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TORIC MORPHISMS AND DIAGONALS OF THE LAURENT SERIES OF RATIONAL FUNCTIONS

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Abstract. We consider the Laurent series of a rational function in n complex variables and the n-dimensional sequence of its coefficients. The diagonal subsequence of this sequence generates the so-called complete diagonal of the Laurent series. We give a new integral representation for the complete diagonal. Based on this representation, we give a sufficient condition for a diagonal to be algebraic.

Keywords: algebraic function, diagonal of Laurent series, generating function, integral representations, toric morphism.

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers, $\mathbb{C}^\times := \mathbb{C} - \{0\}$ be its multiplicative group and $\mathbb{R} \subset \mathbb{C}$ be the subfield of real numbers. Let M be a lattice of the rank n and N be its dual lattice. Consider the n-dimensional complex torus $T^n := M \otimes_{\mathbb{Z}} \mathbb{C}^\times$. We fix a basis $\boldsymbol{e}^1, \ldots, \boldsymbol{e}^n$ of M and the dual basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ of N. Then $M \simeq \mathbb{Z}^n$, $N \simeq (\mathbb{Z}^n)^*$ and the torus can be written in the form

$$T^n_{\boldsymbol{z}} = \mathbb{C}^\times \times \ldots \times \mathbb{C}^\times.$$

It is an abelian group that, also, has the structure of a complex manifold equipped with coordinate functions $\boldsymbol{z} := (z_1, \ldots, z_n)$. Recall that the cohomology group $H^n(T^n_{\boldsymbol{z}}, \mathbb{Z})$ is generated by the class of the differential form $d\boldsymbol{z}/\boldsymbol{z}^{\boldsymbol{e}}$, where $d\boldsymbol{z} := dz_1 \wedge \ldots \wedge dz_n$, $\boldsymbol{z}^{\boldsymbol{e}} := z_1 \cdot \ldots \cdot z_n$ and $\boldsymbol{e} := \boldsymbol{e}_1 + \ldots + \boldsymbol{e}_n = (1, \ldots, 1) \in (\mathbb{Z}^n)^*$.

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A *Laurent polynomial* Q over $\mathbb{C}$ is a finite sum of the form

$$Q(\boldsymbol{z}) := \sum_{\boldsymbol{\alpha}\in A} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}},$$

where A is a finite subset of the dual to $\mathbb{Z}^n$ lattice $(\mathbb{Z}^n)^*$, $q_{\boldsymbol{\alpha}} \in \mathbb{C}$ and $\boldsymbol{z}^{\boldsymbol{\alpha}} := z_1^{\alpha^1} \dots z_n^{\alpha^n}$. Its *Newton polytope* Δ_Q is the convex hull of A in $(\mathbb{R}^n)^*$. The set of all Laurent polynomials over $\mathbb{C}$ in $\boldsymbol{z}$ forms a ring $R_{\boldsymbol{z}}$ of Laurent polynomials. There is an injective homomorphism from $R_{\boldsymbol{z}}$ to the ring of functions on $T^n_{\boldsymbol{z}}$. So the set

$$Z^{\times}(Q) := \{\boldsymbol{z} \in T^n_{\boldsymbol{z}} : Q(\boldsymbol{z}) = 0\}$$

of zeros in $T^n_{\boldsymbol{z}}$ of a Laurent polynomial Q in $R_{\boldsymbol{z}}$ is well-defined. The *amoeba* $\mathcal{A}_Q$ of a Laurent polynomial $Q \in R_{\boldsymbol{z}}$ is the image of $Z^{\times}(Q)$ under the logarithmic mapping

$$\Lambda : T^n_{\boldsymbol{z}} \to \mathbb{R}^n,\ \Lambda(\boldsymbol{z}) := (\log|z_1|, \dots, \log|z_n|),$$

where $\mathbb{R}^n := \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $P(\boldsymbol{z})$, $Q(\boldsymbol{z})$ be irreducible polynomials in $R_{\boldsymbol{z}}$. Consider a Laurent series (centered at the origin)

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{\beta}\in(\mathbb{Z}^n)^*} C_{\boldsymbol{\beta}} \boldsymbol{z}^{\boldsymbol{\beta}} \tag{1}$$

of a rational function $F(\boldsymbol{z}) = P(\boldsymbol{z})/Q(\boldsymbol{z})$. It is well known that the domain of absolute convergence of the series (1) is logarithmically convex. More precisely, such domain has the form $\Lambda^{-1}(E)$, where E is a connected component of $\mathbb{R}^n - \mathcal{A}_Q$ (see Section 2).

Let $\boldsymbol{Q} := (\boldsymbol{q}_1, \dots, \boldsymbol{q}_r)$ be an r-tuple of vectors that generates a sublattice L of the rank r of the lattice $(\mathbb{Z}^n)^*$, and, with a slight abuse of notation, let $\boldsymbol{Q}$ be also the $n \times r$ matrix (q^i_j) with columns $\boldsymbol{q}_1, \dots, \boldsymbol{q}_r$. For $\boldsymbol{k} := (k^1 \dots, k^r) \in (\mathbb{Z}^r)^*$, we define

$$\boldsymbol{Q}\boldsymbol{k} := (\boldsymbol{q}^1 \cdot \boldsymbol{k}, \dots, \boldsymbol{q}^r \cdot \boldsymbol{k}) = k^1\boldsymbol{q}_1 + \dots + k^r\boldsymbol{q}_r,$$

where $\boldsymbol{q}^i$'s are the rows of the matrix $\boldsymbol{Q}$ and the dot product $\boldsymbol{q}^i \cdot \boldsymbol{k} := q^i_1 k^1 + \dots + q^i_r k^r$. Then the *complete $\boldsymbol{Q}$-diagonal* of the Laurent series (1) is the Laurent series

$$d_{\boldsymbol{Q}}(\boldsymbol{t}) = \sum_{\boldsymbol{k}\in(\mathbb{Z}^r)^*} C_{\boldsymbol{Q}\boldsymbol{k}} \boldsymbol{t}^{\boldsymbol{k}} \tag{2}$$

in r variables (r is the *rank* of the diagonal). In other words, the diagonal $d_{\boldsymbol{Q}}(t)$ is a generating function of the r-dimensional subsequence $\{C_{\boldsymbol{Q}\boldsymbol{k}}\}$ of the n-dimensional Laurent coefficients sequence $\{C_{\boldsymbol{\beta}}\}$. The diagonal is called *primitive* if it corresponds to $\boldsymbol{q}_1 = \boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{q}_2 = \boldsymbol{e}_3, \dots, \boldsymbol{q}_{n-1} = \boldsymbol{e}_n$.

Diagonals of rational functions arise naturally in statistical mechanics (see, for example, [1, 2]) and enumerative combinatorics. R. Stanley proposed in [3, Section 6.1] the following natural hierarchy of the most important classes of generating functions in enumerative combinatorics

$$\{\text{rational}\} \subset \{\text{algebraic}\} \subset \{D\text{-finite}\}.$$

The classical result about diagonals states that a diagonal of the Taylor series for a rational function of two complex variables (the case $n = 2$ and $r = 1$) is an algebraic function (see [4, 5, 6] and [3, Section 6.3] for different aspects). It was generalized to the case of Laurent series of two complex variables in [7, Theorem 1]. Primitive diagonals of the Laurent series for rational functions of n complex variables are algebraic too ([7, Theorem 3]). In general, complete diagonals even for Taylor series of rational functions in more than 2 complex variables are not algebraic, see [5, Section 2] and [8, Section 4] for the particular examples.

Nevertheless, the following example shows that such diagonals could be algebraic.

Example 1. Consider the rational function

$$F(\boldsymbol{z}) = \frac{1}{1 - z_1 - (z_2 z_3)^l - z_3}, \ l \in \mathbb{N} := \{1, 2, \ldots\}. \tag{3}$$

It is not difficult to show that the univariate Taylor series

$$d(l;t) = \sum_{k=0}^{\infty} \frac{((l+1)k)!}{(lk)!k!} t^{lk} \tag{4}$$

is the complete $\boldsymbol{e}$-diagonal of the Taylor expansion for $F(\boldsymbol{z})$. It could be represented as the hypergeometric function ${}_lF_{l-1}(\frac{1}{l+1}, \ldots, \frac{l}{l+1}; \frac{1}{l}, \ldots, \frac{l-1}{l}; \frac{(l+1)^{l+1}}{l^l} t^l)$. The series $d(1;t)$ is algebraic, since

$$d(1;t) = \sqrt{1 - 4t}, \ |t| < \frac{1}{4},$$

by the generalized binomial theorem. Also, we have the explicit algebraic expression

$$d(2,t) = \frac{1}{\sqrt{4 - 27t^2}} \left(\left(\sqrt{1 - \frac{27}{4} t^2} + \frac{3\sqrt{3}}{2} \imath t \right)^{-\frac{1}{3}} + \left(\sqrt{1 - \frac{27}{4} t^2} - \frac{3\sqrt{3}}{2} \imath t \right)^{-\frac{1}{3}} \right),$$

when $|t| < \frac{4}{27}$ (see row 4 of Table 7.3.3 in [9, p. 486]). For $l \geq 3$, the algebraicity of $d(l;t)$ follows from Theorem 7.1 [10].

One of the main purposes of the paper is to demonstrate how integral representations help to explain such phenomena. We show in Section 4 that $d_{\boldsymbol{Q}}(\boldsymbol{t})$ could be represented as an integral of a rational differential form ω with parameters $\boldsymbol{t}$ over a $(n-r)$-dimensional cycle of the form $\Lambda^{-1}(\boldsymbol{y}')$, where $\boldsymbol{y}'$ is a point in the component $\tilde{E}' \subset \mathbb{R}^{n-r}$ of the complement to the amoeba of the denominator of ω.

Theorem 1. *Let the r-turple $\boldsymbol{Q}$ generate a saturated r-dimensional sublattice*[1] *of the lattice $(\mathbb{Z}^n)^*$ and p be the dimension of the recession cone of $\tilde{E}'$. Then, if the condition*

$$n - r - p = 1 \tag{5}$$

holds, the complete diagonal $d_{\boldsymbol{Q}}(\boldsymbol{t})$ of the Laurent expansion (1) *for the rational function $F(\boldsymbol{z})$ is an algebraic function.*

We prove the theorem and discuss the example of the rational function (3) in full details in Section 5.

2. Toric Morphisms and Amoebas of Laurent Polynomials

Let $\boldsymbol{A} := (a_i^j)$ be unimodular $n \times n$-matrix and $\boldsymbol{B} := (b_i^j)$ be its inverse. We denote by $\boldsymbol{a}^j := (a_1^j, \ldots, a_n^j)$ and $\boldsymbol{a}_i := (a_i^1, \ldots, a_i^n)$ the j-th row and the i-th column of $\boldsymbol{A}$, correspondingly. Similarly, $\boldsymbol{b}^j := (b_1^j, \ldots, b_n^j)$ and $\boldsymbol{b}_i := (b_i^1, \ldots, b_i^n)$ are the j-th row and the i-th column of $\boldsymbol{B}$.

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[1] Recall that a sublattice L of a lattice N is called *saturated* $\Leftrightarrow$ for any $\boldsymbol{v} \in N$, if $k\boldsymbol{v} \in L$, where k is a positive integer, then $\boldsymbol{v} \in L$.

The matrix $\boldsymbol{A}$ defines the linear transformation of the lattice $(\mathbb{Z}^n)^*$ by

$$\boldsymbol{\alpha} := (\alpha^1, \dots, \alpha^n) \mapsto \boldsymbol{A}\boldsymbol{\alpha} := (\boldsymbol{a}^1 \cdot \boldsymbol{\alpha}, \dots, \boldsymbol{a}^n \cdot \boldsymbol{\alpha}) = \alpha^1 \boldsymbol{a}_1 + \dots \alpha^n \boldsymbol{a}_n.$$

Then we let the *toric morphism*

$$T^n_{\boldsymbol{w}} \to T^n_{\boldsymbol{z}} \tag{6}$$

be defined by $\boldsymbol{w} \mapsto \boldsymbol{z} = \boldsymbol{w}^{\boldsymbol{A}}$, where $\boldsymbol{w}^{\boldsymbol{A}} := (\boldsymbol{w}^{\boldsymbol{a}_1}, \dots, \boldsymbol{w}^{\boldsymbol{a}_n})$ and $\boldsymbol{w}^{\boldsymbol{a}_i} := w_1^{a_i^1} \dots w_n^{a_i^n}$. The inverse morphism

$$T^n_{\boldsymbol{z}} \to T^n_{\boldsymbol{w}} \tag{7}$$

is given by $\boldsymbol{z} \mapsto \boldsymbol{w} = \boldsymbol{z}^{\boldsymbol{B}}$.

The morphism (6) induces the homomorphism $R_{\boldsymbol{z}} \to R_{\boldsymbol{w}}$ of rings of Laurent polynomials over $\mathbb{C}$ in the variables $\boldsymbol{z}$ and $\boldsymbol{w}$ by the formula

$$Q(\boldsymbol{z}) \mapsto \tilde{Q}(\boldsymbol{w}) := Q(\boldsymbol{w}^{\boldsymbol{A}}). \tag{8}$$

Let Q be a Laurent polynomial in $R_{\boldsymbol{z}}$. The amoeba $\mathcal{A}_Q$ of Q is a closed subset of $\mathbb{R}^n$. Then the complement $\mathbb{R}^n - \mathcal{A}_Q$ is open. It consists of a finite number of connected components $E_{\boldsymbol{\nu}}$, which are convex [11, Section 6.1]. These components are in 1-1 correspondence with all possible expansions (1) of an irreducible fraction $F(\boldsymbol{z}) = P(\boldsymbol{z})/Q(\boldsymbol{z})$.

The index $\boldsymbol{\nu}$ emphasizes that each component corresponds to an integer point $\boldsymbol{\nu}$ of the Newton polytope Δ_Q of Q. Recall that the *Newton polytope* Δ_Q is the convex hull of A in $(\mathbb{R}^n)^*$. More precisely, for a point $\boldsymbol{x}$ in E, the integrals

$$\nu^j := \frac{1}{(2\pi\imath)^n} \int\limits_{\Lambda^{-1}(\boldsymbol{x})} z_j \frac{\partial Q/\partial z_j}{Q} \frac{d\boldsymbol{z}}{\boldsymbol{z}^{\boldsymbol{e}}},\ j = 1, \dots n, \tag{9}$$

define on the set of all connected components $\{E\}$ an injective mapping

$$E \mapsto \boldsymbol{\nu} := (\nu^1, \dots, \nu^n) \in (\mathbb{Z}^n)^* \cap \Delta_Q,$$

see for details [12, Section 2]. The image $\boldsymbol{\nu}$ does not depend on the choice of $\boldsymbol{x}$ in E since for all points $\boldsymbol{x}$ in the same connected component cycles

$$\Lambda^{-1}(\boldsymbol{x}) := \{|z_1| = e^{x_1}, \dots, |z_n| = e^{x_n}\}$$

have the same homology class in $H_n(T^n_{\boldsymbol{z}} - Z^\times(Q))$. If $\boldsymbol{p} \in \Lambda^{-1}(\boldsymbol{x})$, then ν^j equals the difference between the number of zeroes and poles of the univariate Laurent polynomial $Q(p_1, \dots, p_{j-1}, z_j, p_{j+1}, \dots, p_n)$ in the circle $|z_j| = e^{x_j}$ by the argument principle.

Moreover, the recession cone of the convex set $E_{\boldsymbol{\nu}}$ coincides with $-C^\vee_{\boldsymbol{\nu}}$, where

$$C^\vee_{\boldsymbol{\nu}} := \{\boldsymbol{s} \in \mathbb{R}^n : \boldsymbol{s} \cdot \boldsymbol{\nu} = \min_{\boldsymbol{\alpha} \in \Delta_Q} \boldsymbol{s} \cdot \boldsymbol{\alpha}\},$$

is the dual cone of Δ_Q at the point $\boldsymbol{\nu} = \boldsymbol{\nu}(E)$. We refer the reader to Figure 1 to observe this fact.

Proposition 1. *Let $Q \in R_{\boldsymbol{z}}$, $\tilde{Q} \in R_{\boldsymbol{w}}$ be its image under the homomorphism* (8)*, $\mathcal{A}$ be the amoeba of Q and $\tilde{\mathcal{A}}$ be the amoeba of $\tilde{Q}$. Then $\tilde{\mathcal{A}}$ is equal to*

$$\mathcal{A}\boldsymbol{B} := \{\boldsymbol{y} \in \mathbb{R}^n : y_j = \boldsymbol{x} \cdot \boldsymbol{b}_j,\ \textit{for all}\ \boldsymbol{x} \in \mathcal{A}\ \textit{and}\ j = 1, \dots, n\}.$$

Moreover, if E is a component of the complement $\mathbb{R}^n - \mathcal{A}$ of order $\boldsymbol{\nu}$, then $\tilde{E} = E\boldsymbol{B}$ is the component of order $\boldsymbol{\mu} = \boldsymbol{A}\boldsymbol{\nu}$ of the complement $\mathbb{R}^n - \tilde{\mathcal{A}}$.

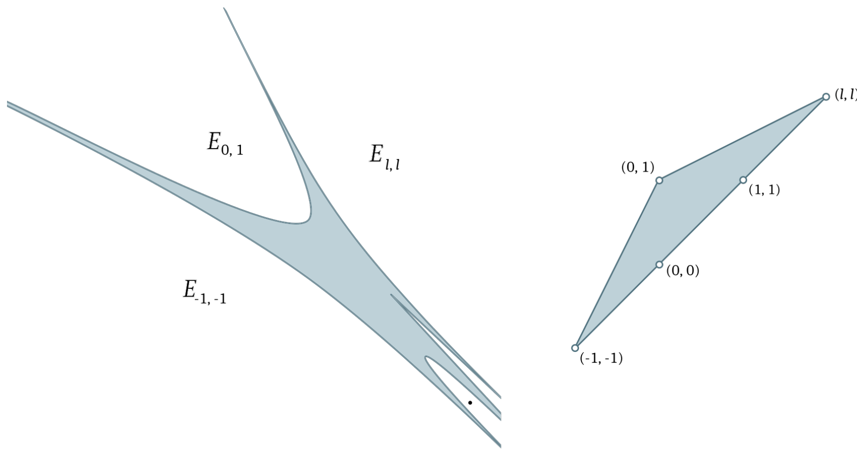

FIG. 1. The amoeba $\mathcal{A}$ (left) and the Newton polytope Δ (right) of the polynomial $1 - \frac{t}{w_2 w_3} - w_2^l w_3^l - w_3$. The black point marks the component of $\mathbb{R}^2 - \mathcal{A}$ of order $(0,0)$; the unlabelled component is of order $(1,1)$.

Proof. Let $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{z} \in T^n_{\boldsymbol{z}}$ such that $\Lambda(\boldsymbol{z}) = \boldsymbol{x}$. Then the image $\boldsymbol{w} = \boldsymbol{z}^{\boldsymbol{B}}$ has coordinates

$$w_j := \boldsymbol{z}^{\boldsymbol{b}_j} = (e^{x_1 + \imath\theta_1})^{b_j^1} \dots (e^{x_n + \imath\theta_n})^{b_j^n} = \exp(\boldsymbol{x} \cdot \boldsymbol{b}_j + \imath\boldsymbol{\theta} \cdot \boldsymbol{b}_j), \ j = 1, \dots, n,$$

where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in (-\pi, \pi]^n$. Thus, the j-th component of $\boldsymbol{y} := \Lambda(\boldsymbol{w})$ equals $\boldsymbol{x} \cdot \boldsymbol{b}_j$.

First, note that $\boldsymbol{w} \in Z^\times(\tilde{Q})$ if and only if $\boldsymbol{z} \in Z^\times(Q)$. So $\boldsymbol{x} \in \mathcal{A}$ if and only if $\boldsymbol{y} \in \tilde{\mathcal{A}}$. Then it follows that $\tilde{\mathcal{A}}$ is the image of the amoeba $\mathcal{A}$ by the linear transform defined by means of the matrix $\boldsymbol{B}$, and the image $E\boldsymbol{B}$ of a component E of $\mathbb{R}^n - \mathcal{A}$ coincides with some component $\tilde{E}$ of the complement $\mathbb{R}^n - \tilde{\mathcal{A}}$.

Next, consider the homomorphism $H_n(T^n_{\boldsymbol{z}} - Z^\times(Q)) \to H_n(T^n_{\boldsymbol{w}} - Z^\times(\tilde{Q}))$ induced by the morphism (7). It maps the cycle of integration $\Lambda^{-1}(\boldsymbol{x})$ in (9) to the cycle $\Lambda^{-1}(\boldsymbol{y})$, where $\boldsymbol{y} = \boldsymbol{x}\boldsymbol{B} := (\boldsymbol{x} \cdot \boldsymbol{b}_1, \dots, \boldsymbol{x} \cdot \boldsymbol{b}_n)$ and $\boldsymbol{x} \in E$. Also, the morphism (6) induces the homomorphism $H^n(T^n_{\boldsymbol{z}} - Z^\times(Q)) \to H^n(T^n_{\boldsymbol{w}} - Z^\times(\tilde{Q}))$ of cohomology groups. It maps the differential form in (9) to the form

$$\sum_{k=1}^n b_k^j w_k \frac{\partial \tilde{Q}/\partial w_k}{\tilde{Q}} \frac{d\boldsymbol{w}}{\boldsymbol{w}^{\boldsymbol{e}}},$$

since the direct calculation shows that

$$\frac{\partial Q}{\partial z_j} = \sum_{k=1}^n \frac{\partial \tilde{Q}}{\partial w_k} \frac{\partial w_k}{\partial z_j} = \sum_{k=1}^n \frac{\partial \tilde{Q}}{\partial w_k} \frac{\partial}{\partial z_j}(\boldsymbol{z}^{\boldsymbol{b}_k}) = \sum_{k=1}^n \frac{\partial \tilde{Q}}{\partial w_k} \frac{b_k^j}{z_j} w_k.$$

Therefore, the relation between $\boldsymbol{\nu}$ and the order $\boldsymbol{\mu} = (\mu^1, \dots, \mu^n)$ of $\tilde{E}$ is given by

$$\nu^j = \frac{1}{(2\pi\imath)^n} \sum_{k=1}^n b_k^j \int\limits_{\Lambda^{-1}(\boldsymbol{y})} w_k \frac{\partial \tilde{Q}/\partial w_k}{\tilde{Q}} \frac{d\boldsymbol{w}}{\boldsymbol{w}^{\boldsymbol{e}}} = \boldsymbol{b}^j \cdot \boldsymbol{\mu}.$$

Since the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are inverse we have that $\mu^j = \boldsymbol{a}^j \cdot \boldsymbol{\nu}$, or equivalently, $\boldsymbol{\mu} = \boldsymbol{A}\boldsymbol{\nu}$. □

3. Integral Representations for Laurent Coefficients

Let E be an unbounded component of order $\boldsymbol{\nu}$ of the complement to $\mathcal{A}$. Then the dual cone $C^\vee_{\boldsymbol{\nu}}$ to the Newton polytope Δ_Q at $\boldsymbol{\nu}$ is generated by p vectors $\boldsymbol{a}^1, \dots \boldsymbol{a}^p \in \mathbb{Z}^n$, where $p \in \{1, \dots, n\}$ since the recession cone of E coincides with $-C^\vee_{\boldsymbol{\nu}}$. The vectors $\boldsymbol{a}^1, \dots \boldsymbol{a}^p$ can be chosen so that they generate the saturated p-dimensional sublattice of $\mathbb{Z}^n$.

Consider the Laurent expansion (1) of the rational function $F(\boldsymbol{z}) = P(\boldsymbol{z})/Q(\boldsymbol{z})$ that converges in the domain $\Lambda^{-1}(E)$. Its coefficient can be represented as

$$C_{\boldsymbol{\beta}} = \frac{1}{(2\pi\imath)^n} \int\limits_{\Lambda^{-1}(\boldsymbol{x})} \frac{P(\boldsymbol{z})}{Q(\boldsymbol{z})} \frac{1}{\boldsymbol{z}^{\boldsymbol{\beta}}} \frac{d\boldsymbol{z}}{\boldsymbol{z}^{\boldsymbol{e}}}, \tag{10}$$

where $\boldsymbol{x} \in E$.

Let $\boldsymbol{A}'$ be a $p \times n$ matrix with rows $\boldsymbol{a}^1, \dots \boldsymbol{a}^p$. Then, since $\boldsymbol{a}^1, \dots \boldsymbol{a}^p$ generate the saturated sublattice, the matrix $\boldsymbol{A}'$ can be extended to an unimodular $n \times n$ matrix $\boldsymbol{A}$ by the Invariant Factor Theorem (see [13, Theorem 16.6]). Now, we consider the toric morphism (6) that corresponds to the matrix $\boldsymbol{A}$. By Proposition 1, $\tilde{\mathcal{A}} := \mathcal{A}\boldsymbol{B}$ is the amoeba of the Laurent polynomial $\tilde{Q}(\boldsymbol{w}) := Q(\boldsymbol{w}^{\boldsymbol{A}})$, $\tilde{E} := E\boldsymbol{B}$ is the component of order $\boldsymbol{\mu} = (\boldsymbol{a}^1 \cdot \boldsymbol{\nu}, \dots, \boldsymbol{a}^n \cdot \boldsymbol{\nu})$ of the complement to $\tilde{\mathcal{A}}$.

Recall that $\boldsymbol{a}_j$ and $\boldsymbol{a}^j$ denote the j-th column and the j-th row of the matrix $\boldsymbol{A}$, correspondingly. We can rewrite the Laurent polynomial $\tilde{Q}$ as

$$\tilde{Q}(\boldsymbol{w}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} (\boldsymbol{w}^{\boldsymbol{a}_1})^{\alpha_1} \cdot \ldots \cdot (\boldsymbol{w}^{\boldsymbol{a}_p})^{\alpha_p} \cdot \ldots \cdot (\boldsymbol{w}^{\boldsymbol{a}_n})^{\alpha_n} =$$

$$= \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} w_1^{\boldsymbol{a}^1 \cdot \boldsymbol{\alpha}} \dots w_p^{\boldsymbol{a}^p \cdot \boldsymbol{\alpha}} \dots w_n^{\boldsymbol{a}^n \cdot \boldsymbol{\alpha}} = w_1^{\boldsymbol{a}^1 \cdot \boldsymbol{\nu}} \dots w_p^{\boldsymbol{a}^p \cdot \boldsymbol{\nu}} \widehat{Q}(\boldsymbol{w}) = w_1^{\mu^1} \cdot \ldots \cdot w_p^{\mu^p} \widehat{Q}(\boldsymbol{w}),$$

since $\mu^j = \boldsymbol{a}^j \cdot \boldsymbol{\nu}$ for $j = 1, \dots, n$. By definition of the dual cone, one has that $\boldsymbol{a}^j \cdot \boldsymbol{\alpha} \geq \mu^j$ for $\boldsymbol{\alpha} \in \Delta_Q$ and $j = 1, \dots, n$. Thus, the quotient

$$\widehat{Q}(\boldsymbol{w}) := \frac{\tilde{Q}(\boldsymbol{w})}{w_1^{\mu^1} \cdot \ldots \cdot w_p^{\mu^p}} = \frac{\tilde{Q}(\boldsymbol{w})}{\boldsymbol{w}^{\boldsymbol{\mu}'}}$$

is a polynomial in variables $w_1, \dots, w_p$, where we set $\boldsymbol{\mu}' := (\mu^1, \dots, \mu^p, 0, \dots, 0)$.

If we clear denominators in $\tilde{P}(\boldsymbol{w})$, we get the polynomial $\hat{P}(\boldsymbol{w}) := \boldsymbol{w}^{\boldsymbol{d}} \tilde{P}(\boldsymbol{w})$, where $\boldsymbol{d} := (d^1, \dots, d^n)$ has non-negative components.

Proposition 2. *Let*

$$\Lambda_p^{-1}(\boldsymbol{y}) := \{|w_{p+1}| = e^{y_{p+1}}\} \times \ldots \times \{|w_n| = e^{y_n}\},$$

where $\boldsymbol{y}$ is a point in $\tilde{E}$. Then the Laurent coefficient

$$C_{\boldsymbol{\beta}} = \frac{1}{(2\pi\imath)^{n-p}} \int\limits_{\Lambda_p^{-1}(\boldsymbol{y})} R(w_{p+1}, \dots, w_n) \frac{1}{\boldsymbol{w}^{\boldsymbol{\gamma}-\boldsymbol{\gamma}'}} \frac{dw_{p+1} \wedge \ldots \wedge dw_n}{w_{p+1} \cdot \ldots \cdot w_n}, \tag{11}$$

where

$$R(w_{p+1}, \dots, w_n) := \frac{\gamma_1! \cdot \ldots \cdot \gamma_p!}{(\gamma_1 + \ldots + \gamma_p)!} \frac{\partial^{\gamma_1 + \ldots + \gamma_p}}{\partial^{\gamma_1} w_1 \ldots \partial^{\gamma_p} w_p} \left(\frac{\hat{P}}{\hat{Q}} \right) (\boldsymbol{0}, w_{p+1}, \dots, w_n)$$

is a rational function in $w_{p+1}, \dots, w_n$ *with the polar set defined by zeroes of the Laurent polynomial* $\hat{Q}(\mathbf{0}, w_{p+1}, \dots, w_n)$*, the vector* $\boldsymbol{\gamma} := \boldsymbol{\mu}' + \boldsymbol{d} + \boldsymbol{A\beta}$*. We use the convention that* $m! = 0$ *for a negative integer* m*.*

Proof. Making a change of variables as in proof of Proposition 1, we arrive at the integral representation

$$C_{\boldsymbol{\beta}} = \frac{1}{(2\pi\imath)^n} \int\limits_{\Lambda^{-1}(\boldsymbol{y})} \frac{\tilde{P}(\boldsymbol{w})}{\tilde{Q}(\boldsymbol{w})} \frac{d\boldsymbol{w}}{\boldsymbol{w}^{\boldsymbol{A\beta}+\boldsymbol{e}}} = \frac{1}{(2\pi\imath)^n} \int\limits_{\Lambda^{-1}(\boldsymbol{y})} \frac{\hat{P}(\boldsymbol{w})}{\hat{Q}(\boldsymbol{w})} \frac{d\boldsymbol{w}}{\boldsymbol{w}^{\boldsymbol{\gamma}+\boldsymbol{e}}}. \tag{12}$$

Since the real n-dimensional torus $\Lambda^{-1}(\boldsymbol{y})$ can be written as $\Lambda^{-1}(\boldsymbol{y}) = \Lambda_p^{-1}(\boldsymbol{y}) \times \Lambda^{-1}(y_1, \dots, y_p)$, we have

$$(2\pi\imath)^n C_{\boldsymbol{\beta}} = \int\limits_{\Lambda_p^{-1}(\boldsymbol{y})} \left(\int\limits_{\Lambda^{-1}(y_1,\dots,y_p)} \frac{\hat{P}(\boldsymbol{w})}{\hat{Q}(\boldsymbol{w})} \frac{1}{\boldsymbol{w}^{\boldsymbol{\gamma}'}} \frac{dw_1 \wedge \dots \wedge dw_p}{w_1 \cdot \ldots \cdot w_p} \right) \frac{1}{\boldsymbol{w}^{\boldsymbol{\gamma}-\boldsymbol{\gamma}'}} \times \\ \times \frac{dw_{p+1} \wedge \dots \wedge dw_n}{w_{p+1} \cdot \ldots \cdot w_n}. \tag{13}$$

The component μ^j of $\boldsymbol{\mu}$ is equal to the number of zeroes minus the number of poles of the univariate Laurent polynomial

$$S_j(w) := \tilde{Q}(e^{y_1}, \dots, e^{y_{j-1}}, w, e^{y_{j+1}}, \dots, e^{y_n})$$

in the disk $\{|w| < e^{y_j}\}$. As it follows from the equality $\tilde{Q}(\boldsymbol{w}) = w_1^{\mu^1} \cdot \ldots \cdot w_p^{\mu^p} \hat{Q}(\boldsymbol{w})$, the polynomials $S_j(w)$ have no poles in $\{|w| < e^{y_j}\}$ for $j = 1, \dots, p$. Therefore, $\widehat{Q}(0, \dots, 0, w_{p+1}, \dots, w_n)$ is a non-zero Laurent polynomial in variables $w_{p+1}, \dots, w_n$.

Now, for each j, $1 \le j \le p$, the integrand in (12) may have only a single pole $w_j = 0$ in the disk $\{|w| < e^{y_j}\}$ with respect to the variable w_j. The order of this pole equals $\gamma_j + 1$, where $\gamma_j := \mu^j + d^j + \boldsymbol{a}^j \cdot \boldsymbol{\beta}$. Then repeated application of the one-dimensional Cauchy formula gives us the equality

$$\frac{1}{(2\pi\imath)^p} \int\limits_{\Lambda^{-1}(y_1,\dots,y_p)} \frac{\hat{P}(\boldsymbol{w})}{\hat{Q}(\boldsymbol{w})} \frac{1}{\boldsymbol{w}^{\boldsymbol{\gamma}'}} \times \frac{dw_1 \wedge \dots \wedge dw_p}{w_1 \dots w_p} = R(w_{p+1}, \dots, w_n).$$

Thus, we are done. □

4. Integral Representations for Complete Diagonals

Let the Laurent expansion (1) converge in $\Lambda^{-1}(E)$, where E is a connected component of $\mathbb{R}^n - \mathcal{A}_Q$ of order $\boldsymbol{\nu}$. We choose $\boldsymbol{t} = (t_1, \dots, t_r)$ so that the amoebas of the polynomials

$$\boldsymbol{z}^{\boldsymbol{q}_1} - t_1, \dots, \boldsymbol{z}^{\boldsymbol{q}_r} - t_r, \tag{14}$$

divide E into 2^r parts $E(\boldsymbol{\varepsilon})$, where $\boldsymbol{\varepsilon} := (\varepsilon_1, \dots, \varepsilon_r)$ exhaust the family of all vertices of the r-dimensional cube $[-1, 1]^r$. The part $E(\boldsymbol{\varepsilon})$ is the intersection of E with the cone

$$\{\boldsymbol{x} \in \mathbb{R}^n : \varepsilon_1(e^{\boldsymbol{x}\cdot\boldsymbol{q_1}} - |t_1|) > 0, \dots, \varepsilon_r(e^{\boldsymbol{x}\cdot\boldsymbol{q_r}} - |t_r|) > 0\}.$$

The locus of all such $\boldsymbol{t}$ in $(\mathbb{C}^\times)^r$ is denoted by T, and it could be be defined by semialgebraic conditions on $|t_1|, \dots, |t_r|$.

Then it is not difficult to show (see the proof of Proposition 2 in [7]) that for $\boldsymbol{t} \in T$, the complete $\boldsymbol{Q}$-diagonal has the integral representation

$$d_{\boldsymbol{Q}}(\boldsymbol{t}) = \frac{1}{(2\pi\imath)^n} \int\limits_{\Gamma} \frac{P(\boldsymbol{z})}{Q(\boldsymbol{z})} \prod_{j=1}^{r} \frac{\boldsymbol{z}^{\boldsymbol{q}_j}}{\boldsymbol{z}^{\boldsymbol{q}_j} - t_j} \frac{d\boldsymbol{z}}{\boldsymbol{z}^{\boldsymbol{e}}}, \tag{15}$$

where $\Gamma = \sum\limits_{\boldsymbol{\varepsilon}} (\varepsilon_1 \cdot \ldots \cdot \varepsilon_r) \Lambda^{-1}(\boldsymbol{x}(\boldsymbol{\varepsilon}))$ and $\boldsymbol{x}(\boldsymbol{\varepsilon})$ is a point in $E(\boldsymbol{\varepsilon})$.

If the vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_r$ generate a saturated r-dimensional sublattice of $(\mathbb{Z}^n)^*$. Then the $n \times r$ matrix $\boldsymbol{Q}$ with columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_r$ can be extended to unimodular $n \times n$ matrix $\boldsymbol{B}$ by the Invariant Factor Theorem (see again [13, Theorem 16.6]). We consider the toric morphism (6) defined by a matrix $\boldsymbol{A}$, which is defined to be the inverse matrix to $\boldsymbol{B}$.

The following proposition generalizes Theorem 1 of [14].

Proposition 3. *Let $\boldsymbol{Q} := (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_r)$ generate a saturated r-dimensional sublattice of $(\mathbb{Z}^n)^*$. Then for $t \in T$, the complete $\boldsymbol{Q}$-diagonal of* (1) *can be represented in the form*

$$d_{\boldsymbol{Q}}(\boldsymbol{t}) = \frac{1}{(2\pi\imath)^{n-r}} \int\limits_{\Lambda_r^{-1}(\boldsymbol{y})} \frac{\tilde{P}(\boldsymbol{t}, w_{r+1}, \ldots, w_n)}{\tilde{Q}(\boldsymbol{t}, w_{r+1}, \ldots, w_n)} \frac{dw_{r+1} \wedge \ldots \wedge dw_n}{w_{r+1} \cdot \ldots \cdot w_n}, \tag{16}$$

where $\boldsymbol{y} := (y_1, \ldots, y_n)$ is a point in the component $\tilde{E}$ of order $\boldsymbol{\mu} = \boldsymbol{A}\boldsymbol{\nu}$ of the complement to the amoeba $\tilde{\mathcal{A}}$ of $\tilde{Q}(\boldsymbol{w})$. Moreover, the point $(y_{r+1}, \ldots, y_n)$ is in the component $\tilde{E}'$ of order $(\mu^{r+1}, \ldots, \mu^n)$ of the complement to the amoeba of $\tilde{Q}(\boldsymbol{t}, w_{r+1}, \ldots, w_n)$.

Proof. Again, we perform the change of variables in the integral representation (15) that corresponds to the morphism (6). Then the differential form in (15) goes to the differential form

$$\frac{\tilde{P}(\boldsymbol{w})}{\tilde{Q}(\boldsymbol{w})} \prod_{j=1}^{r} \frac{w_j}{w_j - t_j} \frac{d\boldsymbol{w}}{\boldsymbol{w}^{\boldsymbol{e}}}.$$

The image of the cycle Γ by the corresponding induced homomorphism is

$$\Gamma_* = \sum_{\boldsymbol{\varepsilon}} (\varepsilon_1 \cdot \ldots \cdot \varepsilon_r) \Lambda^{-1}(\boldsymbol{y}(\boldsymbol{\varepsilon})),$$

where $\boldsymbol{y}(\boldsymbol{\varepsilon}) := (\boldsymbol{x}(\boldsymbol{\varepsilon}) \cdot \boldsymbol{b}_1, \ldots, \boldsymbol{x}(\boldsymbol{\varepsilon}) \cdot \boldsymbol{b}_n)$ is a point in the component $\tilde{E}$ of order $\boldsymbol{\mu}$ of the complement $\mathbb{R}^n - \tilde{\mathcal{A}}$ such that

$$\varepsilon_1(e^{y_1(\boldsymbol{\varepsilon})} - |t_1|) > 0, \ldots, \varepsilon_r(e^{y_r(\boldsymbol{\varepsilon})} - |t_r|) > 0.$$

Since the component $\tilde{E}$ is open, the points $\boldsymbol{y}(\boldsymbol{\varepsilon})$ can be chosen so that their last $n - r$ coordinates coincide for all $\boldsymbol{\varepsilon}$. For all vertices $\boldsymbol{\varepsilon}$ of the cube $[-1, 1]^r$, we fix particular values $(y_{r+1}, \ldots, y_n)$ of these coordinates. Then Γ_* can be represented in the form

$$\Gamma_* = \Gamma_r \times \Lambda_r^{-1}(\boldsymbol{y}),$$

where $\Gamma_r = \sum_{\boldsymbol{\varepsilon}} (\varepsilon_1 \cdot \ldots \cdot \varepsilon_r) \Lambda^{-1}(y_1(\boldsymbol{\varepsilon}), \ldots, y_r(\boldsymbol{\varepsilon}))$, and $\boldsymbol{y}$ is a point in $\tilde{E}$ with the last $n - r$ coordinates $(y_{r+1}, \ldots, y_n)$.

Therefore, the integral (15) can be written as

$$\frac{1}{(2\pi\imath)^n}\int\limits_{\Lambda_r^{-1}(\boldsymbol{y})}\left(\int\limits_{\Gamma_r}\frac{\tilde{P}(\boldsymbol{w})}{\tilde{Q}(\boldsymbol{w})}\frac{dw_1\wedge\ldots\wedge dw_r}{(w_1-t_1)\cdot\ldots\cdot(w_r-t_r)}\right)\frac{dw_{r+1}\wedge\ldots\wedge dw_n}{w_{r+1}\ldots w_n}. \tag{17}$$

The cycle Γ_r is homologically equivalent to the cycle $\{|w_1-t_1|=\delta,\ldots,|w_r-t_r|=\delta\}$ in the complement

$$\mathbb{C}^\times_{w_1}\times\ldots\times\mathbb{C}^\times_{w_r}-Z^\times(\tilde{Q}(w_1,\ldots,w_r,w^0_{r+1},\ldots,w^0_n)(w_1-t_1)\cdot\ldots\cdot(w_r-t_r))$$

for all $(w^0_{r+1},\ldots,w^0_n)\in\Lambda_r^{-1}(\boldsymbol{y})$, where δ is sufficiently small positive number. Then the Cauchy integral formula gives us that

$$\int\limits_{\Gamma_r}\frac{\tilde{P}(\boldsymbol{w})}{\tilde{Q}(\boldsymbol{w})}\frac{dw_1\wedge\ldots\wedge dw_r}{(w_1-t_1)\cdot\ldots\cdot(w_r-t_r)}=(2\pi\imath)^r\frac{\tilde{P}(\boldsymbol{t},w_{r+1},\ldots,w_n)}{\tilde{Q}(\boldsymbol{t},w_{r+1},\ldots,w_n)},$$

and we are done. □

5. Proof of the Main Result

To give an idea of the proof, let us reconsider Example 1 that we encountered previously.

Example 2. Let $F(\boldsymbol{z})$ and $d(l,t)$ be as in Example 1. The series $d(l,t)$ is closely related to the series from Example 6.2.7 and Example 6.3.6 in [3].

Consider the unimodular matrix $\boldsymbol{B}$ and its inverse matrix $\boldsymbol{A}$:

$$\boldsymbol{B}:=\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix},\ \boldsymbol{A}:=\begin{pmatrix}1&0&0\\-1&1&0\\-1&0&1\end{pmatrix}.$$

Then the corresponding homomorphism (8) gives

$$Q(z_1,z_2,z_3)=1-z_1-z_2^lz_3^l-z_3\mapsto\tilde{Q}(w_1,w_2,w_3)=1-\frac{w_1}{w_2w_3}-w_2^lw_3^l-w_3.$$

Note that the Newton polytope Δ_Q is combinatorially equivalent to $\Delta_{\tilde{Q}}$ (see Fig. 2). The vertex $(0,0,0)$ of the polytope Δ_Q corresponds to the vertex $(0,0,0)$ of the polytope $\Delta_{\tilde{Q}}$, and the component $E_{0,0,0}$ of the complement to the amoeba of Q is mapped to the component $\tilde{E}_{0,0,0}$ by Proposition 1.

According to Proposition 3, we can write the complete $\boldsymbol{e}$-diagonal of the Taylor series for $F(\boldsymbol{z})$ as

$$d(l,t)=\frac{1}{(2\pi\imath)^2}\int\limits_{\Lambda^{-1}(y_2,y_3)}\frac{1}{1-\frac{t}{w_2w_3}-w_2^lw_3^l-w_3}\frac{dw_2\wedge dw_3}{w_2w_3},$$

where (y_2,y_3) is a point (black point on Fig. 1) in the component $\tilde{E}'_{0,0}$ of the complement to the amoeba of the polynomial $1-\frac{t}{w_2w_3}-w_2^lw_3^l-w_3$. Fig. 1 depicts the amoeba and the Newton polytope of this polynomial for $l=2$. Even if $l>2$, it still describes the shapes of these objects. The dual cone $C^\vee_{0,0}$ to the Newton polytope at the point $(0,0)$ is generated by the vector $\boldsymbol{a}^1=(-1,1)$. So we choose the unimodular matrices

$$\boldsymbol{A}=\begin{pmatrix}-1&1\\0&1\end{pmatrix},\ \boldsymbol{B}=\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

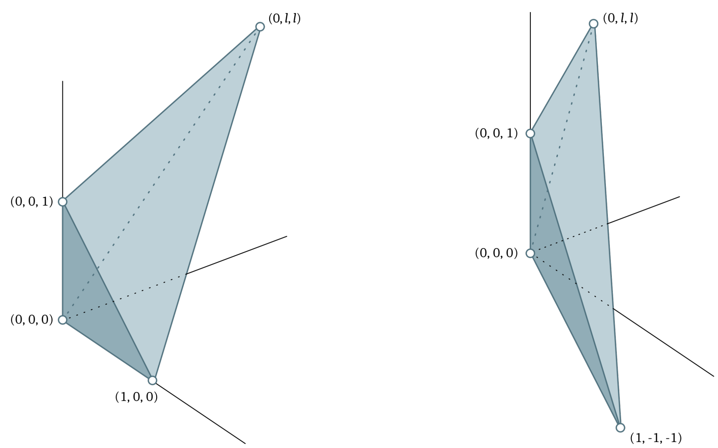

FIG. 2. The Newton polytopes of the polynomials $1-z_1-z_2^l z_3^l-z_3$ (left) and $1-\frac{w_1}{w_2w_3}-w_2^l w_3^l-w_3$ (right)

that define morphisms (6) and (7). Since the latter integral has the form of the integral in (10), applying Proposition 2 to it gives us the integral representation

$$d(l,t)=\frac{1}{2\pi\imath}\int\limits_{|w|=e^y}\frac{1}{1-\frac{t}{w}-w^l}\frac{dw}{w},$$

where y is a point in the component of order 0 of the complement to the amoeba of the Laurent polynomial $Q(w)=1-\frac{t}{w}-w^l$. Since $w=0$ is the only pole of $Q(w)$, the circle $\{|w|=e^y\}$ contains the simple zero $w_1(t)$ of the polynomial $Q(w)$, and the diagonal can be computed as the residue

$$d(l,t)=\underset{w=w_1(t)}{Res}\frac{1}{-w^{l+1}+w-t}.$$

Thus, the diagonal $d(l,t)$ is an algebraic function.

Proof. Our strategy will be to show that if $n-r-p=1$, the complete $\boldsymbol{Q}$-diagonal of the rank r can be represented in the form

$$d_{\boldsymbol{Q}}(\boldsymbol{t})=\frac{1}{2\pi\imath}\int\limits_{|w|=e^y}R(\boldsymbol{t},w)\frac{dw}{w}, \tag{18}$$

where $R(\boldsymbol{t},w)$ is a rational function in w and the parameters $t_1,\ldots,t_r$. Then, the diagonal equals the sum of one-dimensional residues of the integrand at poles inside the disk $\{|w|<e^y\}$. Since such residues are algebraic functions in $t_1,\ldots,t_r$, the diagonal is also algebraic.

If $n-r=1$, then the desired form (18) follows directly from Proposition 3. If $n-r>1$, then we start with the integral representation (16). The integer p is the dimension of the recession cone of $\tilde{E}'$, where $\tilde{E}'$ is the component of order $(\mu_{r+1},\ldots,\mu_n)$ of the complement to the amoeba of the Laurent polynomial $\tilde{Q}(\boldsymbol{t},w_{r+1},\ldots,w_n)$ from (16). We can choose vectors $\boldsymbol{a}^1,\ldots,\boldsymbol{a}^p\in\mathbb{Z}^n$ that generates the dual to cone to its Newton polytope at the point $(\mu_{r+1},\ldots,\mu_n)$ and construct the unimodular $(n-r)\times(n-r)$ matrix $\boldsymbol{A}$ with the first p rows $\boldsymbol{a}^1,\dots.\boldsymbol{a}^p$. Then (18) is given by Proposition 2. □

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