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ON 1-SKELETON OF THE POLYTOPE OF PYRAMIDAL TOURS WITH STEP-BACKS

A.V. NIKOLAEV

Abstract. Pyramidal tours with step-backs are Hamiltonian tours of a special kind: the salesperson starts in city 1, then visits some cities in ascending order, reaches city n, and returns to city 1 visiting the remaining cities in descending order. However, in the ascending and descending direction, the order of neighboring cities can be inverted (a step-back). It is known that on pyramidal tours with step-backs the traveling salesperson problem can be solved by dynamic programming in polynomial time. We define the polytope of pyramidal tours with step-backs $\mathrm{PSB}(n)$ as the convex hull of the characteristic vectors of all possible pyramidal tours with step-backs in a complete directed graph. The 1-skeleton of $\mathrm{PSB}(n)$ is the graph whose vertex set is the vertex set of the polytope, and the edge set is the set of geometric edges or one-dimensional faces of the polytope. We present a linear-time algorithm to verify vertex adjacency in the 1-skeleton of the polytope $\mathrm{PSB}(n)$ and estimate the diameter and the clique number of the 1-skeleton: the diameter is bounded above by 4 and the clique number grows quadratically in the parameter n.

Keywords: pyramidal tour with step-backs, 1-skeleton, vertex adjacency, graph diameter, clique number, pyramidal encoding.

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TABLE 1. Properties of the ATSP(n), PYR(n), and PSB(n) polytopes

	Complexity of TSP problem	Vertex adjacency in 1-skeleton	Diameter of 1-skeleton	Clique number of 1-skeleton
Hamiltonian cycles ATSP(n) and TSP(n)	NP-hard [17]	co-NP-complete [28]	2 for ATSP(n) [27] ≤ 4 for TSP(n) [29]	$\Omega\left(2^{(\sqrt{n}-9)/2}\right)$ [8]
Pyramidal tours PYR(n)	$O(n^2)$ [1]	$O(n)$ [7]	2 [11]	$\Theta(n^2)$ [11]
Pyramidal tours with step-backs PSB(n)	$O(n^2)$ [16]	$\boldsymbol{O(n)}$	$\boldsymbol{\leq 4}$	$\boldsymbol{\Theta(n^2)}$

1. Introduction

The 1-*skeleton* of a polytope P is the graph whose vertex set is the vertex set of P and the edge set is the set of geometric edges or one-dimensional faces of P. In this paper, we consider 3 characteristics of 1-skeleton: vertex adjacency, graph diameter, and clique number.

Two vertices of a graph G are called *adjacent* iff they share a common edge. Vertex adjacency in 1-skeleton is of interest as it can be directly applied to develop simplex-like combinatorial optimization algorithms that move from one feasible solution to another along the edges of the 1-skeleton. This class includes, for example, the blossom algorithm by Edmonds for constructing maximum matchings [15], the set partitioning algorithm by Balas and Padberg [4], Balinski's algorithm for the assignment problem [5], Ikura and Nemhauser's algorithm for the set packing problem [20], etc.

The *diameter* of a graph G is the maximum edge distance between any pair of vertices. The study of 1-skeleton's diameter is motivated by its relationship to the simplex-method and similar edge-following algorithms since the diameter serves as a lower bound for the number of iterations of such algorithms (see [14, 19]), as well as the famous Hirsch conjecture [14, 30].

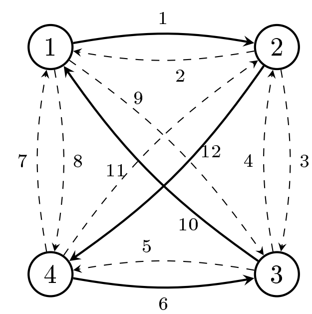
The *clique number* of a graph G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G. It is known that the clique number of 1-skeleton is a lower bound for computational complexity in a class of *direct-type* algorithms based on linear comparisons [8, 9]. Besides, this characteristic is polynomial for known polynomially solvable problems and is superpolynomial for intractable problems (see, for example, [6, 10, 32]).

In this paper we consider three polytopes associated with the traveling salesperson problem: the asymmetric traveling salesperson polytope ATSP(n), the pyramidal tours polytope PYR(n), and the polytope of pyramidal tours with step-backs PSB(n). Properties of their 1-skeletons are summarized in Table 1. The results of the research are highlighted in bold.

2. Traveling salesperson polytope

We consider an asymmetric traveling salesperson problem: given a complete weighted digraph $K_n = (V, E)$ (whose vertices are called *cities*), it is required to find a Hamiltonian tour of minimum weight [17]. With each Hamiltonian tour x in K_n we associate a characteristic vector $\mathbf{v}(x) \in \mathbb{R}^E$ by the following rule:

$$\mathbf{v}(x)_e = \begin{cases} 1, & \text{if an edge } e \in E \text{ is contained in the tour } x, \\ 0, & \text{otherwise.} \end{cases}$$



$$\mathbf{v}(1, 2, 4, 3) = (1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1)$$

FIG. 1. An example of a characteristic vector for a Hamiltonian tour $\langle 1, 2, 4, 3 \rangle$

An example of constructing a characteristic vector $\mathbf{v}(x)$ for a Hamiltonian tour x is shown in Fig. 1.

The polytope

$$\text{ATSP}(n) = \text{conv}\{\mathbf{v}(x) \mid x \text{ is a Hamiltonian tour in } K_n\}$$

is called *the asymmetric traveling salesperson polytope.*

The *symmetric traveling salesperson polytope* TSP(n) is defined similarly as the convex hull of characteristic vectors of all possible Hamiltonian cycles in the complete undirected graph K_n.

The traveling salesperson polytope was introduced by Dantzig, Fulkerson, and Johnson in their classic work on solving the traveling salesperson problem for 49 US cities by integer linear programming [13]. State-of-the-art exact algorithms for the traveling salesperson problem are based on a partial description of the facets of the traveling salesperson polytope and the branch and cut method for integer linear programming [2].

The 1-skeleton of the traveling salesperson polytope has long been the object of close attention in the field of polyhedral combinatorics. The classic result by Papadimitriou [28] states that the question of whether two vertices of the ATSP(n) (or TSP(n)) are not adjacent is NP-complete. It is known that the graph diameter of 1-skeleton equals 2 for ATSP(n) [27] and is at most 4 for TSP(n) [29]. An open conjecture by Grötchel and Padberg states that the diameter is 2 for both polytopes [19]. As for the clique number of ATSP(n) (and TSP(n)), Bondarenko proved that it is superpolynomial in the parameter n [8]. Note that, historically, the traveling salesperson polytope was the first combinatorial polytope for which both the NP-completeness of verifying the vertex non-adjacency and the superpolynomial clique number of the 1-skeleton were established.

Since vertex adjacency is a hard problem for the traveling salesperson polytope, various special cases are of interest. In particular, Sierksma et al. [31] studied the faces of diameter 2, Arthanari [3] considered the pedigree polytope which provided a sufficient condition for non-adjacency in the traveling salesperson polytope, and Bondarenko et al. [7, 11] studied the polytope of pyramidal tours.

In this paper, we consider the polytope associated with Hamiltonian tours of a special kind: pyramidal tours with step-backs.

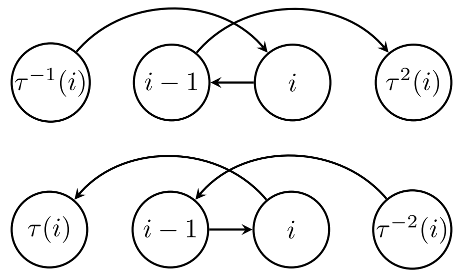


FIG. 2. A step-back in ascending and descending order

3. Pyramidal tours

We suppose that the cities are labeled from 1 to n. Let τ be a Hamiltonian tour. We denote the successor of i-th city as $\tau(i)$, and the predecessor as $\tau^{-1}(i)$. For any natural k, we denote the k-th successor of i as $\tau^k(i)$, the k-th predecessor of i as $\tau^{-k}(i)$. The city i satisfying $\tau^{-1}(i) < i$ and $\tau(i) < i$ is called a *peak*. A *pyramidal tour*, introduced by Aizenshtat and Kravchuk [1], is a Hamiltonian tour with only one peak n. In other words, the salesperson starts in the city 1, then visits some cities in ascending order, reaches city n and returns to the city 1, visiting the remaining cities in descending order.

Enomoto, Oda, and Ota introduced a more general class of pyramidal tours with step-backs [16]. A *step-back peak* (see Fig. 2) is the city i, such that

$$\tau^{-1}(i) < i,\ \tau(i) = i-1,\ \tau^2(i) > i,\ \text{or}\ \tau^{-2}(i) > i,\ \tau^{-1}(i) = i-1,\ \tau(i) < i.$$

A *proper peak* is a peak i which is not a step-back peak. A *pyramidal tour with step-backs* is a Hamiltonian tour with exactly one proper peak n.

Pyramidal tours and pyramidal tours with step-backs are of interest, since, on the one hand, the minimum cost pyramidal tour (with step-backs) can be found in $O(n^2)$ time by dynamic programming, and, on the other hand, there are known restrictions on the distance matrix that guarantee the existence of an optimal tour that is pyramidal (with step-backs). See Klyaus [23] and Gilmore et al. [18] for pyramidal tours, and Enomoto et al. [16] for pyramidal tours with step-backs.

Note that pyramidal tours are among the most studied polynomially solvable special cases of the traveling salesperson problem (see surveys by Burkard et al. [12] and Kabadi [21]). Step-backs allow us to significantly expand the class of considered Hamiltonian cycles. In particular, the complete graph K_n contains 2^{n-2} pyramidal tours and $\Theta((1+\sqrt{3})^{n-1})$ pyramidal tours with step-backs [16].

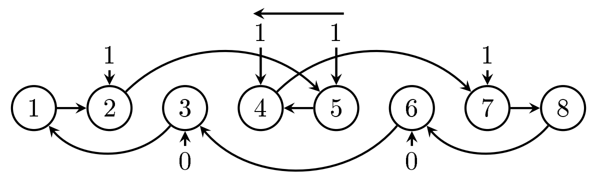
A generalization of pyramidal tours with step-backs is the class of quasi-pyramidal tours, introduced by Oda [26], for which the traveling salesperson problem is fixed-parameter tractable (see also Khachay and Neznakhina [22]).

We consider a complete digraph $K_n = (V, E)$. With each pyramidal tour (with step-backs) x in K_n we associate a characteristic vector $\mathbf{v}(x) \in \mathbb{R}^E$:

$$\mathbf{v}(x)_e = \begin{cases} 1, & \text{if an edge } e \in E \text{ is contained in the tour } x, \\ 0, & \text{otherwise.} \end{cases}$$

The polytope

$$\text{PYR}(n) = \text{conv}\{\mathbf{v}(x) \mid x \text{ is a pyramidal tour in } K_n\}$$



$$\Updownarrow$$

$$\mathbf{x}^{0,1,sb} = \left\langle 1, 0, \overleftarrow{1, 1}, 0, 1 \right\rangle$$

FIG. 3. An example of a tour and the corresponding pyramidal encoding

is called the *polytope of pyramidal tours.*

The polytope

$$\mathrm{PSB}(n) = \mathrm{conv}\{\mathbf{v}(x) \mid x \text{ is a pyramidal tour with step-backs in } K_n\}$$

is called the *polytope of pyramidal tours with step-backs.*

The polytope of pyramidal tours $\mathrm{PYR}(n)$ was introduced in [7] and later considered in [11] by Bondarenko et al. It was established that vertex adjacency in 1-skeleton of the $\mathrm{PYR}(n)$ polytope can be verified in linear time $O(n)$, the diameter of 1-skeleton equals 2, and the asymptotically exact estimate of clique number is $\Theta(n^2)$.

The polytope of pyramidal tours with step-backs was introduced in [24], where a necessary and sufficient condition for vertex adjacency in the 1-skeleton of the polytope is given. Based on this condition, we develop a linear-time algorithm to verify vertex adjacencies in the polytope $\mathrm{PSB}(n)$ and study the diameter and the clique number of the 1-skeleton.

4. PYRAMIDAL ENCODING

Following [24], we introduce a special pyramidal encoding to represent the pyramidal tours with step-backs. With each pyramidal tour with step-backs x in K_n we associate a vector $\mathbf{x}^{0,1,sb}$ of length $n-2$, each coordinate corresponds to a city from 2 to $n-1$, by the following rule:

$$\mathbf{x}_i^{0,1,sb} = \begin{cases} 1, & \text{if } i \text{ is visited by } x \text{ in ascending order,} \\ \overleftarrow{1\,1}, & \text{if } i \text{ is a step-back peak in ascending order,} \\ 0, & \text{if } i \text{ is visited by } x \text{ in descending order,} \\ \overrightarrow{0\,0}, & \text{if } i \text{ is a step-back peak in descending order.} \end{cases}$$

Note that a step-back on i also involves the previous coordinate $i-1$. An example of a pyramidal tour with step-backs and the corresponding encoding vector $\mathbf{x}^{0,1,sb}$ is shown in Fig. 3.

We denote by $\mathbf{x}^{0,1,sb}_{[i,j]}$ a fragment of encoding on coordinates from i to j. The superscript indicates what we consider in the encoding: descending order (0), ascending order (1), or step-backs (sb). For example, $\mathbf{x}^{1,sb}_{[i,j]}$ means a fragment of the encoding only in ascending order taking into account step-backs; $\mathbf{x}^{0,1}_{[i,j]}$ – a fragment of the encoding disregarding step-backs, etc.

5. Vertex adjacency

We consider 12 blocks of the following form (a wavy line means that the corresponding coordinate can either contain a step-back or not):

$$U_{11} = \left\langle\begin{matrix} 1 \\ 1 \end{matrix}\right\rangle, \ U_{00} = \left\langle\begin{matrix} 0 \\ 0 \end{matrix}\right\rangle, \ U_{1111} = \left\langle\begin{matrix} \overleftarrow{1 \quad 1} \\ \overleftarrow{1 \quad 1} \end{matrix}\right\rangle, \ U_{0000} = \left\langle\begin{matrix} \overrightarrow{0 \quad 0} \\ \overrightarrow{0 \quad 0} \end{matrix}\right\rangle,$$

$$L_{1110} = \left\langle\begin{matrix} \overleftarrow{1 \quad 1} \\ 1 \quad \tilde{0} \end{matrix}\right\rangle, \ L_{1011} = \left\langle\begin{matrix} 1 \quad \tilde{0} \\ \overleftarrow{1 \quad 1} \end{matrix}\right\rangle, \ L_{0001} = \left\langle\begin{matrix} \overrightarrow{0 \quad 0} \\ 0 \quad \tilde{1} \end{matrix}\right\rangle, \ L_{0100} = \left\langle\begin{matrix} 0 \quad \tilde{1} \\ \overrightarrow{0 \quad 0} \end{matrix}\right\rangle,$$

$$R_{1101} = \left\langle\begin{matrix} \overleftarrow{1 \quad 1} \\ \tilde{0} \quad 1 \end{matrix}\right\rangle, \ R_{0111} = \left\langle\begin{matrix} \tilde{0} \quad 1 \\ \overleftarrow{1 \quad 1} \end{matrix}\right\rangle, \ R_{0010} = \left\langle\begin{matrix} \overrightarrow{0 \quad 0} \\ \tilde{1} \quad 0 \end{matrix}\right\rangle, \ R_{1000} = \left\langle\begin{matrix} \tilde{1} \quad 0 \\ \overrightarrow{0 \quad 0} \end{matrix}\right\rangle.$$

Theorem 1 (Nikolaev [24]). *Vertices* $\mathbf{v}(x)$ *and* $\mathbf{v}(y)$ *of the polytope* PSB(n) *are not adjacent if and only if the following conditions are satisfied.*

- *There exists a city* i *(called a left block) such that the tours* x *and* y *on the coordinate* i *(coordinates* i *and* $i+1$ *for double blocks) in the pyramidal encoding have the form of* U, L*, or* $i = 1$.
- *There exists a city* j *(called a right block) such that the tours* x *and* y *on the coordinate* j *(coordinates* $j-1$ *and* j *for double blocks) in the pyramidal encoding have the form of* U, R*, or* $j = n$.

  We denote by i_a *the first city after the left block:* $i_a = i + 1$ *for single blocks and* $i_a = i + 2$ *for double blocks. We denote by* j_b *the last city before the right block:* $j_b = i - 1$ *for single blocks and* $j_b = j - 2$ *for double blocks.*

  Two blocks cut the encoding of the tours into three parts: the left (less than i_a*), the central (from* i_a *to* j_b*), and the right (larger than* j_b*).*
- *In the central part, the coordinates of* $\mathbf{x}^{0,1}$ *and* $\mathbf{y}^{0,1}$ *completely coincide:* $\mathbf{x}^{0,1}_{[i_a,j_b]} = \mathbf{y}^{0,1}_{[i_a,j_b]}$.

  We say that two tours
  - *differ in the left part if* $\mathbf{x}^{0,1,sb}_{[1,i_a-1]} \neq \mathbf{y}^{0,1,sb}_{[1,i_a-1]}$,
  - *differ in the right part if* $\mathbf{x}^{0,1,sb}_{[j_b+1,n]} \neq \mathbf{y}^{0,1,sb}_{[j_b+1,n]}$,
  - *differ in the central part in ascending order if* $\mathbf{x}^{1,sb}_{[i_a,j_b]} \neq \mathbf{y}^{1,sb}_{[i_a,j_b]}$,
  - *differ in the central part in descending order if* $\mathbf{x}^{0,sb}_{[i_a,j_b]} \neq \mathbf{y}^{0,sb}_{[i_a,j_b]}$.

  The remaining conditions are divided into four cases depending on the values of $\mathbf{x}^{0,1}_i$ *and* $\mathbf{x}^{0,1}_j$.

  (1) *If* $\mathbf{x}^{0,1}_i = \mathbf{x}^{0,1}_j = 1$*, then the tours differ*
    - *in the central part in ascending order;*
    - *in the left part, or in the central part in descending order, or in the right part.*

  (2) *If* $\mathbf{x}^{0,1}_i = \mathbf{x}^{0,1}_j = 0$*, then the tours differ*
    - *in the central part in descending order;*
    - *in the left part, or in the central part in ascending order, or in the right part.*

  (3) *If* $\mathbf{x}^{0,1}_i = 1, \mathbf{x}^{0,1}_j = 0$*, then the tours differ*
    - *in the central part in ascending order or in the right part;*
    - *in the central part in descending order or in the left part.*

  (4) *If* $\mathbf{x}^{0,1}_i = 0, \mathbf{x}^{0,1}_j = 1$*, then the tours differ*
    - *in the central part in descending order or in the right part;*
    - *in the central part in ascending order or in the left part.*

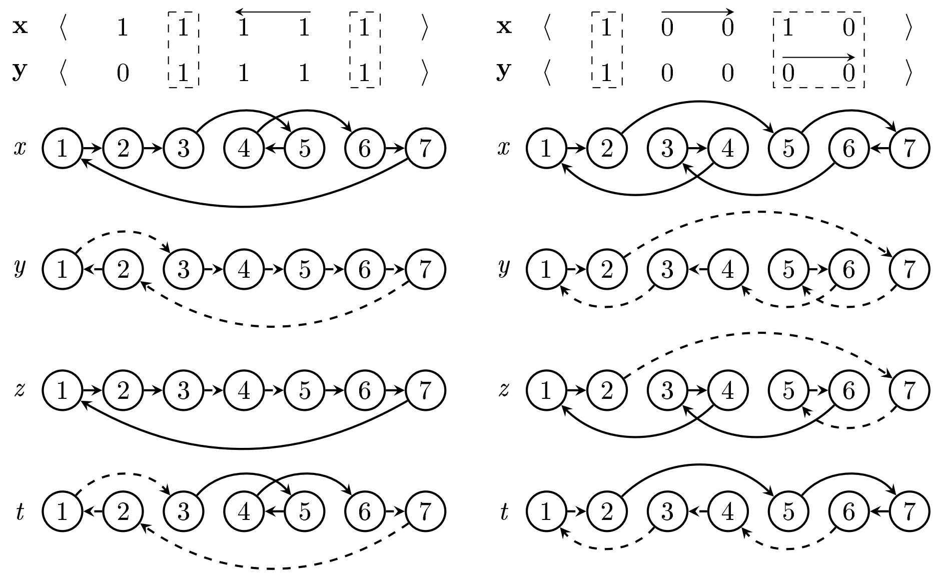


FIG. 4. Examples of first and third sufficient conditions

Cities 1 *and* n *can be considered in the encoding as visited in ascending or descending order if required.*

The idea of sufficient conditions is that if from the edges of the tours x and y we can assemble two complementary pyramidal tours with step-backs z and t, then the segment $[\mathbf{v}(x), \mathbf{v}(y)]$ intersects with the segment $[\mathbf{v}(z), \mathbf{v}(t)]$, and the corresponding vertices of the polytope PSB(n) are not adjacent (see [24]).

The examples of the first and third sufficient conditions for non-adjacency are shown in Fig. 4 (edges of x are solid, edges of y are dashed, left and right blocks in pyramidal encodings are highlighted with dashed boxes).

In [24] it was proved that the necessary and sufficient condition for non-adjacency of Theorem 1 can be verified by exhaustive search in $O(n^3)$ time. We improve this estimate by introducing a linear-time algorithm.

Theorem 2. *The question of whether two vertices of the polytope* PSB(n) *are adjacent can be verified in linear time* $O(n)$.

Proof. We consider two pyramidal tours with step-backs x and y, and the corresponding vertices $\mathbf{v}(x)$ and $\mathbf{v}(y)$ of the polytope PSB(n). Each of the four sufficient non-adjacency conditions of Theorem 1 can be verified in a single pass through the pyramidal encodings $\mathbf{x}^{0,1,sb}$ and $\mathbf{y}^{0,1,sb}$, when we sequentially find the left block, the right block, and check additional conditions. The pseudo-code to construct a tour z (and a tour t as $(x \cup y) \backslash z$) and verify the first and second sufficient conditions is given in the Algorithm 1. The other two sufficient conditions are verified similarly. □

Note that we can consider the problem of finding *a second Hamiltonian decomposition* of a 4-regular multigraph, as described in [25]: find a partition of the edge set of the 4-regular multigraph $x \cup y$ into edge-disjoint Hamiltonian cycles z and t different from the given cycles x and y. The Algorithm 1 solves this problem for

Algorithm 1 Verifying 1st and 2nd sufficient conditions for non-adjacency

```
procedure NonAdjacencyTest(x, y, n)
    LBlock ← TRUE                                    ▷ Consider the city 1 as a left block
    RBlock, zNotx, zNoty ← FALSE
    for i ← 2 to n − 1 do
        if LBlock = TRUE and RBlock = FALSE then                          ▷ Central part
            if z_i (y_i) is different from x_i then
                zNotx ← TRUE                              ▷ z visits i by the edges of y
            end if
            if zNotx = TRUE and we found U or R block then
                RBlock ← TRUE                                ▷ Starting the right part
            end if
            if the conditions of the central part are violated then
                LBlock, zNotx ← FALSE                         ▷ Return to the left part
            end if
        end if
        if LBlock = FALSE then                                               ▷ Left part
            if z_i (x_i) is different from y_i then       ▷ z visits i by the edges of x
                zNoty ← TRUE
            end if
            if we found U or L block then
                LBlock ← TRUE                              ▷ Starting the central part
            end if
        end if
        if RBlock = TRUE then                                               ▷ Right part
            if z_i (x_i) is different from y_i then       ▷ z visits i by the edges of x
                zNoty ← TRUE
            end if
        end if
    end for
    RBlock ← TRUE                                   ▷ Consider the city n as a right block
    if LBlock, RBlock, zNotx, zNoty = TRUE then
        return 1st/2nd sufficient condition for non-adjacency is satisfied
    else
        return 1st/2nd sufficient condition for non-adjacency is not satisfied
    end if
end procedure
```

pyramidal tours with step-backs in linear time $O(n)$. In general, finding a second Hamiltonian decomposition is NP-hard [28].

6. Graph diameter and clique number

Based on Theorem 1, we estimate the diameter of 1-skeleton of $\text{PSB}(n)$.

Theorem 3. *The diameter of 1-skeleton of* $\text{PSB}(n)$ *is bounded above by 4.*

Proof. The idea is as follows. For an arbitrary pyramidal tour with step-backs x we construct a pyramidal tour $\hat{x}$ where

$$\hat{\mathbf{x}}_i^{0,1} = \begin{cases} 0, & \text{if } i \text{ is a part of step-back in } x \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

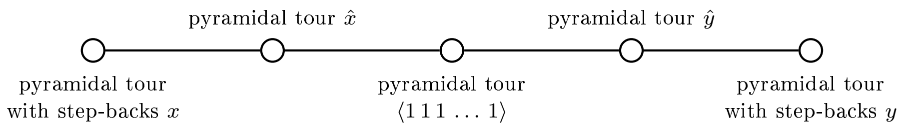


FIG. 5. Path of length 4 between an arbitrary pair of PSB(n) vertices $\mathbf{v}(x)$ and $\mathbf{v}(y)$

For instance:

$$\mathbf{x} : \langle\ 1\ \overleftarrow{1\ \ 1}\ \ 1\ \ 0\ \ 0\ \ 1\ \overrightarrow{0\ \ 0}\ \ 1\rangle,$$
$$\hat{\mathbf{x}} : \langle\ 1\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 1\rangle.$$

By construction, the encodings of the tours x and $\hat{x}$ can only contain blocks U_{11}, which restricts us to the first sufficient condition of Theorem 1. However, by (1) the tours x and $\hat{x}$ cannot differ in the central part in the ascending order. Hence, by Theorem 1, the vertices $\mathbf{v}(x)$ and $\mathbf{v}(\hat{x})$ are adjacent. And for any pyramidal tour $\hat{x}$, the vertex $\mathbf{v}(\hat{x})$ is adjacent to the vertices corresponding to the tours $\langle 1, 1, \dots, 1\rangle$ and $\langle 0, 0, \dots, 0\rangle$ (see [11]). Thus, between any pair of vertices of the polytope PSB(n) we can construct a path of no more than 4 edges. The corresponding scheme is shown in Fig. 5. □

Now we apply the necessary and sufficient condition of Theorem 1 to estimate the clique number of the 1-skeleton of the polytope PSB(n).

Theorem 4. *The clique number of 1-skeleton of the polytope* PSB(n) *is quadratic in the parameter n:*

$$\omega(\mathrm{PSB}(n)) = \Theta\left(n^2\right). \tag{2}$$

Proof. Upper bound. Let $Y^{\mathbf{v}}$ be a set of pairwise adjacent vertices of PSB(n), and Y be the set of corresponding pyramidal tours with step-backs. Let us estimate the cardinality of Y.

Step 1. Consider the pyramidal encodings of the tours. We call a tour $x \in Y$ *unique with respect to a pair of neighboring coordinates* $k, k+1$ if

$$\forall y \in Y \backslash \{x\} :\ \mathbf{x}^{0,1,sb}_{[k,k+1]} \neq \mathbf{y}^{0,1,sb}_{[k,k+1]}.$$

A pair of neighboring coordinates in the pyramidal encoding can take 18 different values. Hence the number of tours in Y that are unique in a pair of neighboring coordinates does not exceed $18(n-3)$. We construct the set W by excluding from Y all tours that are unique in a pair of neighboring coordinates.

Step 2. Let a pyramidal tour with step-backs in a pair of neighboring cities k and $k+1$ visit one of the cities in ascending order, and the other in descending order. Since the city may or may not be a part of step-back, we get 8 possible types of pyramidal encodings at coordinates k and $k+1$:

$$\begin{aligned} &\langle 1\ \ 0\rangle,\ \ \langle 1\ \overleftarrow{0\ \ 0}\rangle,\ \ \langle \overrightarrow{1\ \ 1}\ \ 0\rangle,\ \ \langle \overrightarrow{1\ \ 1}\ \overleftarrow{0\ \ 0}\rangle,\\ &\langle 0\ \ 1\rangle,\ \ \langle 0\ \overrightarrow{1\ \ 1}\rangle,\ \ \langle \overleftarrow{0\ \ 0}\ \ 1\rangle,\ \ \langle \overleftarrow{0\ \ 0}\ \overrightarrow{1\ \ 1}\rangle. \end{aligned} \tag{3}$$

We call such sections of encoding 0/1-*segments.*

We consider a pyramidal tour with step-backs $x \in W$ with 0/1-segment at the coordinates $k, k+1$. By construction, we have excluded from W all tours that are unique in a pair of neighboring coordinates, so there exists a tour $y \in W \setminus \{x\}$ with the same 0/1-segment at coordinates $k, k+1$, i.e. $\mathbf{x}^{0,1,sb}_{[k,k+1]} = \mathbf{y}^{0,1,sb}_{[k,k+1]}$.

Thus, pyramidal encodings of tours x and y on coordinates $k, k+1$ have the form of a pair of blocks U, where the coordinate k is in the left block and $k+1$ is in the right block. Since the corresponding vertices $\mathbf{v}(x)$ and $\mathbf{v}(y)$ of the polytope PSB(n) are adjacent, by Theorem 1, the encodings of the tours x and y coincide either in the left part ($\mathbf{x}^{0,1,sb}_{[1,k]} = \mathbf{y}^{0,1,sb}_{[1,k]}$), or in the right part ($\mathbf{x}^{0,1,sb}_{[k+1,n]} = \mathbf{y}^{0,1,sb}_{[k+1,n]}$). For instance:

$$
\begin{aligned}
\mathbf{x} &: \langle\ \boxed{1\ \overrightarrow{0\ 0}\ 1}\ \overbrace{0\ 1}^{k,k+1}\ \ \overrightarrow{0\ 0}\ 0\ 1\ \rangle,\\
\mathbf{y} &: \langle\ \boxed{1\ \overrightarrow{0\ 0}\ 1}\ \underbrace{0\ 1}_{k,k+1}\ \ 1\ 0\ \overleftarrow{1\ 1}\ \rangle.
\end{aligned}
$$

Note that for any subset of tours in W with a common 0/1-segment, the coinciding parts of the encoding are on the same side of the segment. Indeed, suppose that three tours $x, y, z \in W$ have the same 0/1-segment on the cities $k, k+1$, but the coinciding parts of the encodings are on different sides of the segment. Without loss of generality, let

$$
\begin{aligned}
&\mathbf{x}^{0,1,sb}_{[1,k]} = \mathbf{y}^{0,1,sb}_{[1,k]} \text{ and } \mathbf{x}^{0,1,sb}_{[k+1,n]} = \mathbf{z}^{0,1,sb}_{[k+1,n]},\\
\mathbf{x} &: \langle\ \boxed{\quad L \quad}\ 0\ 1\ \boxed{\quad R \quad}\ \rangle,\\
\mathbf{y} &: \langle\ \boxed{\quad L \quad}\ 0\ 1\ \boxed{\qquad\quad}\ \rangle,\\
\mathbf{z} &: \langle\ \boxed{\qquad\quad}\ 0\ 1\ \boxed{\quad R \quad}\ \rangle.
\end{aligned}
$$

However, since the corresponding vertices $\mathbf{v}(y)$ and $\mathbf{v}(z)$ of the polytope PSB(n) are adjacent, by Theorem 1, the pyramidal encodings of the tours y and z also coincide either in the left part: $\mathbf{y}^{0,1,sb}_{[1,k]} = \mathbf{z}^{0,1,sb}_{[1,k]}$ (in this case $x = z$), or in the right part: $\mathbf{y}^{0,1,sb}_{[k+1,n]} = \mathbf{z}^{0,1,sb}_{[k+1,n]}$ (in this case $x = y$). We got a contradiction.

Thus, for any pyramidal tour with step-backs from the set W, all 0/1-segments can be divided into *left segments* (for which tours with a common segment coincide in the left part) and *right segments* (for which tours coincide in the right part).

Let us show that the left and right 0/1-segments in tours from W are ordered, i.e. if some tour $x \in W$ contains a left 0/1-segment on cities $k, k+1$ and a right 0/1-segment on cities $s, s+1$, then $k < s$.

Assume that $k > s$. Consider a tour $y \in W$ that shares a left 0/1-segment on $k, k+1$ with x, then $\mathbf{x}^{0,1,sb}_{[1,k]} = \mathbf{y}^{0,1,sb}_{[1,k]}$. Hence the tours x and y coincide on the cities $s, s+1$ since $s < k$, and have common right 0/1-segments:

$$
\begin{aligned}
\mathbf{x} &: \langle\ \boxed{*\ \ *\ \ 0\ 1\ \ *\ \ *}\ \overbrace{0\ 1}^{L}\ \ *\ \ *\ \rangle,\\
\mathbf{y} &: \langle\ *\ \ *\ \ \underbrace{0\ 1}_{R}\ \boxed{*\ \ *\ \ 0\ 1\ \ *\ \ *}\ \rangle.
\end{aligned}
$$

It remains to note that

$$
\mathbf{x}^{0,1,sb}_{[1,k]} = \mathbf{y}^{0,1,sb}_{[1,k]} \text{ and } \mathbf{x}^{0,1,sb}_{[s,n]} = \mathbf{y}^{0,1,sb}_{[s,n]},
$$

hence, $x = y$, a contradiction.

Step 3. Consider a pyramidal tour with step-backs $x \in W$. We choose the left 0/1-segment $L_{\max}$ at the largest coordinates $i, i+1$. If there is no such segment, then we set $i = 1$. We choose the right 0/1-segment $R_{\min}$ at the smallest coordinates $j-1, j$. If there is no such segment, then we set $j = n$.

By construction, $i \leq j$ and cities from $i+1$ to $j-1$ are visited in the same direction. Let us call the part of pyramidal encoding $[i+1, j-1]$ a 0-*sequence* if the cities from $i+1$ to $j-1$ are visited in the descending order, and 1-*sequence* if the cities are visited in ascending order.

Since the tours in W coincide to the left of the common left 0/1-segment and to the right of the common right 0/1-segment, each pyramidal tour with step-backs $x \in W$ corresponds to a unique 0-sequence or 1-sequence. For example:

$$\langle\ \underbrace{1\ 0}_{L}\ 0\ \underbrace{\overrightarrow{0\ 0}\ 1}_{L}\ 1\ \underbrace{1\ 0}_{L_{\max}}\ \overrightarrow{0\ 0}\ 0\ \overrightarrow{0\ 0}\ \underbrace{0\ 1}_{R_{\min}}\ \underbrace{\overrightarrow{1\ 1}\ 0}_{R}\ \overrightarrow{0\ 0}\ \rangle. \quad (\text{0-sequence: } 0\ \overrightarrow{0\ 0}\ 0\ \overrightarrow{0\ 0}\ 0)$$

Step 4. We consider in W the subset of all pyramidal tours with step-backs that have a 0-sequence starting at position i. Let's denote this subset as W_i^0.

We consider in the set W_i^0 all tours containing at least one unique 0-coordinate inside the 0-sequence in the pyramidal encoding. There are $n-i$ possible positions of the unique 0-coordinate, which can take 3 different values: start of a step-back, end of a step-back, not a step-back. Thus, the total number of such unique tours in W_i^0 does not exceed $3(n-i) = O(n)$. Let us construct the set $\bar{W}_i^0$ by excluding from W_i^0 all tours with unique coordinates in the 0-sequence.

Consider some tour $x \in \bar{W}_i^0$. By construction, for any coordinate s within the 0-sequence, there exists a second tour $y \in \bar{W}_i^0$ such that $\mathbf{x}_s^{0,sb} = \mathbf{y}_s^{0,sb}$ and s belongs to the 0-sequence of y. Then the tours x and y on the coordinates $i-1$ (segment $L_{\max}$) form a block U_{11} (or U_{1111}), and on the coordinate s – one of the blocks U_{00} or U_{0000}. Therefore, by Theorem 1, the pyramidal encodings of the tours x and y coincide either in the central part between $i-1$ and s in descending order ($\mathbf{x}_{[i,s]}^{0,sb} = \mathbf{y}_{[i,s]}^{0,sb}$) or to the right of s ($\mathbf{x}_{[s,n]}^{0,1,sb} = \mathbf{y}_{[s,n]}^{0,1,sb}$). For instance:

$$\begin{aligned} \mathbf{x} &: \langle\ \boxed{0\ \overleftarrow{1\ 1}}\ \overset{i-1}{1}\ \boxed{0\ \overrightarrow{0\ 0}}\ \overset{s}{0}\ 0\ 0\ 1\ 0\ \overleftarrow{1\ 1}\ \rangle, \\ \mathbf{y} &: \langle\ \boxed{0\ \overleftarrow{1\ 1}}\ \underset{i-1}{1}\ \boxed{0\ \overrightarrow{0\ 0}}\ \underset{s}{0}\ \overrightarrow{0\ 0}\ 0\ 0\ 1\ 1\ \rangle. \end{aligned}$$

Otherwise, the corresponding vertices $\mathbf{v}(x)$ and $\mathbf{v}(y)$ of the polytope $\mathrm{PSB}(n)$ are not adjacent.

Further reasoning completely repeats similar ones for common 0/1-segments. For any subset of tours in $\bar{W}_i^0$ with a common 0-coordinate, the coinciding parts of the encoding must be on the same side of the coordinate. This allows us to divide all 0-sequence coordinates into *left* 0-*coordinates* (tours with a common coordinate coincide on the left side) and *right* 0-*coordinates* (for which tours coincide to the right of the common coordinate). Note also that all coordinates are ordered, i.e. if s is the left 0-coordinate and t is the right 0-coordinate, then $s < t$:

$$\langle\ \boxed{0\ \overleftarrow{1\ 1}}\ \overset{i-1}{1}\ \underbrace{\overbrace{0\ \overrightarrow{0\ 0}\ 0}^{L}\ \overbrace{\overrightarrow{0\ 0}\ 0}^{R}}_{\text{0-sequence}}\ 1\ \overrightarrow{1\ 1}\ 0\ \rangle.$$

We denote by $\bar{W}_{i,s}^0$ the subset of all tours from $\bar{W}_i^0$ for which s is the number of the largest left 0-coordinate, and $s+1$ is the number of the smallest right 0-coordinate.

The key idea is that the coordinates s and $s+1$ of any tour from $\bar{W}_{i,s}^0$ can take only 2 values: $\langle 0 \rangle$ and $\langle \overleftarrow{0 \ 0} \rangle$. Moreover, the values of this pair of coordinates uniquely determine the tour since tours with the same left coordinate coincide to the left and with the same right coordinate – to the right. Thus, the subset $\bar{W}_{i,s}^0$ contains at most 4 tours.

And now we rise back to the original set of pairwise adjacent tours. Firstly,

$$\bar{W}_i^0 = \bigcup_{s=i+1}^{n-1} \bar{W}_{i,s}^0,$$

whence, with the excluded tours with unique 0-coordinates, $|W_i^0| = O(n)$. Similarly, we get that $|W_i^1| = O(n)$.

Secondly,

$$W = \left(\bigcup_{i=2}^{n-1} W_i^0 \right) \cup \left(\bigcup_{i=2}^{n-1} W_i^1 \right),$$

whence we get that $|W| = O(n^2)$.

Finally, returning the excluded tours with unique pairs of neighboring coordinates, we arrive at the upper bound $|Y| = |W| + O(n) = O(n^2)$. Thus,

$$\omega(\mathrm{PSB}(n)) = O(n^2).$$

Lower bound. We construct an example of the set $Z^{\mathbf{v}}$ of pairwise adjacent vertices of the polytope $\mathrm{PSB}(n)$ such that $|Z^{\mathbf{v}}| = \Omega(n^2)$. Let n be even. With each pair of integers k, s such that $0 \leq k, s \leq \frac{n-2}{2}$ we associate a pyramidal tour with step-backs $x(k,s) \in Z$ such that

- $\forall i \ (2 \leq i \leq k+1)$: $\mathbf{x}_i^{0,1,sb} = 1$;
- $\forall j \ (n-s \leq j \leq n-1)$: $\mathbf{x}_j^{0,1,sb} = 1$;
- all other coordinates of $\mathbf{x}^{0,1,sb}$ are equal to 0.

The tours from the set Z do not contain step-backs, therefore, they cannot differ in the central part between the left and right blocks. Moreover, if for two tours $x, y \in Z$ along some coordinate i: $\mathbf{x}_i^{0,1,sb} = \mathbf{y}_i^{0,1,sb} = 1$, then

- if $i \leq \frac{n}{2}$, then the encodings coincide in the left part (i.e. $\mathbf{x}_{[1,i]}^{0,1,sb} = \mathbf{y}_{[1,i]}^{0,1,sb}$);
- if $i \geq \frac{n}{2} + 1$, then the pyramidal encodings coincide in the right part (i.e. $\mathbf{x}_{[i,n]}^{0,1,sb} = \mathbf{y}_{[i,n]}^{0,1,sb}$).

Hence, by Theorem 1, the corresponding vertices $\mathbf{v}(x)$ and $\mathbf{v}(y)$ of the polytope $\mathrm{PSB}(n)$ are adjacent.

The case of odd n is reduced to an even case, it suffices to fix one of the cities in all tours from Z in ascending or descending order. Thus,

$$\omega(\mathrm{PSB}(n)) \geq |Z| = \left\lfloor \frac{n}{2} \right\rfloor^2 = \Omega(n^2).$$

An example of a set Z for $n = 8$ is given in Table 2.

Combining the upper and lower bounds, we obtain the desired asymptotically exact quadratic estimate (2). $\square$

Table 2. An example of pyramidal tours with step-backs with pairwise adjacent vertices in the polytope PSB(8)

$\langle 0\,0\,0 \mid 0\,0\,0 \rangle$	$\langle 0\,0\,0 \mid 0\,0\,1 \rangle$	$\langle 0\,0\,0 \mid 0\,1\,1 \rangle$	$\langle 0\,0\,0 \mid 1\,1\,1 \rangle$
$\langle 1\,0\,0 \mid 0\,0\,0 \rangle$	$\langle 1\,0\,0 \mid 0\,0\,1 \rangle$	$\langle 1\,0\,0 \mid 0\,1\,1 \rangle$	$\langle 1\,0\,0 \mid 1\,1\,1 \rangle$
$\langle 1\,1\,0 \mid 0\,0\,0 \rangle$	$\langle 1\,1\,0 \mid 0\,0\,1 \rangle$	$\langle 1\,1\,0 \mid 0\,1\,1 \rangle$	$\langle 1\,1\,0 \mid 1\,1\,1 \rangle$
$\langle 1\,1\,1 \mid 0\,0\,0 \rangle$	$\langle 1\,1\,1 \mid 0\,0\,1 \rangle$	$\langle 1\,1\,1 \mid 0\,1\,1 \rangle$	$\langle 1\,1\,1 \mid 1\,1\,1 \rangle$

7. Conclusion

We have considered several versions of the traveling salesperson problem and the associated combinatorial polytopes.

The general traveling salesperson problem is NP-hard [17]. The question of whether two vertices of the traveling salesperson polytope are not adjacent is NP-complete [28]. The clique number of 1-skeleton of the polytope ATSP(n) is superpolynomial in n [8].

On the other hand, the traveling salesperson problem on pyramidal tours and pyramidal tours with step-backs is solvable in polynomial time by dynamic programming [16, 18]. The vertex adjacency for the polytopes PYR(n) and PSB(n) can be verified in linear time $O(n)$. The clique numbers of 1-skeletons are quadratic in n.

Thus, the properties of 1-skeletons of the polytopes associated with the traveling salesperson problem directly correlate with the complexity of the problem itself.

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