi)$;
I2. $(\phi \to (\psi \to \theta)) \to ((\phi \to \psi) \to (\phi \to \theta))$;
C1. $\phi \wedge \psi \to \phi$;
C2. $\phi \wedge \psi \to \psi$;
C3. $\phi \to (\psi \to \phi \wedge \psi)$;
D1. $\phi \to \phi \vee \psi$;
D2. $\psi \to \phi \vee \psi$;
D3. $(\phi \to \theta) \to ((\psi \to \theta) \to (\phi \vee \psi \to \theta))$;
N. $\neg\phi \wedge \neg\psi \to \neg\,(\phi \vee \psi)$.

Thus we have the 'positive' axioms of intuitionistic logic plus all instances of N. It also employs two inference rules:

MP. *modus ponens*, i.e.

$$\frac{\phi \qquad \phi \to \psi}{\psi}\ ;$$

CR. the *contraposition rule*, which is rendered as

$$\frac{\phi \to \psi}{\neg\psi \to \neg\phi}\ .$$

---

[1] The writing of this section has been partially motivated by a question of Dick de Jongh (private communication): he asked about extending $\mathsf{N}$ by adding the scheme

$$(\star) \qquad (\phi \leftrightarrow \psi) \to (\neg\phi \leftrightarrow \neg\psi)$$

— which is the same as adding $\neg\phi \wedge \neg\psi \to \neg\,(\phi \vee \psi)$ to the system of 'subminimal logic' studied in [1] (see also [5]). Among other things, we shall derive the contraposition scheme from $(\star)$ over $\mathsf{N}$.

Note that if we think of $\neg$ as an impossibility operator, then `CR` can be viewed as a modal rule. Clearly, $\neg$ is weaker than intuitionistic negation, and even minimal negation.

Now let $\mathsf{N}$ denote the least set of formulas containing the axioms of our calculus and closed under its rules of inference. For each $\Gamma \subseteq \mathrm{Form}$, take

$$\mathrm{Disj}\,(\Gamma) \;:=\; \{\phi_0 \vee \ldots \vee \phi_n \mid n \in \mathbb{N} \text{ and } \phi_0, \ldots, \phi_n \in \Gamma\}.^2$$

Given $\Gamma \subseteq \mathrm{Form}$ and $\Delta \subseteq \mathrm{Form}$, we write $\Gamma \vdash \Delta$ iff some element of $\mathrm{Disj}\,(\Delta)$ can be obtained from elements of $\Gamma \cup \mathsf{N}$ by means of `MP`. As may be expected, $\phi \vdash \Delta$ and $\Gamma \vdash \phi$ abbreviate $\{\phi\} \vdash \Delta$ and $\Gamma \vdash \{\psi\}$ respectively. Exactly as in intuitionistic logic, one can prove:

**Theorem 2.1** (see [3]). *For any* $\Gamma \subseteq \mathrm{Form}$ *and* $\phi, \psi \in \mathrm{Form}$,

$$\Gamma \cup \{\phi\} \vdash \psi \quad \Longleftrightarrow \quad \Gamma \vdash \phi \to \psi.$$

Here is another simple but useful observation.

**Theorem 2.2** (see [3]). *Let* $\{\phi, \psi, \psi'\} \subseteq \mathrm{Form}$*, and suppose that* $\phi'$ *is obtained from* $\phi$ *by replacing some occurrence of* $\psi$ *by* $\psi'$*. Then* $\vdash \psi \leftrightarrow \psi'$ *implies* $\vdash \phi \leftrightarrow \phi'$*.*

*Proof.* By induction on the complexity of $\phi$.

The case where $\phi \in \mathrm{Prop}$ is trivial.

Suppose $\phi = \neg\theta$. The result then follows by the inductive hypothesis and `CR`.

The other cases can be handled as in intuitionistic logic. □

Evidently, for any $\phi, \psi \in \mathrm{Form}$ we have $\vdash (\phi \to \phi) \leftrightarrow (\psi \to \psi)$; thus by Theorem 2.2, $\phi \to \phi$ and $\psi \to \psi$ are practically interchangeable. Denote

$$\top \;:=\; \phi_\circ \to \phi_\circ \quad \text{and} \quad \bot \;:=\; \neg\top$$

where $\phi_\circ$ is a fixed formula. We shall occasionally abbreviate $\phi \to \bot$ to $-\phi$. One may think of $-$ as intuitionistic negation provided that $\bot$ behaves as the falsum.

In this article by a (*normal*) *logic* we mean a superset of $\mathsf{N}$ closed under `MP`, `CR` and substitutions. Given a logic $L$, we define

$$\Gamma \vdash_L \Delta \quad :\Longleftrightarrow \quad L \cup \Gamma \vdash \Delta.$$

Thus Theorems 2.1 and 2.2 generalise readily to extensions of $\mathsf{N}$. If $L$ is a logic and $\mathtt{S}_1, \ldots, \mathtt{S}_n$ are formula schemes, we write $L + \{\mathtt{S}_1, \ldots \mathtt{S}_n\}$ for the least logic containing $L$ and all instances of $\mathtt{S}_1, \ldots, \mathtt{S}_n$. Here are examples of extra schemes:

$\mathtt{N1}^\circ$. $\neg(\phi \to \phi) \to \psi$;
$\mathtt{N2}^\circ$. $\neg\neg(\phi \to \phi)$;
$\mathtt{N}^*$. $\neg(\phi \wedge \psi) \to \neg\phi \vee \neg\psi$;
$\mathtt{N1}^\bullet$. $\phi \to \neg\neg\phi$;
$\mathtt{N2}^\bullet$. $\neg\neg\phi \to \phi$.

They can be used to define three important extensions of $\mathsf{N}$:

$$\begin{aligned} \mathsf{N}^\circ &:= \mathsf{N} + \{\mathtt{N1}^\circ, \mathtt{N2}^\circ\}; \\ \mathsf{N}^* &:= \mathsf{N}^\circ + \{\mathtt{N}^*\}; \\ \mathsf{N}^\bullet &:= \mathsf{N}^* + \{\mathtt{N1}^\bullet, \mathtt{N2}^\bullet\}. \end{aligned}$$

It is known that $\mathsf{N}^\circ$ is the least logic in which $\bot$ behaves as the falsum; see [11]. Next, $\mathsf{N}^*$ was introduced in [2] and studied further in [9, 4]. Finally, $\mathsf{N}^\bullet$ has been

---

[2] When $n = 0$, we have $\phi_0 \vee \ldots \vee \phi_n = \phi_0$. Thus $\mathrm{Disj}\,(\Gamma)$ contains non-empty disjunctions only.

advocated in [6], but it was first described in [7]; consult [8] for discussion.[3] Here are a few useful observations:

- the converses to $\mathtt{N}$ and $\mathtt{N}^*$ are derivable in $\mathsf{N}$, even without $\mathtt{N}$;
- $\mathtt{N}$, $\mathtt{N1}^\circ$, $\mathtt{N2}^\circ$ and $\mathtt{N}^*$ are redundant — i.e. derivable from the other axioms — in $\mathsf{N}^\bullet$.

See [11] for the details.

While the rule $\mathtt{CR}$ is obviously admissible in each extension of $\mathsf{N}$, it is not derivable even in $\mathsf{N}^\bullet$, let alone $\mathsf{N}$ and $\mathsf{N}^*$ — this can be shown using the corresponding possible world semantics; see [9]. The same applies to the rule

$$\frac{\phi \leftrightarrow \psi}{\neg\psi \leftrightarrow \neg\phi}\,.$$

In other words, the following schemes are not derivable in $\mathsf{N}^\bullet$:

$\mathtt{C}$. $(\phi \to \psi) \to (\neg\psi \to \neg\phi)$;
$\mathtt{E}$. $(\phi \leftrightarrow \psi) \to (\neg\psi \leftrightarrow \neg\phi)$.

Interestingly enough, $\mathtt{C}$ and $\mathtt{E}$ turn out to be equivalent, i.e. derivable from each other over the basic logic $\mathsf{N}$.

**Proposition 2.3.** *Let $L$ be a logic. Then $L + \{\mathtt{C}\}$ coincides with $L + \{\mathtt{E}\}$.*

*Proof.* Clearly, it suffices to show that $\mathtt{C}$ is derivable in $\mathsf{N} + \{\mathtt{E}\}$. For convenience, let $\mathsf{L}$ denote $\mathsf{N} + \{\mathtt{E}\}$. Observe that $\phi \to \psi \vdash_{\mathsf{L}} \neg\psi \to \neg\phi$:

| | | |
|---|---|---|
| 1 | $\phi \to \psi$ | hypothesis |
| 2 | $(\phi \to \psi) \to ((\phi \vee \psi) \leftrightarrow \psi)$ | positive intuitionistic logic |
| 3 | $(\phi \vee \psi) \leftrightarrow \psi$ | from 1, 2 |
| 4 | $((\phi \vee \psi) \leftrightarrow \psi) \to (\neg(\phi \vee \psi) \leftrightarrow \neg\psi)$ | E |
| 5 | $\neg(\phi \vee \psi) \leftrightarrow \neg\psi$ | from 3, 4 |
| 6 | $(\neg\phi \wedge \neg\psi) \leftrightarrow \neg(\phi \vee \psi)$ | N |
| 7 | $(\neg\phi \wedge \neg\psi) \leftrightarrow \neg\psi$ | from 6, 5 |
| 8 | $((\neg\phi \wedge \neg\psi) \leftrightarrow \neg\psi) \to (\neg\psi \to \neg\phi)$ | positive intuitionistic logic |
| 9 | $\neg\psi \to \neg\phi$ | from 7, 8. |

By the deduction theorem for $\mathsf{L}$, this gives us $\mathtt{C}$. □

One may wonder what happens if we add $\mathtt{C}$ to a given logic. Some natural examples will be discussed in Section 3.

2.2. **Došen-style semantics.** As in [3], by a *frame* we mean a triple $\mathcal{W} = \langle W, \leqslant, R\rangle$ where $W$ is a non-empty set, $\leqslant$ is a preordering on $W$, and $R$ is a binary relation on $W$ such that

$$\leqslant \circ R \;\subseteq\; R \circ \leqslant^{-1}.[4]$$

Given $\mathcal{W}$, we call $\xi : \mathrm{Prop} \to \mathcal{P}(W)$ a *valuation in* $\mathcal{W}$ iff for any $p \in \mathrm{Prop}$ and $x, y \in W$,

$$x \in \xi(p) \quad \text{and} \quad x \leqslant y \quad \Longrightarrow \quad y \in \xi(p),$$

i.e. the $\xi(p)$'s are upward closed. By a *model* we mean a pair $\mathcal{M} = \langle \mathcal{W}, \xi\rangle$ where $\mathcal{W}$ is a frame and $\xi$ is a valuation in $\mathcal{W}$. Now $\mathcal{M}, x \Vdash \phi$ is defined exactly as in intuitionistic logic, except for the negation clause:

$$\mathcal{M}, x \Vdash \neg\psi \quad :\Longleftrightarrow \quad \mathcal{M}, y \nVdash \psi \quad \text{for all } y \in R(x).[5]$$

---

[3] In [6], $\mathsf{N}^\bullet$ was presented in a slightly different language: $\top$ was treated as primitive, rather than defined. So formally speaking, the system for $\mathsf{N}^\bullet$ as given above is a definitional variant of that in [6].

[4] Here $\circ$ and $\cdot^{-1}$ denote the composition operation and the inverse operation respectively.

When there is no ambiguity, we shall drop $\mathcal{M}$ and write $x \Vdash \phi$ instead of $\mathcal{M}, x \Vdash \phi$. Naturally, $\mathcal{M}, x \Vdash \phi$ is read *$\phi$ is true at $x$ in $\mathcal{M}$*. Also define:

- $\mathcal{M} \Vdash \phi$ iff $\mathcal{M}, x \Vdash \phi$ for all $x \in W$;
- $\mathcal{W} \Vdash \phi$ iff $\mathcal{M} \Vdash \phi$ for all models $\mathcal{M}$ based on $\mathcal{W}$.

These are read *$\phi$ is true in $\mathcal{M}$* and *$\phi$ is valid in $\mathcal{W}$* respectively.

**Lemma 2.4** (see [3]). *Let $\mathcal{M}$ be a model. Then for any $\phi \in \mathrm{Form}$ and $x, y \in W$,*

$$\mathcal{M}, x \Vdash \phi \quad \textit{and} \quad x \leqslant y \quad \Longrightarrow \quad \mathcal{M}, y \Vdash \phi.$$

*More informally, it means that $\Vdash$ is intuitionistically hereditary.*

As in modal logic, some of the formulas correspond to frame properties. For instance, $\bot$ is valid in $\mathcal{W}$ iff $R = \varnothing$. For a more interesting example, consider the principle of weak excluded middle, which can be represented as the formula scheme

`WEM`. $\neg\phi \vee \neg\neg\phi$.

As was shown in [3], it corresponds to a rather complicated property:

$$\mathcal{W} \Vdash \neg p \vee \neg\neg p \quad \Longleftrightarrow \quad \forall x \, \forall y \, \forall z \, (R\,(x, y) \ \& \ R\,(x, z) \Rightarrow \exists u \, (R\,(y, u) \ \& \ z \leqslant u)).$$

Here is yet another example.

**Proposition 2.5.** *For every frame $\mathcal{W}$,*

$$\mathcal{W} \Vdash \neg\,(p \wedge q) \to \neg p \vee \neg q \quad \Longleftrightarrow \quad \forall x \, \forall y \, \forall z \, (R\,(x, y) \ \& \ R\,(x, z) \Rightarrow \exists u \, (R\,(x, u) \ \& \ y \leqslant u \ \& \ z \leqslant u)).$$

*Thus the scheme* `N*` *corresponds to the property on the right-hand side.*

*Proof.* For convenience, denote by $(\star)$ the property on the right-hand side.

$\boxed{\Longleftarrow}$ Assume $(\star)$ holds. Let $\mathcal{M}$ be a model based on $\mathcal{W}$. It suffices to show that for every $x \in W$,

$$\mathcal{M}, x \Vdash \neg\,(p \wedge q) \quad \Longrightarrow \quad \mathcal{M}, x \Vdash \neg p \quad \text{or} \quad \mathcal{M}, x \Vdash \neg q.$$

Suppose $x \nVdash \neg p$ and $x \nVdash \neg q$. So there exist $y, z \in R\,(x)$ such that

$$y \Vdash p \quad \text{and} \quad z \Vdash q$$

Then by $(\star)$, there exists $u \in R\,(x)$ such that $y \leqslant u$ and $z \leqslant u$. Consequently, $u \Vdash p$ and $u \Vdash q$, i.e. $u \Vdash p \wedge q$. Hence $x \nVdash \neg\,(p \wedge q)$.

$\boxed{\Longrightarrow}$ Assume $(\star)$ fails. So there exist $x \in W$ and $y, z \in R\,(x)$ such that for every $u \in R\,(x)$ we have $y \nleqslant u$ or $z \nleqslant u$. Consider a model $\mathcal{M}$ based on $\mathcal{W}$ such that

$$\xi\,(p) \ := \ \{u \in W \mid y \leqslant u\} \quad \text{and} \quad \xi\,(q) \ := \ \{u \in W \mid z \leqslant u\}.$$

It is straightforward to check that $\mathcal{M}, x \Vdash \neg\,(p \wedge q)$ but $\mathcal{M}, x \nVdash \neg p$ and $\mathcal{M}, x \nVdash \neg q$. $\square$

For more examples, see the table below.

---

[5] Here $R\,(u)$ denotes the image of $\{u\}$ under $R$, i.e. $\{v \in W \mid uRv\}$.

| **Property** | **Scheme** | **Reference** |
|---|---|---|
| $\forall x\, \exists y\, R\,(x,y)$ | $\mathtt{N1}^{\circ}$ | [3] |
| $\forall x\, (\exists y\, R\,(y,x) \Rightarrow \exists z\, R\,(x,z))$ | $\mathtt{N2}^{\circ}$ | [3] |
| $\forall x\, \forall y\, \forall z\, (R\,(x,y)\ \&\ R\,(x,z) \Rightarrow \exists u\, (R\,(x,u)\ \&\ y \leqslant u\ \&\ z \leqslant u))$ | $\mathtt{N}^{*}$ | This article |
| $\forall x\, \forall y\, (\exists u\, (R\,(x,u)\ \&\ y \leqslant u) \Rightarrow \exists u\, (R\,(y,u)\ \&\ x \leqslant u))$ | $\mathtt{N1}^{\bullet}$ | [3] |
| $\forall x\, \exists y\, (R\,(x,y)\ \&\ \forall z\, (R\,(y,z) \rightarrow z \leqslant x))$ | $\mathtt{N2}^{\bullet}$ | [3] |
| $\forall x\, \forall y\, (R\,(x,y) \Rightarrow \exists z\, (R\,(x,z)\ \&\ x \leqslant z\ \&\ y \leqslant z))$ | $\mathtt{C}$ | [3] |

Table 1. Properties vs. Schemes.

As was shown in [3], the canonical model method can be adapted to $\mathsf{N}$ and its extensions. Let $L$ be a logic. Call $\Gamma \subseteq \mathrm{Form}$ a *prime $L$-theory* iff:

(i) $\{\phi \in \mathrm{Form} \mid \Gamma \vdash_L \phi\} \subseteq \Gamma$;

(ii) for every $\phi \vee \psi \in \Gamma$ we have $\phi \in \Gamma$ or $\psi \in \Gamma$.

Thus (i) and (ii) say that $\Gamma$ is closed under $\vdash_L$ and has the disjunction property. The following is proved in the usual way.

**Lemma 2.6** (see [3])**.** *Let $L$ be a logic. Suppose $\Gamma \subseteq \mathrm{Form}$ and $\Delta \sqsubseteq \mathrm{Form}$ are such that $\Gamma \nvdash_L \Delta$. Then there exists a prime $L$-theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash_L \Delta$.*

Given $\Gamma \subseteq \mathrm{Form}$, we write $\underline{\Gamma}$ for $\{\phi \mid \neg\phi \in \Gamma\}$.

**Proposition 2.7** (see [3, 11])**.** *Let $\Gamma \subseteq \mathrm{Form}$ be such that $\{\phi \in \mathrm{Form} \mid \Gamma \vdash \phi\} \subseteq \Gamma$ and $\underline{\Gamma} \neq \varnothing$. Then:*

(i) *$\{\phi \in \mathrm{Form} \mid \phi \vdash \underline{\Gamma}\} \subseteq \underline{\Gamma}$;*

(ii) *for any $\phi, \psi \in \underline{\Gamma}$ we have $\phi \vee \psi \in \underline{\Gamma}$.*

Now take $W^L$ to be the collection of all prime $L$-theories. By the *canonical frame for $L$* we mean the triple $\mathcal{W}^L = \langle W^L, \leqslant^L, R^L \rangle$ where

$$\begin{aligned} \leqslant^L \ &:= \ \big\{(\Gamma, \Delta) \in W^L \times W^L \mid \Gamma \subseteq \Delta\big\} \quad \text{and} \\ R^L \ &:= \ \big\{(\Gamma, \Delta) \in W^L \times W^L \mid \underline{\Gamma} \cap \Delta = \varnothing\big\}. \end{aligned}$$

By the *canonical model for $L$* we mean the pair $\mathcal{M}^L = \langle \mathcal{W}^L, \xi^L \rangle$ where $\xi^L$ is given by

$$\xi^L\,(p) \ := \ \big\{\Gamma \in W^L \mid p \in \Gamma\big\}.$$

One readily verifies that $\mathcal{W}^L$ is a frame, and $\mathcal{M}^L$ is a model. Moreover,

$$\leqslant^L \circ\, R^L \circ \leqslant^{-1} \ \subseteq \ R^L.$$

Hence $\mathcal{W}^L$ is *strictily condensed*, in the terminology of [3].[6] In fact, we might limit ourselves to strictly condensed frames if needed.

**Lemma 2.8** (see [3])**.** *Let $L$ be a logic. Then for any $\Gamma \in W^L$ and $\phi \in \mathrm{Form}$,*

$$\mathcal{M}^L, \Gamma \Vdash \phi \quad \Longleftrightarrow \quad \phi \ \in \ \Gamma.$$

Given $\Gamma \subseteq \mathrm{Form}$ and $\Delta \sqsubseteq \mathrm{Form}$, we write $\Gamma \vDash \Delta$ iff for any model $\mathcal{M}$ and $w \in W$,

$$\mathcal{M}, w \Vdash \phi \quad \text{for all} \ \ \phi \in \Gamma \quad \Longrightarrow \quad \mathcal{M}, w \Vdash \psi \quad \text{for some} \ \ \psi \in \Delta.$$

For each logic $L$, denote by $\vDash_L$ the relativization of $\vDash$ to $\{\mathcal{W} \mid \mathcal{W} \Vdash L\}$.[7] Further, call a logic $L$ *canonical* iff $\mathcal{W}^L \Vdash L$.

---

[6] There are several different but equivalent ways of defining this notion; see Definition 6 in [3] together with the comments after it. In particular, for every frame $\mathcal{W}$,

$$\leqslant \circ\, R \circ \leqslant^{-1} \ \subseteq \ R \quad \Longleftrightarrow \quad R \circ \leqslant^{-1} \ \subseteq \ R.$$

[7] Here $\mathcal{W} \Vdash L$ means that $\mathcal{W} \Vdash \phi$ for all $\phi \in L$.

**Theorem 2.9** (see [3]). *Let $L$ be a canonical logic. Then for any $\Gamma \subseteq \mathrm{Form}$ and $\Delta \sqsubseteq \mathrm{Form}$,*

$$\Gamma \vdash_L \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_L \Delta.$$

*In particular, since $\mathsf{N}$ is canonical, $\vdash$ coincides with $\vDash$.*

We conclude with two technical remarks.

(I) In [3], Došen employed single-succedent derivability and semantical consequence relations — this, in a sense, forced him to utilize Zorn's lemma for the canonical model lemma. As has been shown in [11], this is not essential.

(II) Došen emphasized that Form should be treated as a prime theory in his canonical model construction. In most cases this is not necessary; see [11]. In particular, if $L$ contains at least one negated formula, we may assume that all prime $L$-theories are non-trivial.

In view of (II), we shall adopt the convention that Form is a prime $L$-theory iff no negated formula belongs to $L$. So if $L$ contains some negated formulas, then each element of $W_L$ must be non-trivial; otherwise Form is also an element of $W_L$.

Next we turn to a more elegant semantics appropriate for logics containing $\mathsf{N}^*$.

2.3. **Routley-style semantics.** Following [9], by a *Routley frame* we mean a triple $\mathbf{W} = \langle W, \leqslant, * \rangle$ where $W$ is a non-empty set, $\leqslant$ is a preordering on $W$, and $*$ is an anti-monotone function from $W$ to $W$. Obviously, $*$ may be viewed as a binary relation on $W$, and moreover, it is easy to verify that

$$\leqslant \circ * \;\subseteq\; * \circ \leqslant^{-1}.$$

So Routley frames are frames.[8] Models based on Routley frames are called *Routley models*. By definition, for any Routley model $\mathbf{M}$, $w \in W$ and $\psi \in \mathrm{Form}$,

$$\mathbf{M}, x \Vdash \neg\psi \quad \Longleftrightarrow \quad \mathbf{M}, x^* \nVdash \psi$$

where $x^*$ stands for $*\,(x)$. This kind of semantics is suitable for $\mathsf{N}^*$ and its extensions.

Given $\Gamma \subseteq \mathrm{Form}$, denote $\mathrm{Form} \setminus \underline{\Gamma}$ — i.e. $\{\phi \in \mathrm{Form} \mid \neg\phi \notin \Gamma\}$ — by $\Gamma^*$. Notice that if $L$ is an extension of $\mathsf{N}^*$, then it contains some negated formulas, and hence all prime $L$-theories are non-trivial by the above convention. The following is straightforward.

**Proposition 2.10** (see [9]). *Let $\Gamma$ be a prime $L$-theory, where $L$ is an extension of $\mathsf{N}^*$. Then $\Gamma^*$ is also a prime $L$-theory.*

Let $L$ be an extension of $\mathsf{N}^*$. Observe that for every $\Gamma \in W^L$,

$$\Gamma^* \;=\; \text{the greatest element of } \{\Delta \in W^L \mid \underline{\Gamma} \cap \Delta = \varnothing\}$$

(with respect to inclusion). By the *canonical Routley frame for $L$* we mean

$$\mathbf{W}^L \;=\; \langle W^L, \leqslant^L, *^L \rangle$$

where $W^L$ and $\leqslant^L$ are as before, and $*^L$ maps each $\Gamma$ in $W^L$ to $\Gamma^*$. By the *canonical Routley model for $L$* we mean

$$\mathbf{M}^L \;=\; \langle \mathbf{W}^L, \xi^L \rangle$$

where $\xi^L$ is defined in the usual way. Clearly, $\mathbf{W}^L$ and $\mathbf{M}^L$ are a Routley frame and a Routley model respectively.

**Lemma 2.11** (see [9]). *Let $L$ be an extension of $\mathsf{N}^*$. Then for any $\Gamma \in W^L$ and $\phi \in \mathrm{Form}$,*

$$\mathbf{M}^L, \Gamma \Vdash \phi \quad \Longleftrightarrow \quad \phi \;\in\; \Gamma.$$

---

[8] Semantically, the class of Routley frames plays the same role as the class of all frames $\mathcal{W}$ such that for each $w \in W$, $R(w)$ has a greatest element with respect to $\leqslant$; cf. [9].

For each extension $L$ of $\mathsf{N}^*$, define $\models^*_L$ exactly as $\models_L$ but with 'frames' replaced by 'Routley frames'. Further, we call an extension $L$ of $\mathsf{N}^*$ *Routley canonical* iff $\mathbf{W}^L \Vdash L$.

**Theorem 2.12** (see [9]). *Let $L$ be a Routley canonical extension of $\mathsf{N}^*$. Then for any $\Gamma \subseteq \mathrm{Form}$ and $\Delta \sqsubseteq \mathrm{Form}$,*

$$\Gamma \vdash_L \Delta \quad \Longleftrightarrow \quad \Gamma \models^*_L \Delta.$$

*In particular, since $\mathsf{N}^*$ is Routley canonical, $\vdash_{\mathsf{N}^*}$ coincides with $\models^*_{\mathsf{N}^*}$.*

## 3. Adding the contraposition axiom

Naturally, one may wonder what happens when we add the scheme $\mathtt{C}$ — which is equivalent to $\mathtt{E}$ by Proposition 2.3 — to a given logic, e.g. $\mathsf{N}^\circ$, $\mathsf{N}^*$ or $\mathsf{N}^\bullet$. Here we focus on logics containing at least one negated formula. As will become clear from what follows, this leads us to consider logics that include $\mathtt{N2}^\circ$.[9] Note that $\mathtt{N2}^\circ$ is semantically weaker than $\mathtt{N1}^\circ$; see Table 1. We shall write $\mathsf{CL}$ for classical logic, $\mathsf{IL}$ for intuitionistic logic, and $\mathsf{JL}$ for Johansson's minimal logic.

**Lemma 3.1.** *$\mathsf{N} + \{\mathtt{N2}^\circ, \mathtt{E}\}$ coincides with $\mathsf{JL}$, and hence includes $\mathtt{N1}^\bullet$.*[10]

*Proof.* For convenience, let $L$ denote $\mathsf{N} + \{\mathtt{N2}^\circ, \mathtt{E}\}$. Clearly, $L \subseteq \mathsf{JL}$. For the other inclusion, it suffices to show that

$$\neg\phi \leftrightarrow \underbrace{(\phi \to \neg(\phi \to \phi))}_{-\phi}$$

is derivable in $L$. The implication from left to right can be obtained as follows:

| | | |
|---|---|---|
| 1 | $\phi \to ((\phi \to \phi) \to \phi)$ | |
| 2 | $((\phi \to \phi) \to \phi) \to (\neg\phi \to \neg(\phi \to \phi))$ | `C (see Proposition 2.3)` |
| 3 | $\phi \to (\neg\phi \to \neg(\phi \to \phi))$ | `from 1, 2` |
| 4 | $\neg\phi \to (\phi \to \neg(\phi \to \phi))$ | `from 3.` |

On the other hand, observe that $\phi \to \neg(\phi \to \phi) \vdash_L \neg\phi$:

| | | |
|---|---|---|
| 1 | $\phi \to \neg(\phi \to \phi)$ | `hypothesis` |
| 2 | $(\phi \to \neg(\phi \to \phi)) \to (\neg\neg(\phi \to \phi) \to \neg\phi)$ | `C (see Proposition 2.3)` |
| 3 | $\neg\neg(\phi \to \phi) \to \neg\phi$ | `from 1, 2` |
| 4 | $\neg\neg(\phi \to \phi)$ | $\mathtt{N2}^\circ$ |
| 5 | $\neg\phi$ | `from 4, 3.` |

By the deduction theorem for $L$, this gives us the implication from right to left. □

Before proceeding, a few observations from [11] are worth recalling here.

**Proposition 3.2** (see [11]). (i) *$\mathtt{N2}^\circ$ is derivable in $\mathsf{N} + \{\mathtt{N1}^\bullet\}$.*
(ii) *$\mathtt{N1}^\circ$ and $\mathtt{N}^*$ are derivable in $\mathsf{N} + \{\mathtt{N2}^\bullet\}$.*

So in particular, $\mathtt{N2}^\circ$ is deductively weaker than $\mathtt{N1}^\bullet$, which implies the following.

**Corollary 3.3.** *$\mathsf{N} + \{\mathtt{N1}^\bullet, \mathtt{E}\}$ coincides with $\mathsf{JL}$.*

*Proof.* Since $\mathtt{N2}^\circ$ and $\mathtt{N1}^\bullet$ are derivable in $\mathsf{N} + \{\mathtt{N1}^\bullet\}$ and $\mathsf{JL}$ respectively, we have

$$\mathsf{N} + \{\mathtt{N1}^\bullet, \mathtt{E}\} \;=\; \mathsf{N} + \{\mathtt{N2}^\circ, \mathtt{N1}^\bullet, \mathtt{E}\} \;=\; \mathsf{JL} + \{\mathtt{N1}^\bullet\} \;=\; \mathsf{JL}$$

(using Lemma 3.1 for the second equality). □

---

[9] In particular, $\mathsf{N} + \{\mathtt{E}\}$ will not be considered because of Proposition 3.4 below.

[10] In [3], Došen notes that $\mathsf{JL}$ coincides with $\mathsf{N} + \{\mathtt{N1}^\bullet, \mathtt{C}\}$.

Interestingly enough, $\mathtt{N1}^\circ$ is stronger than $\mathtt{N2}^\circ$ semantically but not deductively, as the next result shows.

**Proposition 3.4** (cf. [9]). *No formula beginning with* $\neg$ *can be derived in* $\mathsf{N} + \{\mathtt{N2}^\bullet, \mathtt{E}\}$. *In particular,* $\mathtt{N2}^\circ$ *is not derivable in* $\mathsf{N} + \{\mathtt{N2}^\bullet, \mathtt{E}\}$.[11]

*Proof.* The analogous result for $\mathsf{N} + \{\mathtt{N1}^\circ, \mathtt{N}^*\}$ was proved in [9], using a 'Kleene slash', and the same argument applies to $\mathsf{N} + \{\mathtt{N2}^\bullet, \mathtt{E}\}$. □

Finally, we turn to the extensions of $\mathsf{N}^\circ$, $\mathsf{N}^*$ and $\mathsf{N}^\bullet$ obtained by adding $\mathtt{E}$.

**Theorem 3.5.** (i) $\mathsf{N}^\circ + \{\mathtt{E}\} = \mathsf{IL}$.
(ii) $\mathsf{N}^* + \{\mathtt{E}\} = \mathsf{IL} + \{\mathtt{WEM}\}$.
(iii) $\mathsf{N}^\bullet + \{\mathtt{E}\} = \mathsf{CL}$.

*Proof.* $\boxed{\text{i}}$ By Lemma 3.1, $\mathsf{N}^\circ + \{\mathtt{E}\}$ coincides with $\mathsf{JL} + \{\mathtt{N1}^\circ\}$, i.e. with $\mathsf{IL}$.

$\boxed{\text{ii}}$ By (i), $\mathsf{N}^* + \{\mathtt{E}\}$ coincides with $\mathsf{IL} + \{\mathtt{N}^*\}$. Thus it remains to show that $\mathtt{N}^*$ and $\mathtt{WEM}$ are equivalent over $\mathsf{IL}$.[12] Notice that $\mathtt{WEM}$ is easily derivable in $\mathsf{IL} + \{\mathtt{N}^*\}$:

| | | |
|---|---|---|
| 1 | $\neg(\phi \wedge \neg\phi)$ | JL |
| 2 | $\neg(\phi \wedge \neg\phi) \to \neg\phi \vee \neg\neg\phi$ | $\mathtt{N}^*$ |
| 3 | $\neg\phi \vee \neg\neg\phi$ | from 1, 2. |

On the other hand, $\neg\phi \vee \neg\psi$ can be derived from $\neg(\phi \wedge \psi)$ in $\mathsf{IL} + \{\mathtt{WEM}\}$ as follows:

| | | |
|---|---|---|
| 1 | $\neg\phi \vee \neg\neg\phi$ | WEM |
| 2 | $\neg\psi \vee \neg\neg\psi$ | WEM |
| 3 | $(\neg\phi \vee \neg\neg\phi) \wedge (\neg\psi \vee \neg\neg\psi)$ | from 1, 2 |
| 4 | $(\neg\phi \wedge \neg\psi) \vee (\neg\phi \wedge \neg\neg\psi) \vee (\neg\neg\phi \wedge \neg\psi) \vee (\neg\neg\phi \wedge \neg\neg\psi)$ | from 3 |
| 5 | $\neg\phi \wedge \neg\psi \to \neg\phi \vee \neg\psi$ | |
| 6 | $\neg\phi \wedge \neg\neg\psi \to \neg\phi \vee \neg\psi$ | |
| 7 | $\neg\neg\phi \wedge \neg\psi \to \neg\phi \vee \neg\psi$ | |
| 8 | $\neg(\phi \wedge \psi)$ | hypothesis |
| 9 | $\neg\neg\phi \wedge \neg\neg\psi \to \neg(\phi \wedge \psi)$ | from 8 |
| 10 | $\neg\neg\phi \wedge \neg\neg\psi \to \neg\neg(\phi \wedge \psi)$ | JL |
| 11 | $\neg\neg\phi \wedge \neg\neg\psi \to \neg(\phi \wedge \psi) \wedge \neg\neg(\phi \wedge \psi)$ | from 9, 10 |
| 12 | $\neg(\phi \wedge \psi) \wedge \neg\neg(\phi \wedge \psi) \to \neg\phi \vee \neg\psi$ | IL |
| 13 | $\neg\neg\phi \wedge \neg\neg\psi \to \neg\phi \vee \neg\psi$ | from 11, 12 |
| 14 | $\neg\phi \vee \neg\psi$ | from 4-7, 13. |

By the deduction theorem for $\mathsf{IL} + \{\mathtt{WEM}\}$, this gives us $\mathtt{N}^*$.

$\boxed{\text{iii}}$ By (ii), $\mathsf{N}^\bullet + \{\mathtt{E}\}$ coincides with $\mathsf{IL} + \{\mathtt{N2}^\bullet\}$, i.e. with $\mathsf{CL}$. □

## 4. CERTAIN CANONICAL EXTENSIONS

Note that every canonical logic containing $\mathtt{N1}^\circ$ must contain $\mathtt{N2}^\circ$. So in particular, we have the following negative result.

**Proposition 4.1.** *Let $L$ be a logic between* $\mathsf{N} + \{\mathtt{N1}^\circ\}$ *and* $\mathsf{N} + \{\mathtt{N2}^\bullet, \mathtt{E}\}$. *Then $L$ is not canonical.*

*Proof.* For any frame $\mathcal{W}$, if $\mathcal{W} \Vdash L$, then $\mathcal{W} \Vdash \mathtt{N1}^\circ$ and therefore $\mathcal{W} \Vdash \mathtt{N2}^\circ$ (see Table 1). Thus $\vDash_L \neg\neg(p \to p)$. On the other hand, we have $\nvdash_L \neg\neg(p \to p)$ by Proposition 3.4. Hence $L$ is not canonical by Theorem 2.9. □

---

[11] Bear in mind that $\mathsf{N} + \{\mathtt{N2}^\bullet, \mathtt{E}\}$ coincides with $\mathsf{N} + \{\mathtt{N1}^\circ, \mathtt{N}^*, \mathtt{N2}^\bullet, \mathtt{E}\}$.

[12] In fact, it is straightforward to prove this using the possible world semantics for $\mathsf{IL}$ (see e.g. [10, Section 3]). Here a syntactic proof is provided.

Obviously, $\mathsf{N}$ and $\mathsf{N}^*$ are canonical and Routley canonical respectively. Also, one can check that $\mathsf{N}^\circ$ is canonical and $\mathsf{N}^\bullet$ is Routley canonical; cf. [11, 8]. Further examples can be obtained by using so-called 'canonical schemes'.

Let $L$ be a logic and $\mathtt{S}$ be a scheme. We call $\mathtt{S}$ *canonical over* $L$ iff $\mathcal{W}^{L'} \Vdash \mathtt{S}$ for every extension $L'$ of $L + \{\mathtt{S}\}$. Similarly with 'Routley canonical' in place of 'canonical'. For instance, since the formulas of the form $\phi \to \phi$ are valid in all frames, $\mathtt{N2}^\circ$ turns out to be canonical over $\mathsf{N}$.

For the purposes of the next proof note that if $L$ includes $\mathtt{N2}^\circ$, then some negated formulas belong to $L$, and hence for every $\Gamma \in W^L$ we have $\underline{\Gamma} \neq \varnothing$.

**Theorem 4.2.** *$\mathtt{N1}^\circ$, $\mathtt{N}^*$, $\mathtt{N1}^\bullet$ and $\mathtt{N2}^\bullet$ are canonical over $\mathsf{N} + \{\mathtt{N2}^\circ\}$.*

*Proof.* Let $\mathtt{S}$ be one of the schemes above, and let $L$ be an extension of $\mathsf{N} + \{\mathtt{N2}^\circ, \mathtt{S}\}$. We want to show that $\mathcal{W}^L$ has the property corresponding to $\mathtt{S}$.

$\boxed{\mathtt{N1}^\circ}$ Let $\Gamma \in W^L$. We need to find $\Delta \in W^L$ such that $\underline{\Gamma} \cap \Delta = \varnothing$. It suffices to show that $\nvdash_L \underline{\Gamma}$ — because a suitable $\Delta$ can then be obtained by applying Lemma 2.6. Now assume, by way of contradiction, that $\vdash_L \underline{\Gamma}$. So $\top \vdash_L \underline{\Gamma}$; thus $\top \in \underline{\Gamma}$ by Proposition 2.7, i.e. $\neg\top \in \Gamma$. Hence we obtain $\Gamma = \mathrm{Form}$ (using $\mathtt{N1}^\circ$), a contradiction.

$\boxed{\mathtt{N}^*}$ Let $\Gamma, \Delta, \Sigma \in W^L$ be such that $\underline{\Gamma} \cap \Delta = \varnothing$ and $\underline{\Gamma} \cap \Sigma = \varnothing$. We need to find $\Pi \in W^L$ such that

$$\underline{\Gamma} \cap \Pi \;=\; \varnothing \quad \text{and} \quad \Delta \cup \Sigma \;\subseteq\; \Pi.$$

It suffices to show that $\Delta \cup \Sigma \nvdash_L \underline{\Gamma}$ — because a suitable $\Pi$ can then be obtained by applying Lemma 2.6. Now assume, by way of contradiction, that $\Delta \cup \Sigma \vdash_L \underline{\Gamma}$. Since $\Delta$ and $\Sigma$ are closed under conjunction, while $\underline{\Gamma}$ is closed under disjunction by Proposition 2.7, we have

$$\{\phi, \psi\} \vdash_L \theta \quad \text{for some } \phi \in \Delta,\ \psi \in \Sigma \text{ and } \theta \in \underline{\Gamma}.$$

So $\phi \wedge \psi \to \theta \in L$. Therefore $\neg\theta \to \neg(\phi \wedge \psi) \in L$ (by $\mathtt{CR}$). This implies $\neg(\phi \wedge \psi) \in \Gamma$ (since $\neg\theta \in \Gamma$). Thus $\neg\phi \vee \neg\psi \in \Gamma$ (using $\mathtt{N}^*$), which gives $\neg\phi \in \Gamma$ or $\neg\psi \in \Gamma$, i.e. $\phi \in \underline{\Gamma}$ or $\psi \in \underline{\Gamma}$. Hence we obtain $\underline{\Gamma} \cap \Delta \neq \varnothing$ or $\underline{\Gamma} \cap \Sigma \neq \varnothing$, a contradiction.

$\boxed{\mathtt{N1}^\bullet}$ Since the composition of $R_L$ and $\leqslant_L^{-1}$ coincides with $R_L$, it suffices to prove that $R_L$ is symmetric. Let $\Gamma, \Delta \in W^L$ be such that $\underline{\Gamma} \cap \Delta = \varnothing$. We need to show that $\underline{\Delta} \cap \Gamma = \varnothing$. This is easy: if $\phi \in \Gamma$, then $\neg\neg\phi \in \Gamma$ (using $\mathtt{N1}^\bullet$), i.e. $\neg\phi \in \underline{\Gamma}$, which implies $\neg\phi \notin \Delta$, i.e. $\phi \notin \underline{\Delta}$.

$\boxed{\mathtt{N2}^\bullet}$ Assume, by way of contradiction, that $\mathcal{W}^L$ does not have the property corresponding to $\mathtt{N2}^\bullet$, i.e. there exists $\Gamma \in W^L$ such that for every $\Delta \in W^L$,

$$\underline{\Gamma} \cap \Delta \;=\; \varnothing \quad \Longrightarrow \quad \begin{array}{c} \text{there exists } \Sigma \in W^L \text{ such that} \\ \underline{\Delta} \cap \Sigma = \varnothing \text{ and } \Sigma \nsubseteq \Gamma. \end{array}$$

For each $\Delta \in W^L$ such that $\underline{\Gamma} \cap \Delta = \varnothing$, choose $\Sigma_\Delta \in W^L$ and $\phi_\Delta \in \mathrm{Form}$ satisfying

$$\underline{\Delta} \cap \Sigma_\Delta \;=\; \varnothing \quad \text{and} \quad \phi_\Delta \;\in \Sigma_\Delta \setminus \Gamma.$$

Take $\Pi_0$ to be $\left\{\neg\phi_\Delta \mid \Delta \in W^L \text{ and } \underline{\Gamma} \cap \Delta = \varnothing\right\}$. Observe that $\Pi_0 \neq \varnothing$:

> Assume that $\Pi_0 = \varnothing$, i.e. there exists no $\Delta \in W^L$ such that $\underline{\Gamma} \cap \Delta = \varnothing$. Then $\phi \vdash_L \underline{\Gamma}$ for all $\phi \in \mathrm{Form}$. So $\underline{\Gamma} = \mathrm{Form}$ by Proposition 2.7. Hence we obtain $\Gamma = \mathrm{Form}$ (using $\mathtt{N2}^\bullet$), a contradiction.

Next, it is not hard to show that $\Pi_0 \nvdash_L \underline{\Gamma}$:

> Assume that $\Pi_0 \vdash_L \Gamma$. So $\neg\phi_{\Delta_0} \wedge \ldots \wedge \neg\phi_{\Delta_n} \vdash_L \underline{\Gamma}$ for some $\neg\phi_{\Delta_0}, \ldots, \neg\phi_{\Delta_n} \in \Pi_0$. Thus $\neg\phi_{\Delta_0} \wedge \ldots \wedge \neg\phi_{\Delta_n} \in \underline{\Gamma}$ by Proposition 2.7, i.e.
>
> $$\neg(\neg\phi_{\Delta_0} \wedge \ldots \wedge \neg\phi_{\Delta_n}) \;\in\; \Gamma.$$

> From this we obtain $\neg\neg\phi_{\Delta_0} \vee \ldots \vee \neg\neg\phi_{\Delta_n} \in \Gamma$ (using $\mathtt{N}^*$), and hence $\phi_{\Delta_0} \vee \ldots \vee \phi_{\Delta_n} \in \Gamma$ (using $\mathtt{N2}^\bullet$).[13] Therefore one of $\phi_{\Delta_0}, \ldots, \phi_{\Delta_n}$ must be in $\Gamma$, a contradiction.

Finally, let $\Pi \in W^L$ be such that $\Pi_0 \subseteq \Pi$ and $\Pi \nvdash_L \underline{\Gamma}$; the latter implies $\underline{\Gamma} \cap \Pi = \varnothing$, of course. Then $\phi_\Pi \in \Sigma_\Pi \subseteq \text{Form} \setminus \underline{\Pi}$, which contradicts $\neg\phi_\Pi \in \Pi_0 \subseteq \Pi$. ☐

**Corollary 4.3.** *Let* $S \subseteq \{\mathtt{N1}^\circ, \mathtt{N}^*, \mathtt{N1}^\bullet, \mathtt{N2}^\bullet\}$. *Then* $(\mathsf{N} \cup \{\mathtt{N2}^\circ\}) + S$ *is canonical.*

*Proof.* Immediate. ☐

**Theorem 4.4.** $\mathtt{N1}^\bullet$ *and* $\mathtt{N2}^\bullet$ *are Routley canonical over* $\mathsf{N}^*$.

*Proof.* Let $\mathtt{S}$ be one of the schemes above, and let $L$ be an extension of $\mathsf{N}^* + \{\mathtt{S}\}$. We want to show that $\mathcal{W}^L$ has the property corresponding to $\mathtt{S}$.

$\boxed{\mathtt{N1}^\bullet}$ It suffices to show that for any $\Gamma \in W^L$ we have $\Gamma \subseteq \Gamma^{**}$. This is easy: if $\phi \in \Gamma$, then $\neg\neg\phi \in \Gamma$ (using $\mathtt{N1}^\bullet$), i.e. $\neg\phi \notin \Gamma^*$, i.e. $\phi \in \Gamma^{**}$.

$\boxed{\mathtt{N2}^\bullet}$ Similarly to $\mathtt{N1}^\bullet$. ☐

**Corollary 4.5.** $\mathsf{N}^* + \{\mathtt{N1}^\bullet\}$, $\mathsf{N}^* + \{\mathtt{N2}^\bullet\}$ *and* $\mathsf{N}^\bullet$ *are Routley canonical.*

*Proof.* Immediate. ☐

## 5. THE METHOD OF FILTRATION REVISITED

The decidability of $\mathsf{N}$ and $\mathsf{N}^*$ can be established using the method of filtration as presented in [4, Section 4]. We are going to develop a somewhat more flexible approach to filtrations, which will lead to further decidability results.

### 5.1. General filtrations.

Fix a model $\mathcal{M} = \langle \mathcal{W}, \xi \rangle$. Let $\Phi \subseteq \text{Form}$ be closed under subformulas. Take

$$\equiv_\Phi \;:=\; \{(x,y) \in W^2 \mid \text{for all } \phi \in \Phi,\ \mathcal{M}, x \Vdash \phi \text{ iff } \mathcal{M}, y \Vdash \phi\}.$$

For every $x \in W$, denote by $[x]_\Phi$ the equivalence class of $x$ under $\equiv_\Phi$. We shall often omit the subscript $_\Phi$ if it is clear from the context. By a *$\Phi$-filtration of $\mathcal{M}$* we mean a model

$$\mathcal{M}' \;=\; \langle \langle W_\Phi, \leqslant', R' \rangle, \xi_\Phi \rangle$$

where:

- $W_\Phi$ is $\{[x] \mid x \in W\}$;
- $\leqslant'$ is such that:
  - for all $x, y \in W$, if $x \leqslant y$, then $[x] \leqslant' [y]$;
  - for all $x, y \in W$ and $\phi \in \Phi$, if $[x] \leqslant' [y]$ and $\mathcal{M}, x \Vdash \phi$, then $\mathcal{M}, y \Vdash \phi$;
- $R'$ is such that:
  - for all $x, y \in W$, if $xRy$, then $[x]\, R'\, [y]$;
  - for all $x, y \in W$ and $\neg\phi \in \Phi$, if $[x]\, R'\, [y]$ and $\mathcal{M}, x \Vdash \neg\phi$, then $\mathcal{M}, y \nVdash \phi$;
- $\xi_\Phi$ is the function mapping each $p \in \text{Prop}$ to $\{[x] \mid x \in \xi(p) \text{ and } p \in \Phi\}$.

Obviously, $W_\Phi$ and $\xi_\Phi$ are both uniquely determined by $\mathcal{M}$ and $\Phi$, unlike $\leqslant'$ and $R'$. Define

$$\begin{aligned} \leqslant_\Phi \;&:=\; \text{the transitive closure of } \{([x],[y]) \mid x \leqslant y\}, \\ R_\Phi \;&:=\; \text{the composition of } \leqslant_\Phi,\ \{([x],[y]) \mid xRy\} \text{ and } \leqslant_\Phi^{-1}. \end{aligned}$$

To make this definition easier to handle, denote by $\sqsubseteq$ the transitive closure of $\leqslant \cup \equiv$, i.e. $x \sqsubseteq y$ iff there exist $x_0, \ldots, x_n \in W$ such that:

---

[13] By Proposition 3.2, $\mathtt{N}^*$ is derivable in $L$.

- $x_0 = x$ and $x_n = y$;
- for every $i \in \{0, \dots, n-1\}$ we have $x_i \leqslant x_{i+1}$ or $x_i \equiv x_{i+1}$.

Now, using $\sqsubseteq$, the relations $\leqslant_\Phi$ and $R_\Phi$ can be described as follows:

$$\begin{aligned} [x] \leqslant_\Phi [y] \quad &\Longleftrightarrow \quad x \sqsubseteq y; \\ [x]\, R_\Phi\, [y] \quad &\Longleftrightarrow \quad x \sqsubseteq u\, R\, v \sqsupseteq y \quad \text{for some } u, v \in W. \end{aligned}$$

On the other hand, following [4, Section 4], one may consider

$$\begin{aligned} \leqslant^\Phi \; &:= \; \{([x],[y]) \mid \text{for all } \phi \in \Phi, \text{ if } \mathcal{M}, x \Vdash \phi, \text{ then } \mathcal{M}, y \Vdash \phi\}, \\ R^\Phi \; &:= \; \{([x],[y]) \mid \text{for all } \neg\phi \in \Phi, \text{ if } \mathcal{M}, x \Vdash \neg\phi, \text{ then } \mathcal{M}, y \nVdash \phi\}. \end{aligned}$$

Substituting $\leqslant_\Phi$, $R_\Phi$ and $\leqslant^\Phi$, $R^\Phi$ for $\leqslant'$, $R'$, we get

$$\mathcal{M}_\Phi \; := \; \langle\langle W_\Phi, \leqslant_\Phi, R_\Phi\rangle, \xi_\Phi\rangle \quad \text{and} \quad \mathcal{M}^\Phi \; := \; \langle\langle W_\Phi, \leqslant^\Phi, R^\Phi\rangle, \xi_\Phi\rangle.^{14}$$

Naturally, $\langle W_\Phi, \leqslant_\Phi, R_\Phi\rangle$ and $\langle W_\Phi, \leqslant^\Phi, R^\Phi\rangle$ are abbreviated to $\mathcal{W}_\Phi$ and $\mathcal{W}^\Phi$ respectively. It is easy to see that for every $\Phi$-filtration $\mathcal{M}'$ of $\mathcal{M}$,

$$\leqslant_\Phi \subseteq \leqslant' \subseteq \leqslant^\Phi \quad \text{and} \quad R' \subseteq R^\Phi;$$

furthermore, in the case where $\mathcal{M}'$ is strictly condensed we also have $R_\Phi \subseteq R'$.

**Proposition 5.1.** *Let $\mathcal{M}$ and $\Phi$ be as above. Then $\mathcal{W}_\Phi$ is a strictly condensed frame and $\mathcal{M}_\Phi$ is a $\Phi$-filtration of $\mathcal{M}$.*

*Proof.* Obviously, $\leqslant_\Phi$ is a preordering on $W_\Phi$. We also have $\leqslant_\Phi \circ R_\Phi \circ \leqslant_\Phi^{-1} \subseteq R_\Phi$:

> Let $x, y \in W$. Suppose there exist $u, v \in W$ such that
>
> $$[x] \;\leqslant_\Phi\; [u], \quad [u]\, R_\Phi\, [v] \quad \text{and} \quad [y] \;\leqslant_\Phi\; [v].$$
>
> Then $x \sqsubseteq u$, $y \sqsubseteq v$ and there are $s, t \in W$ such that $u \sqsubseteq s\, R\, t \sqsupseteq v$. Hence $x \sqsubseteq s\, R\, t \sqsupseteq y$, which implies $[x]\, R_\Phi\, [y]$.

Thus $\mathcal{W}_\Phi$ is a strictly condensed frame. The rest is routine. □

**Proposition 5.2** (see [4, Section 4])**.** *Let $\mathcal{M}$ and $\Phi$ be as above. Then $\mathcal{W}^\Phi$ is a strictly condensed frame and $\mathcal{M}^\Phi$ is a $\Phi$-filtration of $\mathcal{M}$.*

*Proof.* Evidently, $\leqslant_\Phi$ is a preordering on $W_\Phi$. We also have $\leqslant^\Phi \circ R^\Phi \circ \left(\leqslant^\Phi\right)^{-1} \subseteq R_\Phi$:

> Let $x, y \in W$. Suppose there exist $u, v \in W$ such that
>
> $$[x] \;\leqslant^\Phi\; [u], \quad [u]\, R^\Phi\, [v] \quad \text{and} \quad [y] \;\leqslant^\Phi\; [v].$$
>
> Observe that for all $\neg\phi \in \Phi$, if $\mathcal{M}, x \Vdash \neg\phi$, then $\mathcal{M}, u \Vdash \neg\phi$, and hence $\mathcal{M}, v \nVdash \phi$, which implies $\mathcal{M}, y \nVdash \phi$.

Thus $\mathcal{W}^\Phi$ is a strictly condensed frame. The rest is clear. □

**Lemma 5.3.** *Let $\mathcal{M}$, $\Phi$ and $\mathcal{M}'$ be as above. Then for any $x \in W$ and $\phi \in \Phi$,*

$$\mathcal{M}, x \Vdash \phi \quad \Longleftrightarrow \quad \mathcal{M}', [x] \Vdash \phi.$$

*Proof.* By induction on the complexity of $\phi$.

The case where $\phi \in \mathrm{Prop}$ is trivial.

Suppose $\phi = \neg\psi$. Consider each of the two implications separately.

$\boxed{\Longrightarrow}$ Assume $x \Vdash \phi$. Let $y \in W$ be such that $[x]\, R'\, [y]$. Then $y \nVdash \psi$. So we have $[y] \nVdash \psi$ by the inductive hypothesis.

$\boxed{\Longleftarrow}$ Assume $[x] \Vdash \phi$. Let $y \in W$ be such that $xRy$. Then $[x]\, R'\, [y]$, and hence $[y] \nVdash \psi$. So we have $y \nVdash \psi$ by the inductive hypothesis.

---

[14] In [4], only filtrations of the form $\mathcal{M}^\Phi$ (which are, in a sense, rather 'syntactic') were considered. Since our approach allows other kinds of filtration, it appears to be more flexible.

The other cases can be handled as in intuitionistic logic. ☐

By a standard argument, this gives the following.

**Theorem 5.4** (see [4, Section 4]). $\mathsf{N}$ *has the finite model property and is decidable.*

Further applications can be obtained by studying what happens to a given frame property when we pass from $\mathcal{M}$ to $\mathcal{M}_\Phi$ or $\mathcal{M}^\Phi$ for a suitable $\Phi$.

**Lemma 5.5.** *Let* $\mathcal{M}$, $\Phi$ *and* $\mathcal{M}'$ *be as above, and let* $\mathtt{S} \in \{\mathtt{N1}^\circ, \mathtt{N2}^\circ\}$. *Suppose* $\mathcal{W} \Vdash \mathtt{S}$. *Then* $\mathcal{W}' \Vdash \mathtt{S}$.[15]

*Proof.* $\boxed{\mathtt{N1}^\circ}$ Since $\mathcal{W}$ is serial (see Table 1), so is $\mathcal{W}'$. Thus $\mathcal{W}' \Vdash \mathtt{N1}^\circ$.

$\boxed{\mathtt{N2}^\circ}$ Immediate from Lemma 5.3 — because $\mathtt{N2}^\circ$ may be treated as variable-free. ☐

**Theorem 5.6.** $\mathsf{N} + \{\mathtt{N2}^\circ\}$ *and* $\mathsf{N}^\circ$ *have the finite model property and are decidable.*

Concerning more complex schemes:

**Lemma 5.7.** *Let* $\mathcal{M}$ *and* $\Phi$ *be as above. Suppose* $\mathcal{W} \Vdash \mathtt{N1}^\bullet$. *Then* $\mathcal{W}_\Phi \Vdash \mathtt{N1}^\bullet$.

*Proof.* Note the composition of $R_\Phi$ and $\leqslant_\Phi^{-1}$ coincides with $R_\Phi$. So it suffices to show that $R_\Phi$ is symmetric (see Table 1). Let $x, y \in W$ be such that $[x]\, R_\Phi\, [y]$. Then $x \sqsubseteq u\ R\ v \sqsupseteq y$ for some $u, v \in W$. Since $R \circ \leqslant^{-1}$ is symmetric, there exists $t \in W$ such that $v\ R\ t \geqslant u$. Hence

$$y \ \sqsubseteq\ v\ \ R\ \ t\ \sqsupseteq\ u\ \sqsupseteq\ x.$$

Therefore $[y]\, R_\Phi\, [x]$. ☐

**Theorem 5.8.** $\mathsf{N} + \{\mathtt{N1}^\bullet\}$ *and* $\mathsf{N}^\circ + \{\mathtt{N1}^\bullet\}$ *have the finite model property and are decidable.*

For each $\Phi \subseteq \mathrm{Form}$, denote by $N(\Phi)$ the closure of $\Phi$ under negation, i.e. the least set $\Psi$ of formulas such that $\Phi \subseteq \Psi$ and $\{\neg\phi \mid \phi \in \Psi\} \subseteq \Psi$. Notice that the scheme $\neg\neg\phi \leftrightarrow \neg\neg\neg\neg\phi$ can be derived in $\mathsf{N} + \{\mathtt{N1}^\bullet\}$ as follows:

| | | |
|---|---|---|
| 1 | $\neg\phi \to \neg\neg\neg\phi$ | $\mathtt{N1}^\bullet$ |
| 2 | $\neg\neg\neg\neg\phi \to \neg\neg\phi$ | `from 1 (by CR)` |
| 3 | $\neg\neg\phi \to \neg\neg\neg\neg\phi$ | $\mathtt{N1}^\bullet$ |
| 4 | $\neg\neg\phi \leftrightarrow \neg\neg\neg\neg\phi$ | `from 2, 3.` |

Hence if $\Phi \subseteq \mathrm{Form}$ is finite, then $N(\Phi)$ may be treated as finite modulo $\mathsf{N} + \{\mathtt{N1}^\bullet\}$, so $N(\Phi)$ is finitely based over any model whose frame validates $\mathtt{N1}^\bullet$.[16]

**Lemma 5.9.** *Let* $\mathcal{M}$ *and* $\Phi$ *be as above. Suppose* $\mathcal{W} \Vdash \mathtt{N1}^\bullet$. *Then* $\mathcal{W}^{N(\Phi)} \Vdash \mathtt{N1}^\bullet$.

*Proof.* For convenience, denote $N(\Phi)$ by $\Psi$. Let $x, y \in W$ be such that $[x]\, R^\Psi\, [y]$. For every $\phi \in \Psi$, if $\mathcal{M}, x \Vdash \phi$, then $\mathcal{M}, x \Vdash \neg\neg\phi$ (by $\mathtt{N1}^\bullet$), and therefore $\mathcal{M}, y \nVdash \neg\phi$ (because $\neg\phi \in \Psi$ and $[x]\, R^\Psi\, [y]$). Thus $[y]\, R^\Psi\, [x]$. ☐

This gives us another way of proving Theorem 5.8.

For each $\Phi \subseteq \mathrm{Form}$ we denote by $C(\Phi)$ the closure of $\Phi$ under conjunction, disjunction and negation. Evidently, the scheme $\neg\phi \leftrightarrow \neg\neg\neg\phi$ can be derived in $\mathsf{N} + \{\mathtt{N2}^\bullet\}$:

| | | |
|---|---|---|
| 1 | $\neg\neg\phi \to \phi$ | $\mathtt{N2}^\bullet$ |
| 2 | $\neg\phi \to \neg\neg\neg\phi$ | `from 1 (by CR)` |
| 3 | $\neg\neg\neg\phi \to \neg\phi$ | $\mathtt{N2}^\bullet$ |
| 4 | $\neg\phi \leftrightarrow \neg\neg\neg\phi$ | `from 2, 3.` |

---

[15] Here $\mathcal{W}'$ abbreviates $\langle W_\Phi, \leqslant', R' \rangle$.

[16] It follows that for any $\mathcal{M}$ and $\Phi$ as above, if $\mathcal{W} \Vdash \mathtt{N1}^\bullet$ and $\Phi$ is finite, then $W_{N(\Phi)}$ is finite — though $N(\Phi)$ is infinite, provided that $\Phi \neq \varnothing$.

Also, it is known that the following schemes are derivable in $\mathsf{N} + \{\mathtt{N}^*\}$, and hence in $\mathsf{N} + \{\mathtt{N2}^\bullet\}$ (by Proposition 3.2):

- $\neg(\phi \vee \psi) \leftrightarrow (\neg\phi \wedge \neg\psi)$;
- $\neg(\phi \wedge \psi) \leftrightarrow (\neg\phi \vee \neg\psi)$.

Consequently, if $\Phi \subseteq \text{Form}$ is finite, then $C(\Phi)$ may be treated as finite modulo $\mathsf{N} + \{\mathtt{N2}^\bullet\}$, so $C(\Phi)$ is finitely based over any model whose frame validates $\mathtt{N2}^\bullet$.

**Lemma 5.10.** *Let $\mathcal{M}$ and $\Phi$ be as above, with $\Phi$ finite. Suppose $\mathcal{W} \Vdash \mathtt{N2}^\bullet$. Then $\mathcal{W}^{C(\Phi)} \Vdash \mathtt{N2}^\bullet$.*

*Proof.* For convenience, take $\Psi := C(\Phi)$. Assume, by way of contradiction, that $\mathcal{W}^\Psi$ does not have the property corresponding to $\mathtt{N2}^\bullet$, i.e. there exists $x \in W$ such that for every $y \in W$,

$$[x]\, R^\Psi\, [y] \quad \Longrightarrow \quad \begin{array}{c}\text{there exists } z \in W \text{ such that}\\ [y]\, R^\Psi\, [z] \text{ and } [z] \not\leqslant^\Psi [x].\end{array}$$

For each $y \in W$ such that $[x]\, R^\Psi\, [y]$, choose $z_y \in W$ and $\phi_y \in \Psi$ satisfying

$$[y]\, R^\Psi\, [z_y], \quad \mathcal{M}, z_y \Vdash \phi_y \quad \text{and} \quad \mathcal{M}, x \nVdash \phi_y.$$

Take $\Theta$ to be $\{\phi_y \mid y \in W \text{ and } [x]\, R^\Psi\, [y]\}$. Obviously, since $\mathcal{W}$ is serial (recall Proposition 3.2), $\Theta$ is non-empty. Moreover, it can be treated as a finite set (see the comments made just before this lemma). Consider

$$\theta \;:=\; \bigvee \Theta.$$

We have $\mathcal{M}, x \nVdash \theta$, which implies $\mathcal{M}, x \nVdash \neg\neg\theta$ (by $\mathtt{N2}^\bullet$), so $\mathcal{M}^\Psi, [x] \nVdash \neg\neg\theta$ by Lemma 5.3. On the other hand, for every $y \in W$, if $[x]\, R^\Phi\, [y]$, then $\mathcal{M}, z_y \Vdash \theta$, so $\mathcal{M}^\Psi, [z_y] \Vdash \theta$ by Lemma 5.3, and therefore $\mathcal{M}^\Psi, [y] \nVdash \neg\theta$. Thus $\mathcal{M}^\Psi, [x] \Vdash \neg\neg\theta$, a contradiction. □

**Theorem 5.11.** *$\mathsf{N}^* + \{\mathtt{N2}^\bullet\}$ and $\mathsf{N}^\bullet$ have the finite model property and are decidable.*

5.2. **More specific filtrations.** Now we are going to present a different way of proving Theorem 5.11. It uses a special kind of filtration suitable for Routley models whose frames validate $\mathtt{N1}^\bullet$ or $\mathtt{N2}^\bullet$.

Fix a Routley model $\mathbf{M}$. Let $\Phi \subseteq \text{Form}$ be closed under subformulas and under negation — so in particular, $N(\Phi) = \Phi$. Define the *special $\Phi$-filtration of* $\mathbf{M}$ to be

$$\mathbf{M}_\Phi \;=\; \langle\langle W_\Phi, \leqslant_\Phi, *_\Phi\rangle, \xi_\Phi\rangle$$

where $W_\Phi$, $\leqslant_\Phi$ and $\xi_\Phi$ are as before, and $*_\Phi$ maps each $[x]$ in $W_\Phi$ to $[x^*]$. Note that since $\Phi$ is closed under negation, the definition of $*_\Phi$ is correct; furthermore, one can easily check that $*_\Phi$ is anti-monotone with respect to $\leqslant_\Phi$. Thus $\mathbf{M}_\Phi$ is a Routley model. Naturally, we shall abbreviate $\langle W_\Phi, \leqslant_\Phi, *_\Phi\rangle$ to $\mathbf{W}_\Phi$.

**Lemma 5.12.** *Let $\mathbf{M}$ and $\Phi$ be as above. Then for any $x \in W$ and $\phi \in \Phi$,*

$$\mathbf{M}, x \Vdash \phi \quad \Longleftrightarrow \quad \mathbf{M}_\Phi, [x] \Vdash \phi.$$

*Proof.* By induction on the complexity of $\phi$.

The case where $\phi \in \text{Prop}$ is trivial.

Suppose $\phi = \neg\psi$. Then

$$\begin{aligned}\mathbf{M}, x \Vdash \phi \quad &\Longleftrightarrow \quad \mathbf{M}, x^* \nVdash \psi\\ &\Longleftrightarrow \quad \mathbf{M}_\Phi, [x^*] \nVdash \psi\\ &\Longleftrightarrow \quad \mathbf{M}_\Phi, [x]^* \nVdash \psi\\ &\Longleftrightarrow \quad \mathbf{M}_\Phi, [x] \Vdash \phi\end{aligned}$$

where $[x]^*$ stands for $*_\Phi([x])$, of course.

The other cases can be handled as in intuitionistic logic. $\square$

**Lemma 5.13.** *Let* $\mathbf{M}$ *and* $\Phi$ *be as above, and let* $\mathtt{S} \in \{\mathtt{N1}^\bullet, \mathtt{N2}^\bullet\}$. *Suppose* $\mathbf{W} \Vdash \mathtt{S}$. *Then* $\mathbf{W}_\Phi \Vdash \mathtt{S}$.

*Proof.* It is easy to check that for every Routley frame $\mathbf{W}'$:

$$\mathbf{W}' \Vdash \mathtt{N1}^\bullet \quad \Longleftrightarrow \quad x \leqslant x^{**} \text{ for all } x \in W';$$
$$\mathbf{W}' \Vdash \mathtt{N2}^\bullet \quad \Longleftrightarrow \quad x^{**} \leqslant x \text{ for all } x \in W'.$$

In particular, this holds for $\mathbf{W}' \in \{\mathbf{W}, \mathbf{W}_\Phi\}$.

$\boxed{\mathtt{N1}^\bullet}$ For every $x \in W$ we have $[x] \leqslant_\Phi [x^{**}] = [x^*]^* = [x]^{**}$. Thus $\mathbf{W}_\Phi \Vdash \mathtt{N1}^\bullet$.

$\boxed{\mathtt{N2}^\bullet}$ Similarly to $\mathtt{N1}^\bullet$. $\square$

This leads to a refinement of Theorem 5.11:

**Theorem 5.14.** $\mathsf{N}^* + \{\mathtt{N1}^\bullet\}$, $\mathsf{N}^* + \{\mathtt{N2}^\bullet\}$ *and* $\mathsf{N}^\bullet$ *have the finite model property in terms of Routley models and are decidable.*

## 6. FURTHER DISCUSSION

One may wish to look at $\mathsf{N}$ and its extensions from a somewhat more general point of view. In particular, following [12], Došen's semantics can be modified by replacing $\langle W, \leqslant, R\rangle$ by $\langle W, \leqslant, R, N\rangle$ where $N$ is a subset of $W$ such that for any $x, y \in W$,

$$x \in N \text{ and } x \leqslant y \quad \Longrightarrow \quad y \in N$$

(the elements of $N$ are called *normal worlds*). Then $\mathcal{M}, x \Vdash \phi$ is defined as before, except that the negation clause becomes a bit more complicated:

$$\mathcal{M}, x \Vdash \neg\psi \quad :\Longleftrightarrow \quad (\mathcal{M}, y \nVdash \psi \text{ for all } y \in R(x)) \text{ and } x \in N.$$

Naturally, some of the claims made with Došen's semantics in mind may fail when we pass to the modified semantics. One may proceed to study the problems discussed above in this more general setting.

## REFERENCES

[1] N. Bezhanishvili, A. Colacito, D. de Jongh, *A study of subminimal logics of negation and their modal companions*, In: A. Silva et al., eds., *Language, Logic, and Computation: Proceedings of the 12th International Tbilisi Symposium*, LNCS **11456**, Springer, 2019, 21–41.

[2] P. Cabalar, S.P. Odintsov, D. Pearce, *Logical foundations of well-founded semantics*, In: P. Doherty (ed.) et al., *Proceedings KR 2006*, AAAI Press, 2006, 25–35.

[3] K. Došen, *Negation as a modal operator*, Rep. Math. Logic, **20** (1986), 15–28. Zbl 0626.03006

[4] S.A. Drobyshevich, S.P. Odintsov, *Finite model property for negative modalities*, Sib. Élektron. Mat. Izv., **10** (2013), 1–21. Zbl 1330.03043

[5] A.P. Hazen, *Is even minimal negation constructive?*, Analysis, Oxf., **55**:2 (1995), 105–107. Zbl 0943.03611

[6] H. Leitgeb, *HYPE: a system of hyperintensional logic (with an application to semantic paradoxes)*, J. Philos. Log., **48**:2 (2019), 305–405. Zbl 1457.03057

[7] G.C. Moisil, *Logique modale*, Disqu. Math. Phys., Bucuresti, **2** (1942), 3–98. Zbl 0063.04064

[8] S.P. Odintsov, H. Wansing, *Routley star and hyperintensionality*, J. Philos. Log., **50**:1 (2021), 33–56. Zbl 7345775

[9] S.P. Odintsov, *Combining intuitionistic connectives and Routley negation*, Sib. Èlektron. Mat. Izv., **7** (2010), 21–41. Zbl 1329.68058

[10] K. Segerberg, *Propositional logics related to Heyting's and Johansson's*, Theoria, **34**:1 (1968), 26–61.

[11] S.O. Speranski, *Negation as a modality in a quantified setting*, J. Log. Comput., **31**:5 (2021), 1330–1355. Zbl 7398787

[12] D. Vakarelov, *Consistency, completeness and negation*, In: G. Priest (ed.) et al., *Paraconsistent Logics. Essays on the Inconsistent*, Philosophia Verlag, München etc., 1989, 238–363. Zbl 0692.03016

Stanislav O. Speranski
Steklov Mathematical Institute of Russian Academy of Sciences,
8, Gubkina str.,
Moscow, 119991, Russia
*Email address*: `katze.tail@gmail.com`