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A FASTER ALGORITHM FOR COUNTING THE INTEGER POINTS NUMBER IN Δ-MODULAR POLYHEDRA

D.V. GRIBANOV, D.S. MALYSHEV

Abstract. Let a polytope $\mathcal{P}$ be defined by a system $Ax \leq b$. We consider the problem to count a number of integer points inside $\mathcal{P}$, assuming that $\mathcal{P}$ is Δ-modular. The polytope $\mathcal{P}$ is Δ-modular if all the rank sub-determinants of A are bounded by Δ in the absolute value.

We present a new FPT-algorithm, parameterized by Δ and by the number of simple cones in the normal fun triangulation of $\mathcal{P}$, which is more efficient for Δ-modular problems, than the approach of A. Barvinok et al. [1, 2, 3, 4, 5]. To this end, we do not directly compute the short rational generating function for $\mathcal{P} \cap \mathbb{Z}^n$, which is commonly used for the considered problem. We compute its particular representation in the form of exponential series that depends on one variable, using the dynamic programming principle. We completely do not use the A. Barvinok's unimodular sign decomposition technique.

Using our new complexity bound, we consider different special cases that may be of independent interest. For example, we give FPT-algorithms for counting the integer points number in Δ-modular simplicies and similar polytopes that have $n + O(1)$ facets. For any fixed m, we give an FPT-algorithm to count solutions of the unbounded m-dimensional Δ-modular knapsack problem. For the case, when Δ grows slowly with

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respect to n, we give a counting algorithm, which is more effective, than the state of the art ILP feasibility algorithm due to [6, 7].
Keywords: integer linear programming, short rational generating function, bounded sub-determinants, multidimensional knapsack problem, subset-sum problem, counting problem.

1. Introduction

1.1. **Brief discussion of our results.** Let a polytope $\mathcal{P}$ be defined by one of the following ways:

(i) **System in the canonical form:** $\mathcal{P} = \{x \in \mathbb{R}^n \colon Ax \leq b\}$, where $A \in \mathbb{Z}^{(n+m)\times n}$, $b \in \mathbb{Q}^{(n+m)}$, $\operatorname{rank}(A) = n$ and $d := \dim(\mathcal{P}) = n$;
(ii) **System in the standard form:** $\mathcal{P} = \{x \in \mathbb{R}^n_+ \colon Ax = b\}$, where $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, $\operatorname{rank}(A) = m$ and $d := \dim(\mathcal{P}) = n - m$;

and let all the rank-order sub-determinants of A be bounded by Δ in the absolute values. We show that $|\,\mathcal{P} \cap \mathbb{Z}^n\,|$ can be computed with an algorithm, having the arithmetic complexity bound

$$O\big(\nu \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)\big),$$

where ν is the maximal possible number of vertices in a d-dimensional polytope $\mathcal{P}$, defined by one of the systems above.

1.2. **Basic definitions and notations.** Let $A \in \mathbb{Z}^{m\times n}$. We denote by A_{ij} its ij-th element, by A_{i*} its i-th row, and by A_{*j} its j-th column. For subsets $I \subseteq \{1,\dots,m\}$ and $J \subseteq \{1,\dots,n\}$, the symbol A_{IJ} denote the sub-matrix of A, which is generated by all the rows with indices in I and all the columns with indices in J. If I or J is replaced by $*$, then all the rows or columns are selected, respectively. Sometimes, we simply write A_I instead of A_{I*} and A_J instead of A_{*J}, if this does not lead to confusion.

The maximum absolute value of entries of a matrix A is denoted by $\|A\|_{\max} = \max\limits_{i,j}|A_{i\,j}|$. The l_p-norm of a vector x is denoted by $\|x\|_p$. The number of non-zero components of a vector x is denoted by $\|x\|_0 = \big|\{i \colon x_i \neq 0\}\big|$.

For $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$, $\{x\}$, and $\lceil x \rceil$ the floor, fractional part, and ceiling of x, respectively.

For $c, x \in \mathbb{R}^n$, by $\langle c, x\rangle$ we denote the standard scalar product of two vectors. In other words, $\langle c, x\rangle = c^\top x$.

Let $S \in \mathbb{Z}^{n\times n}_{\geq 0}$ be a diagonal matrix and $v \in \mathbb{Z}^n$. We denote by $v \bmod S$ the vector, whose i-th component equals $v_i \bmod S_{ii}$. For $\mathcal{M} \subseteq \mathbb{Z}^n$, we denote $\mathcal{M} \bmod S = \{v \bmod S \colon v \in \mathcal{M}\}$. For example, the set $\mathbb{Z}^n \bmod S$ consists of $\det(S)$ elements.

Definition 1. *For a matrix $A \in \mathbb{Z}^{m\times n}$, by*

$$\Delta_k(A) = \max\left\{\big|\det(A_{IJ})\big| : I \subseteq \{1,\dots,m\},\ J \subseteq \{1,\dots,n\},\ |I| = |J| = k\right\},$$

we denote the maximum absolute value of determinants of all the $k \times k$ sub-matrices of A. By $\Delta_{\gcd}(A,k)$ we denote the greatest common divisor of determinants of all the $k \times k$ sub-matrices of A. Additionally, let $\Delta(A) = \Delta_{\operatorname{rank}(A)}(A)$ and $\Delta_{\gcd}(A) = \Delta_{\gcd}(A, \operatorname{rank}(A))$.

If $\Delta(A) \leq \Delta$, for some $\Delta > 0$, then A is called Δ-modular.

Definition 2. *For $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, we denote*

$$\mathcal{P}(A,b) = \{x \in \mathbb{R}^n \colon Ax \le b\},$$

$$\mathbf{x}^z = x_1^{z_1} \cdot \ldots \cdot x_n^{z_n}, \quad \text{and} \quad \mathfrak{f}(\mathcal{P};\mathbf{x}) = \sum_{z \in \mathcal{P} \cap \mathbb{Z}^n} \mathbf{x}^z .$$

1.3. **The lattice points counting problem and the detailed description of our results.** In this paper, we consider the problem to count integer points in a polyhedron, which is defined as follows:

Problem 1. *Let $\mathcal{P}$ be a rational polytope defined by one of the following ways:*

(1) *The polytope $\mathcal{P}$ is defined by* a system in the canonical form*: $\mathcal{P} = \{x \in \mathbb{R}^n \colon Ax \le b\}$, where $A \in \mathbb{Z}^{(n+m) \times n}$, $b \in \mathbb{Q}^{n+m}$, and $\dim(\mathcal{P}) = \operatorname{rank}(A) = n$;*

(2) *The polytope $\mathcal{P}$ is defined by* a system in the standard form*: $\mathcal{P} = \{x \in \mathbb{R}^n_{\ge 0} \colon Ax = b\}$, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Q}^m$, $\operatorname{rank}(A) = m$, $\dim(\mathcal{P}) = n - m$ and $\Delta_{\mathrm{gcd}}(A) = 1$.*

The problem at state is to compute the value of $|\,\mathcal{P} \cap \mathbb{Z}^n\,|$.

Theorem 1. *Problem 1 can be solved with an algorithm, having the arithmetic complexity bound*

$$O\big(\nu(d,m,\Delta) \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)\big),$$

where $\Delta = \Delta(A)$, $d = \dim(\mathcal{P})$ ($d = n$, for the canonical form, and $d = n - m$, for the standard form) and $\nu(d,m,\Delta)$ is the maximal possible number of vertices in a d-dimensional polytope of problem 1.

Using this theorem and results of the papers [8, 9] that can help to bound the value of $\nu(d,m,\Delta)$, we present new complexity bounds for problem 1. Additionally, we show how to handle the case of unbounded polyhedron.

Corollary 1. *The arithmetic complexity of an algorithm by Theorem 1 can be bounded with the following relations:*

(1) *The bound $O\big(\frac{d}{m}+1\big)^m \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)$ that is polynomial in d and Δ, for any fixed m;*

(2) *The bound $O\big(\frac{m}{d}+1\big)^{\frac{d}{2}} \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)$ that is polynomial in m and Δ, for any fixed d;*

(3) *The bound $O(d)^{3+\frac{d}{2}} \cdot \Delta^{4+d} \cdot \log(\Delta)$ that is polynomial in Δ, for any fixed d.*

To handle the case, when $\mathcal{P}$ is an unbounded polyhedron, we need to pay an additional factor of $O(\frac{d}{m}+1) \cdot d^4$ in the first bound and $O(d^4)$ in the second bound. The third bound stays unchanged.

Proofs of Theorem 1 and Corollary 1 will be given in Section 2 and Subsection 2.4, respectively.

Taking $m = 1$, the first bound can be used to count the number of integer points in a simplex or the number of solutions of *the unbounded subset-sum problem* $w^\top x = w_0$, $x \in \mathbb{Z}^n_{\ge 0}$. For both problems, it gives the arithmetic complexity bound $O\big(n^4 \cdot \Delta^4 \cdot \log(\Delta)\big)$, where $\Delta = \|w\|_\infty$ for the subset-sum problem.

The second and third bounds can be used to obtain a faster algorithm for the ILP feasibility problem, when the parameters m and Δ are relatively small. For example, taking $m = O(d)$ and $\Delta = 2^{O(d)}$ in the second bound, it becomes $2^{O(d)}$, which is faster, than the state of the art algorithm, due to [6, 7] (see also [10, 11, 12], for a bit more general setting) that has the complexity bound $O(d)^d \cdot \operatorname{poly}(\operatorname{size}(A,b))$. Substituting $\Delta = (\log d)^{O(1)}$ to the third bound, it gives $O(d)^{\frac{d}{2}+o(d)}$, which again is better, than the general case bound $O(d)^d \cdot \operatorname{poly}(\operatorname{size}(A,b))$.

Remark 1. *We are interested in development of algorithms that will be polynomial, when we bound some of the parameters d, m, and Δ. Due to* [13, Corollary 3]*, the problem in the standard form can be polynomially reduced to the problem in the canonical form maintaining values of m and Δ, see also* [14, Lemmas 4 and 5] *and* [15] *for a more general reduction. Hence, in the proofs we will only consider polytopes defined by systems in the canonical form.*

Remark 2. *To simplify analysis, we assume that $\Delta_{\text{gcd}}(A) = 1$ for ILP problems in the standard form. It can be done without loss of generality, because the original system $Ax = b$, $x \geq \mathbf{0}$ can be polynomially transformed to the equivalent system with $\Delta_{\text{gcd}}(A) = 1$. For the justification see* [13, Remark 3].

Good surveys on the related Δ-modular ILP problems and parameterised ILP complexity are given in [16, 14, 13, 17].

1.4. **Auxiliary facts from the polyhedral algebra.** In this Subsection, we mainly follow [1, 2]. Let $\mathcal{V}$ be a d-dimensional real vector space and $\mathcal{L} \subset \mathcal{V}$ be a lattice.

Definition 3. *Let $\mathcal{A} \subseteq \mathcal{V}$ be a set. The* indicator $[\mathcal{A}]$ *of $\mathcal{A}$ is the function $[\mathcal{A}]\colon \mathcal{V} \to \mathbb{R}$ defined by* $[\mathcal{A}](x) = \begin{cases} 1, \text{ if } x \in \mathcal{A} \\ 0, \text{ if } x \notin \mathcal{A} \end{cases}$.

The algebra of polyhedra $\mathscr{P}(\mathcal{V})$ *is the vector space defined as the span of the indicator functions of all the polyhedra $\mathcal{P} \subset \mathcal{V}$.*

Definition 4. *A linear transformation $\mathcal{T}\colon \mathscr{P}(\mathcal{V}) \to \mathcal{W}$, where $\mathcal{W}$ is a vector space, is called a* valuation. *We consider only* $\mathcal{L}$-valuations *or* lattice valuations *that satisfy*

$$\mathcal{T}([\mathcal{P} + u]) = \mathcal{T}([\mathcal{P}]), \quad \text{for all rational polytopes } \mathcal{P} \text{ and } u \in \mathcal{L},$$

see [38, pp. 933–988]*,* [39].

We are mainly interested in two valuations, the first is the counting valuation $\mathcal{E}([\mathcal{P}]) = |\,\mathcal{P} \cap \mathbb{Z}^d\,|$ and the second valuation $\mathcal{F}([\mathcal{P}])$, which will be significantly used in our paper, is defined by the following theorem, proved by J. Lawrence [40], and, independently, by A. Khovanskii and A. Pukhlikov [41]. We borrowed the formulation from [1, Section 13]:

Theorem 2 ([40, 41])**.** *Let $\mathscr{R}(\mathbb{C}^d)$ be the space of rational functions on $\mathbb{C}^d$ spanned by the functions of the type*

$$\frac{\mathbf{x}^v}{(1-\mathbf{x}^{u_1})\dots(1-\mathbf{x}^{u_d})},$$

where $v \in \mathbb{Z}^d$ and $u_i \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, for any $i \in \{1, \dots, d\}$. Then there exists a linear transformation (a valuation) $\mathcal{F}\colon \mathscr{P}(\mathbb{Q}^d) \to \mathscr{R}(\mathbb{C}^d)$ such that the following properties hold:

(1) *Let $\mathcal{P} \subset \mathbb{R}^d$ be a non-empty rational polyhedron without lines, and let $\mathcal{C}$ be its recession cone. Let $\mathcal{C}$ be generated by rays $w_1, \dots, w_n$, for some $w_i \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, and let us define*

$$\mathcal{W}_{\mathcal{C}} = \bigl\{\mathbf{x} \in \mathbb{C}^d \colon |\, \mathbf{x}^{w_i} \,| < 1, \text{ for any } i \in \{1, \dots, n\}\bigr\}.$$

Then, $\mathcal{W}_{\mathcal{C}}$ is a non-empty open set and, for all $\mathbf{x} \in \mathcal{W}_{\mathcal{C}}$, the series

$$\mathfrak{f}(\mathcal{P}; \mathbf{x}) = \sum_{z \in \mathcal{P} \,\cap\, \mathbb{Z}^d} \mathbf{x}^z$$

converges absolutely and uniformly on compact subsets of $\mathcal{W}_{\mathcal{C}}$ to the function $f(\mathcal{P}; \mathbf{x}) = \mathcal{F}([\mathcal{P}]) \in \mathscr{R}(\mathbb{C}^d)$.

(2) *If P contains a line, then $f(\mathcal{P}; \mathbf{x}) = 0$.*

If $\mathcal{P}$ is a rational polyhedron, then $f(\mathcal{P}; \mathbf{x})$ is called its *short rational generating function*.

Definition 5. *Let $\mathcal{P} \subset \mathcal{V}$ be a non-empty polyhedron, and let $v \in \mathcal{P}$ be a point. We define the* tangent cone *of $\mathcal{P}$ at v by*

$$\operatorname{tcone}(\mathcal{P}, v) = \bigl\{v + y \colon v + \varepsilon y \in \mathcal{P}, \quad \text{for some } \varepsilon > 0\bigr\}.$$

If an n-dimensional polyhedron $\mathcal{P}$ is defined by a system $Ax \le b$, then, for any $v \in \mathcal{P}$, it holds

$$\operatorname{tcone}(\mathcal{P}, v) = \{x \in \mathcal{V} \colon A_{\mathcal{J}(v)*} x \le b_{\mathcal{J}(v)}\}, \quad \text{where } \mathcal{J}(v) = \{j \colon A_{j*} v = b_j\}.$$

It is widely known that a slight perturbation in the right-hand sides of a system $Ax \le b$ can transform the polyhedron $\mathcal{P}(A, b)$ to a simple one. Here, we need an algorithmic version of this fact, presented in the following technical theorem.

Theorem 3. *Let $A \in \mathbb{Z}^{k \times n}$, $\operatorname{rank}(A) = n \le k$, $b \in \mathbb{Q}^k$, $\gamma = \max\{\|A\|_{\max}, \|b\|_\infty\}$, and $\mathcal{P} = \mathcal{P}(A, b)$ be the n-dimensional polyhedron.*

Then, for $1/\varepsilon = 1 + 2n \cdot n^{\lceil n/2 \rceil} \cdot \gamma^n$ and the vector $t \in \mathbb{Q}^k$, with $t_i = \varepsilon^{i-1}$, the polyhedron $\mathcal{P}' = \mathcal{P}(A, b + t)$ is simple.

Proof. Let us suppose by the contrary that there exists a vertex v of $\mathcal{P}'$ and a set of indices $\mathcal{J}$ such that $A_{\mathcal{J}} v = (b+t)_{\mathcal{J}}$, $|\,\mathcal{J}\,| = n+1$ and $\operatorname{rank}(A_{\mathcal{J}}) = n$. The last is possible iff $\det(M) = 0$, where $M = \bigl(A_{\mathcal{J}}\, (b+t)_{\mathcal{J}}\bigr)$. Note that $M = B + D$, where $B = \bigl(A_{\mathcal{J}}\, b_{\mathcal{J}}\bigr)$ and $D = \bigl(\mathbf{0}_{(n+1) \times n}\; t_{\mathcal{J}}\bigr)$. We have,

$$\det(M) = \det(B) + \sum_{i=1}^{n+1} \det(B[i, t_{\mathcal{J}}]) =$$
$$= \det(B) + \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (-1)^{i+j} \cdot (t_{\mathcal{J}})_j \cdot \det(B_{\mathcal{J} \setminus \{j\}\, \mathcal{I} \setminus \{i\}}),$$

where $\mathcal{I} = \{1, \dots, n+1\}$ and $B[i, t_{\mathcal{J}}]$ is the matrix induced by the substitution of the column $t_{\mathcal{J}}$ instead of i-th column of B.

Let us assume that $(t_{\mathcal{J}})_j = \varepsilon^{d_j}$, for $j \in \mathcal{I}$, where $d_j \in \mathbb{Z}$ and $0 \le d_1 < d_2 < \dots < d_{n+1} \le k-1$. Consequently, the condition $\det(M) = 0$ is equivalent to the following condition:

$$\det(B) + \sum_{j=1}^{n+1} \varepsilon^{d_j} \cdot \left(\sum_{i=1}^{n} (-1)^{i+j} \cdot \det(B_{\mathcal{J} \setminus \{j\}\, \mathcal{I} \setminus \{i\}}) \right) = 0. \tag{1}$$

Note that the polynomial (1) is not zero. Definitely, since $\operatorname{rank}(A_{\mathcal{J}}) = n$, we can assume that the first n rows of $A_{\mathcal{J}}$ are linearly independent. Consequently, there exists a unique vector $y \in \mathbb{Q}^n_{\neq 0}$ such that the last row of $A_{\mathcal{J}}$ is a linear combination of the first rows with the coefficients vector y. Since $\forall \varepsilon\colon \det(M) = 0$, we have $\binom{y}{-1}^\top M = \mathbf{0}$ and, consequently, $\binom{y}{-1}^\top (b_{\mathcal{J}} + t) = \mathbf{0}$. But, the last may hold only for a finite number of ε. That is the contradiction.

Using the well known Cauchy's bound, we have that $|\varepsilon^*| \geq \frac{1}{1+\alpha_{\max}/\beta} = \frac{\beta}{\beta + \alpha_{\max}}$, where ε^* is any root of (1), $\alpha_{\max}$ is the maximal absolute value of the coefficients, and β is the absolute value of the leading coefficient.

Finally, $1/|\varepsilon^*| \leq 2\alpha_{\max} \leq 2n \cdot n^{n/2} \cdot \gamma^n$, which contradicts to the Theorem's condition on ε. $\square$

2. Proof of Theorem 1

2.1. A recurrent formula for the generating function of a group polyhedron. Let $\mathcal{G}$ be a finite Abelian group and $g_1, \dots, g_n \in \mathcal{G}$. Let, additionally, $r_i = \bigl|\langle g_i \rangle\bigr|$ be the order of g_i, for $i \in \{1, \dots, n\}$, and $r_{\max} = \max\limits_i r_i$. For $g_0 \in \mathcal{G}$ and $k \in \{1, \dots, n\}$, let $\mathcal{P}_{\mathcal{G}}(k, g_0)$ be the polyhedron induced by the convex hull of solutions of the following system:

$$\begin{cases} \sum\limits_{i=1}^{k} x_i g_i = g_0 \\ x \in \mathbb{Z}^k_{\geq 0}\,. \end{cases} \tag{2}$$

Let us consider the formal power series $\mathfrak{f}_k(g_0; \mathbf{x}) = \sum_{z \in \mathcal{P}_{\mathcal{G}}(k,g_0) \cap \mathbb{Z}^k} \mathbf{x}^z$.

For $k = 1$, we clearly have

$$\mathfrak{f}_1(g_0; \mathbf{x}) = \frac{x_1^s}{1 - x_1^{r_1}}, \quad \text{where } s = \min\{x_1 \in \mathbb{Z}_{\geq 0} \colon x_1 g_1 = g_0\}.$$

If such s does not exist, then we put $\mathfrak{f}_1(g_0; \mathbf{x}) = 0$.

Note that, for any value of $x_k \in \mathbb{Z}_{\geq 0}$, the system (2) can be rewritten as

$$\begin{cases} \sum\limits_{i=1}^{k-1} x_i g_i = g_0 - x_k g_k \\ x \in \mathbb{Z}^{k-1}_{\geq 0}\,. \end{cases}$$

Hence, for $k \geq 1$, we have

$$\begin{aligned} &\mathfrak{f}_k(g_0; \mathbf{x}) = \\ &= \frac{\mathfrak{f}_{k-1}(g_0; \mathbf{x}) + x_k \cdot \mathfrak{f}_{k-1}(g_0 - g_k; \mathbf{x}) + \dots + x_k^{r_k - 1} \cdot \mathfrak{f}_{k-1}(g_0 - g_k \cdot (r_k - 1); \mathbf{x})}{1 - x_k^{r_k}} = \\ &= \frac{1}{1 - x_k^{r_k}} \cdot \sum_{i=0}^{r_k - 1} x_k^i \cdot \mathfrak{f}_{k-1}(g_0 - i \cdot g_k; \mathbf{x}). \end{aligned} \tag{3}$$

$$\text{Consequently,} \quad \mathfrak{f}_k(g_0; \mathbf{x}) = \frac{\sum\limits_{i_1=0}^{r_1-1} \dots \sum\limits_{i_k=0}^{r_k-1} \epsilon_{i_1,\dots,i_k} x_1^{i_1} \dots x_k^{i_k}}{(1 - x_1^{r_1})(1 - x_2^{r_2}) \dots (1 - x_k^{r_k})}, \tag{4}$$

where the numerator is a polynomial with coefficients $\epsilon_{i_1,\dots,i_k} \in \{0,1\}$ and degree at most $(r_1-1)\dots(r_k-1)$. Additionally, the formal power series $\mathfrak{f}_k(g_0;\mathbf{x})$ converges absolutely to the given rational function if $|x_i^{r_i}| < 1$, for each $i \in \{1,\dots,k\}$.

2.2. **Simple Δ-modular polyhedral cone and its generating function.** Let $A \in \mathbb{Z}^{n\times n}$, $b \in \mathbb{Z}^n$, $\Delta = |\det(A)| > 0$, $\mathcal{P} = \mathcal{P}(A,b)$, and let us consider the formal power series

$$\mathfrak{f}(\mathcal{P};\mathbf{x}) = \sum_{z \in \mathcal{P}\,\cap\,\mathbb{Z}^n} \mathbf{x}^z .$$

Let $A = P^{-1}SQ^{-1}$ and $\sigma = S_{nn} = \Delta/\Delta_{\gcd}(A, n-1)$, where $S \in \mathbb{Z}^{n\times n}$ is the SNF of A and $P, Q \in \mathbb{Z}^{n\times n}$ are unimodular matrices. After the unimodular map $x = Qx'$ and introducing slack variables y, the system $Ax \le b$ becomes

$$\begin{cases} Sx + Py = Pb \\ x \in \mathbb{Z}^n \\ y \in \mathbb{Z}^n_{\ge 0} . \end{cases}$$

Since P is unimodular, the last system is equivalent to the system

$$\begin{cases} Py = Pb \pmod{S\cdot\mathbb{Z}^n} \\ y \in \mathbb{Z}^n_{\ge 0} . \end{cases} \tag{5}$$

Note that points of $\mathcal{P}\cap\mathbb{Z}^n$ and the system (5) are connected by the bijective map $x = A^{-1}(b-y)$.

The system (5) can be interpreted as a group system (2), where $\mathcal{G} = \mathbb{Z}^n \bmod S$ with an addition modulo S, $k = n$, $g_0 = Pb \bmod S$ and $g_i = P_{*i} \bmod S$, for $i \in \{1,\dots,n\}$. Clearly, $\mathcal{G}$ is isomorphic to $\mathbb{Z}^n/S\cdot\mathbb{Z}^n$, $|\mathcal{G}| = |\det(S)| = \Delta$ and $r_{\max} \le \sigma$.

Following the previous Subsection, for $k \in \{1,\dots,n\}$ and $g_0 \in \mathcal{G}$, let $\mathcal{M}_k(g_0)$ be the solutions set of the system

$$\begin{cases} \sum\limits_{i=1}^{k} y_i g_i = g_0 \\ y \in \mathbb{Z}^k_{\ge 0}, \end{cases} \quad \text{and} \quad \mathfrak{f}_k(g_0;\mathbf{x}) = \sum_{y \in \mathcal{M}_k(g_0)} \mathbf{x}^{-\sum\limits_{i=1}^{k} h_i y_i},$$

where h_i is the i-th column of the matrix $A^* = \Delta \cdot A^{-1}$.

Note that

$$\begin{aligned} \mathfrak{f}(\mathcal{P};\mathbf{x}) &= \sum_{z\in\mathcal{P}\,\cap\,\mathbb{Z}^n} \mathbf{x}^z = \sum_{y \in \mathcal{M}_n(Pb \bmod S)} \mathbf{x}^{A^{-1}(b-y)} = \\ &= \mathbf{x}^{A^{-1}b} \cdot \sum_{y\in\mathcal{M}_n(Pb \bmod S)} \mathbf{x}^{-\frac{1}{\Delta}A^* y} = \mathbf{x}^{A^{-1}b}\cdot \mathfrak{f}_n\big(Pb \bmod S; \mathbf{x}^{\frac{1}{\Delta}}\big). \end{aligned} \tag{6}$$

Next, we will use the formulas (3) and (4) after the substitution $x_i \to \mathbf{x}^{-h_i}$, for $i \in \{1,\dots,n\}$. For $k = 1$, we have

$$\mathfrak{f}_1(g_0;\mathbf{x}) = \frac{\mathbf{x}^{-sh_1}}{1-\mathbf{x}^{-r_1h_1}}, \quad \text{where } s = \min\{y_1 \in \mathbb{Z}_{\ge 0} \colon y_1 g_1 = g_0\}. \tag{7}$$

For $k \geq 2$, we have

$$\mathfrak{f}_k(g_0; \mathbf{x}) = \frac{1}{1 - \mathbf{x}^{-r_k h_k}} \cdot \sum_{i=0}^{r_k - 1} \mathbf{x}^{-i h_k} \cdot \mathfrak{f}_{k-1}(g_0 - i \cdot g_k; \mathbf{x}) \text{ and} \tag{8}$$

$$\mathfrak{f}_k(g_0; \mathbf{x}) = \frac{\sum\limits_{i_1=0}^{r_1-1} \dots \sum\limits_{i_k=0}^{r_k-1} \epsilon_{i_1,\dots,i_k} \, \mathbf{x}^{-(i_1 h_1 + \dots + i_k h_k)}}{(1 - \mathbf{x}^{-r_1 h_1})(1 - \mathbf{x}^{-r_2 h_2}) \dots (1 - \mathbf{x}^{-r_k h_k})}, \tag{9}$$

where the numerator is a Laurent polynomial with coefficients $\epsilon_{i_1,\dots,i_k} \in \{0, 1\}$. Clearly, the power series $\mathfrak{f}_k(g_0; \mathbf{x})$ converges absolutely to the given function if $|\,\mathbf{x}^{-r_i h_i}\,| < 1$, for each $i \in \{1, \dots, k\}$.

Due to the formulae (9) and (6), we have

$$\mathfrak{f}(\mathcal{P}; \mathbf{x}) = \frac{\sum\limits_{i_1=0}^{r_1-1} \dots \sum\limits_{i_n=0}^{r_n-1} \epsilon_{i_1,\dots,i_n} \, \mathbf{x}^{\frac{1}{\Delta} A^* (b - (i_1, \dots, i_n)^\top)}}{\left(1 - \mathbf{x}^{-\frac{r_1}{\Delta} h_1}\right)\left(1 - \mathbf{x}^{-\frac{r_2}{\Delta} h_2}\right) \dots \left(1 - \mathbf{x}^{-\frac{r_n}{\Delta} h_n}\right)}. \tag{10}$$

Note that $\frac{r_i}{\Delta} h_i$ is an integer vector, for any $i \in \{1, \dots, n\}$, and $\frac{1}{\Delta} A^*(b - (i_1, \dots, i_n)^\top)$ is an integer vector, for any $(i_1, \dots, i_n)$, such that $\epsilon_{i_1,\dots,i_n} \neq 0$. Indeed, by definition of r_i, we have $r_i P_{*i} \equiv \mathbf{0} \pmod{S \cdot \mathbb{Z}^n}$, so $\frac{r_i}{\Delta} h_i = (r_i A^{-1})_{*i} = (Q S^{-1} P r_i)_{*i}$, which is an integer vector. Vectors $(i_1, \dots, i_n)^\top$ correspond to solutions y of the system (5), and $\frac{1}{\Delta} A^*(b - (i_1, \dots, i_n)^\top) = A^{-1}(b - y)$ is an integer vector.

Additionally, note that the vectors $-\frac{r_i}{\Delta} h_i$ represent extreme rays of the recession cone of $\mathcal{P}$.

Let $c \in \mathbb{Z}^n$ be chosen, such that $(c^\top A^*)_i \neq 0$, for any i. Let us consider the exponential sum

$$\hat{\mathfrak{f}}_k(g_0; \tau) = \sum_{y \in \mathcal{M}_k(g_0)} e^{-\tau \cdot \langle c, \sum_{i=1}^k h_i y_i \rangle}$$

that is induced by $\mathfrak{f}_k(g_0; \mathbf{x})$, substituting $x_i = e^{\tau \cdot c_i}$.

The formulae (7), (8), and (9) become

$$\hat{\mathfrak{f}}_1(g_0; \tau) = \frac{e^{-\langle c, s h_1 \rangle \cdot \tau}}{1 - e^{-\langle c, r_1 h_1 \rangle \cdot \tau}}, \tag{11}$$

$$\hat{\mathfrak{f}}_k(g_0; \tau) = \frac{1}{1 - e^{-\langle c, r_k h_k \rangle \cdot \tau}} \cdot \sum_{i=0}^{r_k - 1} e^{-\langle c, i h_k \rangle \cdot \tau} \cdot \hat{\mathfrak{f}}_{k-1}(g_0 - i \cdot g_k; \tau), \tag{12}$$

$$\hat{\mathfrak{f}}_k(g_0; \tau) = \frac{\sum\limits_{i_1=0}^{r_1-1} \dots \sum\limits_{i_k=0}^{r_k-1} \epsilon_{i_1,\dots,i_k} e^{-\langle c, i_1 h_1 + \dots + i_k h_k \rangle \cdot \tau}}{\left(1 - e^{-\langle c, r_1 h_1 \rangle \cdot \tau}\right)\left(1 - e^{-\langle c, r_2 h_2 \rangle \cdot \tau}\right) \dots \left(1 - e^{-\langle c, r_k h_k \rangle \cdot \tau}\right)}. \tag{13}$$

Let $\chi = \max\limits_{i \in \{1, \dots, n\}} \{|\langle c, h_i \rangle|\}$. Since $\langle c, h_i \rangle \in \mathbb{Z}_{\neq 0}$, for each i, the number of terms $e^{-\langle c, \cdot \rangle \cdot \tau}$ is bounded by $1 + 2 \cdot k \cdot r_{\max} \cdot \chi \leq 1 + 2 \cdot k \cdot \sigma \cdot \chi$. So, after combining similar terms, the numerator's length becomes $O(k \cdot \sigma \cdot \chi)$.

In other words, there exist coefficients $\epsilon_i \in \mathbb{Z}_{\geq 0}$, such that

$$\hat{\mathfrak{f}}_k(g_0; \tau) = \frac{\sum\limits_{i=-k \cdot \sigma \cdot \chi}^{k \cdot \sigma \cdot \chi} \epsilon_i \cdot e^{-i \cdot \tau}}{\left(1 - e^{-\langle c, r_1 \cdot h_1 \rangle \tau}\right)\left(1 - e^{-\langle c, r_2 h_2 \rangle \cdot \tau}\right) \dots \left(1 - e^{-\langle c, r_k h_k \rangle \cdot \tau}\right)}. \tag{14}$$

Let us discuss the group-operations complexity issues to find the representation (14) of $\hat{\mathfrak{f}}_k(g_0;\tau)$, for any $k \in \{1,\dots,n\}$ and $g_0 \in \mathcal{G}$.

Clearly, to find the desired representation of $\hat{\mathfrak{f}}_1(g_0;\tau)$, for all $g_0 \in \mathcal{G}$, we need $r_1 \cdot \Delta$ group operations.

Fix $g_0 \in \mathcal{G}$ and $k \in \{1,\dots,n\}$. To find $\hat{\mathfrak{f}}_k(g_0;\tau)$, for $k \geq 2$, we can use the formula (12). Each numerator of the term $e^{-\langle c, i h_k\rangle\cdot\tau} \cdot \hat{\mathfrak{f}}_{k-1}(g_0 - i g_k;\tau)$ contains at most $1 + 2\cdot(k-1)\cdot\sigma\cdot\chi$ non-zero terms of the type $\epsilon \cdot e^{-\langle c,\cdot\rangle\cdot\tau}$. Hence, the summation can be done with $O(k\cdot\sigma^2\cdot\chi)$ group operations. Consequently, the total group-operations complexity can be expressed by the formula

$$O(\Delta\cdot n^2\cdot\sigma^2\cdot\chi).$$

Finally, since the diagonal matrix S can have at most $\log_2(\Delta)$ terms that are not equal to 1, the arithmetic complexity of one group operation is $O(\log(\Delta))$. Hence, the total arithmetic complexity is

$$O\big(\Delta\cdot\log(\Delta)\cdot n^2\cdot\sigma^2\cdot\chi\big).$$

Finally, let us show how to find the exponential form

$$\hat{\mathfrak{f}}(\mathcal{P};\tau) = \sum_{z \in \mathcal{P}\cap\mathbb{Z}^n} e^{\langle c,z\rangle\cdot\tau}$$

of the power series $\mathfrak{f}(\mathcal{P};\mathbf{x})$ induced by the map $x_i = e^{c_i\cdot\tau}$.

Due to the formula (6), we have

$$\hat{\mathfrak{f}}(\mathcal{P};\tau) = e^{\langle c, A^{-1}b\rangle\cdot\tau}\cdot\hat{\mathfrak{f}}_n\Big(Pb \bmod S; \frac{\tau}{\Delta}\Big).$$

Due to the last formula and the formulae (10) and (14), we have

$$\hat{\mathfrak{f}}(\mathcal{P};\tau) = \frac{\sum\limits_{i=-n\cdot\sigma\cdot\chi}^{n\cdot\sigma\cdot\chi} \epsilon_i \cdot e^{\frac{1}{\Delta}(\langle c, A^* b\rangle - i)\cdot\tau}}{\big(1 - e^{-\langle c, \frac{r_1}{\Delta}\cdot h_1\rangle\cdot\tau}\big)\big(1 - e^{-\langle c, \frac{r_2}{\Delta}\cdot h_2\rangle\cdot\tau}\big)\dots\big(1 - e^{-\langle c, \frac{r_n}{\Delta}\cdot h_n\rangle\cdot\tau}\big)}.$$

Again, due to (10), we have $\langle c, \frac{r_i}{\Delta} h_i\rangle \in \mathbb{Z}_{\neq 0}$, for any $i \in \{1,\dots,n\}$, and $\frac{1}{\Delta}(\langle c, A^* b\rangle - i) \in \mathbb{Z}$, for any i, such that $\epsilon_i > 0$.

We have proven the following:

Theorem 4. *Let $A \in \mathbb{Z}^{n\times n}$, $b \in \mathbb{Z}^n$, $\Delta = |\det(A)| > 0$, and $\mathcal{P} = \mathcal{P}(A,b)$. Let, additionally, $\sigma = S_{nn}$, where S is the SNF of A, and $\chi = \max\limits_{i\in\{1,\dots,n\}}\big\{|\langle c, h_i\rangle|\big\}$, where h_i is the i-th column of $A^* = \Delta\cdot A^{-1}$.*

Then, the formal exponential series $\hat{\mathfrak{f}}(\mathcal{P};\tau)$ can be represented as

$$\hat{\mathfrak{f}}(\mathcal{P};\tau) = \frac{\sum\limits_{i=-n\cdot\sigma\cdot\chi}^{n\cdot\sigma\cdot\chi} \epsilon_i\cdot e^{\alpha_i\cdot\tau}}{\big(1 - e^{\beta_1\cdot\tau}\big)\big(1 - e^{\beta_2\cdot\tau}\big)\dots\big(1 - e^{\beta_n\cdot\tau}\big)},$$

where $\epsilon_i \in \mathbb{Z}_{\geq 0}$, $\beta_i \in \mathbb{Z}_{\neq 0}$, and $\alpha_i \in \mathbb{Z}$.

This representation can be found with an algorithm having the arithmetic complexity bound

$$O\big(T_{\mathrm{SNF}}(n) + \Delta\cdot\log(\Delta)\cdot n^2\cdot\sigma^2\cdot\chi\big),$$

where $T_{SNF}(n)$ is the arithmetical complexity of computing the SNF for $n\times n$ integer matrices.

2.3. **Handling the general case.** Following Remark 1, we will only work with systems in the canonical form. Let $A \in \mathbb{Z}^{(n+m)\times n}$, $b \in \mathbb{Q}^{n+m}$, $\operatorname{rank}(A) = n$, and $\Delta = \Delta(A)$. Let us consider the polytope $\mathcal{P} = \mathcal{P}(A, b)$.

Let us choose $\gamma = \max\{\|A\|_{\max}, \|b\|_\infty\}$, $\beta = \min\limits_{i \in \{1,\dots,n+m\}}\{\lceil b_i \rceil - b_i \colon b_i \notin \mathbb{Z}\}$, and $\varepsilon = \min\{\beta/2,\, (1 + 2n \cdot n^{\lceil n/2 \rceil} \cdot \gamma)^{-1}\}$. If all b_i are integer, we put $\beta = +\infty$, so the formula for ε remains correct. Then, by Theorem 3, the polytope $\mathcal{P}' = \mathcal{P}(A, b + t)$ is simple, where the vector t is chosen, such that $t_i = \varepsilon^{i-1}$, for $i \in \{1, \dots, n + m\}$. By the construction, $\mathcal{P} \cap \mathbb{Z}^n = \mathcal{P}' \cap \mathbb{Z}^n$. From this moment, we assume that $\mathcal{P} = \mathcal{P}(A, b)$ is a simple polytope.

Using Definition 5 for tangent cones, the Brion's Theorem [42] (see also [1, Chapter 6]) gives:

$$
\begin{aligned}
[\mathcal{P}] = &\sum_{v \in \operatorname{vert}(\mathcal{P})} \bigl[\operatorname{tcone}(\mathcal{P}, v)\bigr] \\
&\qquad = \sum_{v \in \operatorname{vert}(\mathcal{P})} \bigl[\mathcal{P}(A_{\mathcal{J}(v)}, b_{\mathcal{J}(v)})\bigr] \quad \textit{modulo polyhedra with lines}.
\end{aligned}
$$

Due to the seminal work [43], all vertices of the simple polyhedron $\mathcal{P}$ can be enumerated with $O\bigl((m + n) \cdot n \cdot |\operatorname{vert}(\mathcal{P})|\bigr)$ arithmetic operations.

Denote $f(\mathcal{P}; \mathbf{x}) = \mathcal{F}([\mathcal{P}]) \in \mathscr{R}(\mathbb{Q}^n)$, for any rational polyhedron $\mathcal{P}$, where $\mathcal{F}$ is the evaluation considered in Theorem 2.

Note that $f(\mathcal{P}(B, u); \mathbf{x}) = f(\mathcal{P}(B, \lfloor u \rfloor); \mathbf{x})$, for any $B \in \mathbb{Z}^{n\times n}$ and $u \in \mathbb{Q}^n$. So, due to Theorem 2, we can write

$$
f(\mathcal{P}; \mathbf{x}) = \sum_{v \in \operatorname{vert}(\mathcal{P})} f\Bigl(\mathcal{P}\bigl(A_{\mathcal{J}(v)}, \lfloor b_{\mathcal{J}(v)} \rfloor\bigr); \mathbf{x}\Bigr).
$$

Due to results of the previous Subsection, each term $f\Bigl(\mathcal{P}\bigl(A_{\mathcal{J}(v)}, \lfloor b_{\mathcal{J}(v)} \rfloor\bigr); \mathbf{x}\Bigr)$ has the form (10).

To find the value of $|\,\mathcal{P} \cap \mathbb{Z}^n\,| = \lim\limits_{\mathbf{x} \to \mathbf{1}} f(\mathcal{P}; \mathbf{x})$, we follow Chapters 13 and 14 of [1]. Let us choose $c \in \mathbb{Z}^n$, such that any element of the row-vector $c^\top (A_{\mathcal{J}(v)})^{-1}$ is non-zero, for each $v \in \operatorname{vert}(\mathcal{P})$. Substituting $x_i = e^{c_i \cdot \tau}$, let us consider the exponential function

$$
\hat{f}(\mathcal{P}; \tau) = \sum_{v \in \operatorname{vert}(\mathcal{P})} \hat{f}\Bigl(\mathcal{P}\bigl(A_{\mathcal{J}(v)}, \lfloor b_{\mathcal{J}(v)} \rfloor\bigr); \tau\Bigr).
$$

Due to [1, Chapter 14], the value $|\,\mathcal{P} \cap \mathbb{Z}^n\,|$ is a constant term in the Tailor series of the function $\hat{f}(\mathcal{P}; \tau)$, so we just need to compute it.

Let us fix some term $\hat{f}\bigl(\mathcal{P}(B, u); \tau\bigr)$ of the previous formula. Due to Theorem 4, it can be represented as

$$
\hat{f}\bigl(\mathcal{P}(B, u); \tau\bigr) = \frac{\sum\limits_{i=-n\cdot\sigma\cdot\chi}^{n\cdot\sigma\cdot\chi} \epsilon_i \cdot e^{\alpha_i \cdot \tau}}{\bigl(1 - e^{\beta_1 \cdot \tau}\bigr)\bigl(1 - e^{\beta_2 \cdot \tau}\bigr)\dots\bigl(1 - e^{\beta_n \cdot \tau}\bigr)},
$$

where $\epsilon_i \in \mathbb{Z}_{\geq 0}$, $\beta_i \in \mathbb{Z}_{\neq 0}$, and $\alpha_i \in \mathbb{Z}$.

Again, due to [1, Chapter 14], we can see that the constant term in Tailor series for $\hat{f}\big(\mathcal{P}(B,u);\tau\big)$ is exactly

$$\sum_{i=-n\cdot\sigma\cdot\chi}^{n\cdot\sigma\cdot\chi} \frac{\epsilon_i}{\beta_1\dots\beta_n} \sum_{j=0}^{n} \frac{\alpha_i^j}{j!} \cdot \mathrm{td}_{n-j}(\beta_1,\dots,\beta_n), \tag{15}$$

where $\mathrm{td}_j(\beta_1,\dots,\beta_n)$ is a homogeneous polynomial of degree j, called the *j-th Todd polynomial* on $\beta_1,\dots,\beta_n$. Due to [20, Theorem 7.2.8, p. 137], the values of $\mathrm{td}_j(\beta_1,\dots,\beta_n)$, for $j \in \{1,\dots,n\}$, can be computed with an algorithm that is polynomial in n, and the bit-encoding length of $\beta_1,\dots,\beta_n$. Moreover, it follows from the theorem's proof that the arithmetical complexity can be bounded by $O(n^3)$.

Since $\sigma \le \Delta$, due to Theorem 4, the total arithmetic complexity to find the value of (15) can be bounded by

$$O\big(n^3 + T_{SNF}(n) + \Delta^3 \cdot \log(\Delta) \cdot n^2 \cdot \chi\big).$$

Due to [44], $T_{SNF}(n) = O(n^3)$. Assuming that $O(n^2 \cdot \chi)$ dominates $O(n^3)$, the last bound can be rewritten to $O\big(\Delta^3 \cdot \log(\Delta) \cdot n^2 \cdot \chi\big)$.

The constant term in Tailor series for the complete function $\hat{f}(\mathcal{P};\tau)$ can be found just by summation. It gives the arithmetic complexity bound

$$O\big(\nu(n,m,\Delta) \cdot n^2 \cdot \Delta^3 \cdot \log(\Delta) \cdot \chi\big).$$

Finally, we choose $c^\top$ as the sum of rows of some non-degenerate $n \times n$ sub-matrix of A. Note that elements of the matrix $A \cdot A^*_{\mathcal{J}(v)}$ are included in the set of all $n \times n$ sub-determinants of A, where $A^*_{\mathcal{J}(v)} = \Delta \cdot A^{-1}_{\mathcal{J}(v)}$, for all $v \in \mathrm{vert}(\mathcal{P})$. Hence, $\chi \le n\Delta$, and the total arithmetic complexity bound becomes

$$O\big(\nu(n,m,\Delta) \cdot n^3 \cdot \Delta^4 \cdot \log(\Delta)\big). \quad \text{It finishes the proof of Theorem 1.}$$

2.4. **Proof of Corollary 1.** The presented complexity bounds follow from the different ways to estimate the value $\nu(m,n,\Delta)$. The first bound trivially follows from the inequalities $\nu(m,n,\Delta) \le \binom{n+m}{n} = \binom{n+m}{m} \lesssim \frac{e^m \cdot (n+m)^m}{m^m} = O\big(\frac{n}{m}+1\big)^m$.

To obtain the second bound, we refer to the seminal result, due to P. McMullen [9]. Together with the formula from [45, Section 4.7] for the number of facets of a cyclic polytope, it follows that the maximal number of vertices in an n-dimensional polyhedron with k facets is bounded by

$$\xi(n,k) = \begin{cases} \frac{k}{k-s}\binom{k-s}{s}, & \text{for } n = 2s \\ 2\binom{k-s-1}{s}, & \text{for } n = 2s+1 \end{cases} = O\left(\frac{k}{n}\right)^{n/2}.$$

Clearly, $\nu(m,n,\Delta) \le \xi(n,n+m)$, and $\nu(m,n,\Delta) = O\big(\frac{n+m}{n}\big)^{\frac{n}{2}}$. So, the second bound holds.

Due to [8], we can assume that $n+m = O(n^2 \cdot \Delta^2)$. Substituting the last formula to the second bound, we obtain $\nu(m,n,\Delta) = O(n^{\frac{n}{2}} \cdot \Delta^n)$, and the third bound holds.

Finally, let us show how to handle the case, when $\mathcal{P}$ is an unbounded n-dimensional polyhedron. Clearly, we need to distinguish between two possibilities: $|\,\mathcal{P} \cap \mathbb{Z}^n\,| = 0$ and $|\,\mathcal{P} \cap \mathbb{Z}^n\,| = \infty$. Let us choose any vertex v of $\mathcal{P}$ and consider a set of indices $\mathcal{J}$, such that $|\,\mathcal{J}\,| = n$, $A_{\mathcal{J}} v = b_{\mathcal{J}}$ and $\mathrm{rank}(A_{\mathcal{J}}) = n$. For the first and second bounds, we add a new inequality $c^\top x \le c_0$ to the system $Ax \le b$, where $c^\top = \sum_{i=1}^{n} (A_{\mathcal{J}})_{i*}$

and $c_0 = c^\top v + \|c\|_1 \cdot n\Delta + 1$. Let $A'x \leq b'$ be the new system. Due to [46], $|\,\mathcal{P} \cap \mathbb{Z}^n\,| = 0$ iff $|\,\mathcal{P}(A', b') \cap \mathbb{Z}^n\,| = 0$. Since $\mathcal{P}(A', b')$ is a polytope and $\Delta(A') \leq n\Delta$, we just need to add an additional multiplicative factor of $O(\frac{d}{m} + 1) \cdot n^4$ to the first bound and $O(n^4)$ to the second bound.

To deal with third bound, we just need to add additional inequalities $A_{\mathcal{J}} x \geq b_{\mathcal{J}} - \|A_{\mathcal{J}}\|_{\max} \cdot n^2 \Delta \cdot \mathbf{1}$ to the system $Ax \leq b$. The polyhedron becomes bounded and the sub-determinants stay unchanged, and we follow the original scenario.

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