S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 19, №2, стр. 724–740 (2022) УДК 517.956
DOI 10.33048/semi.2022.19.060 MSC 35K70, 35D35, 35Q35, 35Q79

THE ONE-DIMENSIONAL IMPULSIVE BARENBLATT–ZHELTOV–KOCHINA EQUATION WITH A TRANSITION LAYER

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Abstract. The initial-boundary value problem for the one-dimensional impulsive pseudoparabolic equation is studied. As a coefficient in the second-order diffusion term, this equation contains the smoothed Dirac delta-function concentrated at some time moment. From a physical viewpoint, such term allows to describe impulsive pressure drop phenomena in filtration problems. Existence and uniqueness of solutions for fixed values of the small parameter of smoothing is proved. After this, the limiting passage as the small parameter tends to zero is fulfilled and rigorously justified. As the result, the limit instantaneous impulsive microscopic-macroscopic model is established. This model is well-posed and involves the additional equation on a transition layer posed on a 'very fast' timescale.

Key words: pseudoparabolic equation, impulsive equation, strong solution, Fourier series, transition layer

1. Introduction

The classical Barenblatt–Zheltov–Kochina equation

$$\partial_t u = \chi \partial^3_{xxt} u + \nu \partial^2_{xx} u + f \quad (\chi, \nu > 0) \tag{1}$$

(here, in one-dimensional case) describes nonstationary filtration of a viscous fluid in a cracky-porous ground [8]. In this framework, the sought function u in (1) has a

Kuznetsov I.V. and Sazhenkov S.A. The one-dimensional impulsive Barenblatt–Zheltov–Kochina equation with a transition layer.

The work was supported by the State Assignment of the Russian Ministry of Science and Higher Education entitled "Modern methods of hydrodynamics for environmental management, industrial systems and polar mechanics"(Govt. contract code: FZMW-2020-0008, 24 January 2020).

Received April, 26, 2022, published November, 11, 2022.

physical sense of the distribution of pressure in the gallery of cracks. Equation (1) also arises in studies of non-equilibrium processes in the heat transfer [11], where u plays the role of one of the temperatures in a two-temperature continuum. As well, some dynamical problems regarding non-Newtonian second-order fluid reduce to boundary value problems for equation (1), and in this context u means some expression of velocity components [43].

If processes under study are rather regular, we can confine ourselves to consideration of rather regular coefficients $\chi = \chi(x,t)$ and $\nu = \nu(x,t)$. The relevant theory of equation (1) and of its multi-dimensional version is vast and deep; its systematic exhibition can be found, for example, in [14]. On the other hand, irregular processes, like very fast pressure drop in the cracky gallery in the problem of filtration, lead to the cases when coefficients become highly non-smooth. More certainly, in this case coefficient ν may behave closely to $\nu_0 + \alpha\delta_{(t=\tau)}$, where $\nu_0 = \text{const} > 0$, $\alpha = \text{const} \neq 0$, and $\delta_{(t=\tau)}$ is the Dirac delta concentrated at the time moment $t = \tau$ of the (instant) pressure drop. For example, such quick capillary pressure drops were experimentally observed in [34] in a two-phase flow. We can also note that the additional term $\alpha\delta_{(t=\tau)}$ may correspond to pressure fluctuations linked with 'impulsive' fast diffusion in porous media. In this view, we remark that fast diffusion can be simulated also with the help of nonstandard gradient growth [10].

In the present paper, we approximate $\delta_{(t=\tau)}$ on the so-called transition ε-layer. To this end, instead of $\delta_{(t=\tau)}$, we put into equation (1) the sequence $\{K_\varepsilon^\tau\}_{\varepsilon\to 0+}$, which converges weakly* to $\delta_{(t=\tau)}$. We observe that the similar approach was applied in [3, 21], where the Dirac delta-function $\delta_{(t=\tau)}$ was encompassed in the minor (source) term, not in the term containing a derivative.

Now, in addition to the modification of (1) described above regarding the approximation of the Dirac delta, we also set $\nu_0 = 1$, $\chi = 1$ and $f = 0$ for simplicity and formulate the following initial-boundary value problem for the Barenblatt–Zheltov–Kochina equation with the homogeneous boundary conditions, which will be studied further in the article:

$$\partial_t u_\varepsilon = \partial^3_{xxt} u_\varepsilon + (1 + \alpha K_\varepsilon^\tau(t))\, \partial^2_{xx} u_\varepsilon, \quad (x,t) \in \Pi_T, \tag{2a}$$

$$u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, \quad t \in (0,T), \tag{2b}$$

$$u_\varepsilon(x,0) = g(x), \quad x \in (0,1). \tag{2c}$$

In (2), $\Pi_T = (0,1)\times(0,T)$ is a space-time rectangle, $T = \text{const} > 0$ and $\tau \in (0,T)$ are given fixed time moments, $\alpha \in \mathbb{R}$ is a given fixed value, $u_\varepsilon = u_\varepsilon(x,t)$ is the sought function, $g = g(x)$ is a given function, and K_ε^τ is a given smooth kernel. For a fixed value τ, kernel $K_\varepsilon^\tau = K_\varepsilon^\tau(t)$ is supported on $\left\{\tau - \frac{\varepsilon}{2} \leqslant t \leqslant \tau + \frac{\varepsilon}{2}\right\}$ and is defined by the formula

$$K_\varepsilon^\tau(t) = \frac{1}{\varepsilon} K\left(\frac{t-\tau}{\varepsilon}\right), \quad t \in [0,T],$$

where $K = K(\vartheta)$ is a nonnegative smooth even function supported on segment $\left\{-\frac{1}{2} \leqslant \vartheta \leqslant \frac{1}{2}\right\}$, with the mean value equal to unity, i.e., $\int_{-\frac{1}{2}}^{\frac{1}{2}} K(\vartheta)\, d\vartheta = 1$, and $\varepsilon > 0$ is a small parameter. Thus, K_ε^τ approximates the Dirac delta-function $\delta_{(t=\tau)}$ in weak* sense as $\varepsilon \to 0+$, i.e., the limiting relation $\lim\limits_{\varepsilon\to 0+} \int_{\mathbb{R}} \phi(t) K_\varepsilon^\tau(t)\, dt = \phi(\tau)$

holds for any integrable in a neighborhood of $\{t = \tau\} \subset \mathbb{R}$ function ϕ having the trace at the point $t = \tau$. Note that

$$\varepsilon K_\varepsilon^\tau(\tau + \varepsilon\vartheta) = K(\vartheta), \quad \forall\, \vartheta \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{3}$$

By analogy with the theory of generalized ordinary differential equations [16, 20, 35], we may call (2a) *the generalized Barenblatt–Zheltov–Kochina equation* corresponding to the pressure of a liquid in cracks.

Further, in Sec. 2, for any fixed rather small ε (say, $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \ll 1$), we construct a strong generalized solution to problem (2) in the form of Fourier series (12) and establish that this solution is unique and satisfies the first and the second energy estimates uniform in ε, see Proposition 1. In Sec. 3, we formulate Theorems 1 and 2, which are the main results of the paper and which relate to the limiting transition in problem (2), as $\varepsilon \to 0+$. Theorem 1 asserts the compactness of the family $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ as $\varepsilon \to 0+$ and the fact that the limit function $u = \lim\limits_{\varepsilon\to 0+} u_\varepsilon$ satisfies the homogeneous Barenblatt–Zheltov–Kochina equation outside the section $\{t = \tau\}$. The proof of Theorem 1 is given in Secs. 4 and 6. In order to prove compactness of $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$, we apply the Aubin–Lions–Simon lemma [38, Theorem 3], see also [4, 12, 24, 25, 32]. Theorem 2 deals with the study of the family $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ near the section $\{t = \tau\}$ and is proved in Sec. 5. More certainly, in Sec. 5, in order to link the one-sided traces of the limit solution at $t = \tau$, we rescale u_ε on $\left(0, \tau - \frac{\varepsilon}{2}\right)$, $\left(\tau - \frac{\varepsilon}{2}, \tau + \frac{\varepsilon}{2}\right)$, and $\left(\tau + \frac{\varepsilon}{2}, T\right)$ and apply the Aubin–Lions lemma [26, Lemma 2.48, p. 36], [38, Corollary 6] to the rescaled solution. As the result, for the rescaled solution we derive the so-called *equation on the transition layer* between $t = \tau - 0$ and $t = \tau + 0$, see (17b), which incorporates not the 'macroscopic' ('slow') time variable t, but the 'microscopic' ('fast') time variable $\bar{t}$, so that $\bar{t} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $t = \tau - 0$ corresponds to $\bar{t} = -\frac{1}{2} + 0$, and $t = \tau + 0$ corresponds to $\bar{t} = \frac{1}{2} - 0$.

In aggregation, the results of Theorems 1 and 2 give us the well-posed limit two-scale microscopic-macroscopic model (15b)–(15d), (17b)–(17e) describing the process in which an instantaneous impulsive phenomenon manifests itself.

The main difference between the aforementioned results from [3, 21] and the results from the present paper is that, in the present paper, the dissipative term persists on a transition layer $\left\{-\frac{1}{2} \leqslant \bar{t} \leqslant \frac{1}{2}\right\}$ after rescaling, since $\varepsilon\partial_{xx}^2 \bar{u}_\varepsilon$ does not vanish, as $\varepsilon \to 0+$ in equation (16). This is one of the novelties in the present paper.

We should emphasize the difference between impulsive differential equations with and without a *transition layer*. We address the reader to the monographs [5, 6, 7, 22, 33, 40] devoted to fairly extensive studies of impulsive differential equations without a *transition layer*. In turn, impulsive differential equations with a *transition layer* are studied poorer, although there is a number of notable works as well, see [13, 15, 16, 17, 18, 20, 27, 29, 35]. Moreover, the scaling on a transition layer is linked with the theory of boundary transition layer [19, 28, 45]. Also, it is important to mention a rather new type of equations, namely, non-instantaneous differential equations [1, 46], where a transition layer is given a priori, without a limiting passage.

Remark that the straightforward substitution of $\delta_{(t=\tau+0)} = w^\star\text{-}\lim_{\varepsilon\to 0} K_\varepsilon^\tau$ for K_ε^τ into (2a) yields the *instantaneous impulsive* pseudoparabolic equation

$$\partial_t u = \partial^3_{xxt} u + \big(1 + \alpha\delta_{(t=\tau+0)}(t)\big)\,\partial^2_{xx} u, \quad (x,t) \in \Pi_T,$$

which is equivalent in the sense of distributions to the system consisting of the homogeneous pseudoparabolic equation

$$\partial_t u = \partial^3_{xxt} u + \partial^2_{xx} u \quad \text{in } \Pi_T \setminus \{t=\tau\} \tag{4}$$

and the *impulsive condition* on the section $\{t=\tau\}$:

$$u(x,\tau+0) - (1+\alpha)\partial^2_{xx} u(x,\tau+0) = u(x,\tau-0) - \partial^2_{xx} u(x,\tau-0) \quad \text{for } x \in (0,1). \tag{5}$$

One can notice that the system (4)–(5) does not coincide with the system (15b), (17b), (17d), (17e).

Let us now give a simple example [13] demonstrating the same feature, which manifests the difference between impulsive differential equations with a *transition layer* and instantaneous ones. Consider the first-order ordinary differential equation

$$\frac{df_\varepsilon(x)}{dx} = \alpha f_\varepsilon(x)\delta_\varepsilon(x), \quad x \in \mathbb{R}, \tag{6}$$

with

$$\alpha = \text{const}, \quad \delta_\varepsilon(x) = \begin{cases} \dfrac{1}{\varepsilon} & \text{if} \quad x \in \left(-\dfrac{\varepsilon}{2}, \dfrac{\varepsilon}{2}\right), \\ 0 & \text{if} \quad |x| \geqslant \dfrac{\varepsilon}{2}, \end{cases} \qquad \varepsilon > 0.$$

It is easy to deduce that the function δ_ε weakly* approximates the Dirac delta function $\delta_{(x=0)}$ concentrated at the origin ($x=0$) and to calculate explicitly the solution of (6) for every fixed $\varepsilon > 0$:

$$f_\varepsilon(x) = C\exp\left(\alpha\int_0^x \delta_\varepsilon(y)\,dy\right) = \begin{cases} C\exp\left(-\dfrac{\alpha}{2}\right) & \text{if} \quad x \leqslant -\dfrac{\varepsilon}{2}, \\ C\exp\left(\dfrac{\alpha x}{\varepsilon}\right) & \text{if} \quad -\dfrac{\varepsilon}{2} < x < \dfrac{\varepsilon}{2}, \\ C\exp\left(\dfrac{\alpha}{2}\right) & \text{if} \quad x \geqslant \dfrac{\varepsilon}{2}. \end{cases}$$

If $C \neq 0$, one establishes that the limit function $f = \lim_{\varepsilon\to 0+} f_\varepsilon$ meets the jump condition

$$f(0+) - f(0-) = \tanh\left(\frac{\alpha}{2}\right)\big[f(0+) + f(0-)\big]. \tag{7}$$

At the same time, the solution of equation $\dfrac{df(x)}{dx} = \alpha f(x)\delta_{(x=0)}$ meets the standard impulsive condition

$$f(0+) - f(0-) = \frac{\alpha}{2}\big[f(0+) + f(0-)\big], \tag{8}$$

and we immediately notice the discrepancy between (7) and (8), since these two conditions coincide only for $\alpha = 0$, i.e., in the trivial case.

Concluding this introduction, let us remark that pseudoparabolic equations are of Sobolev type [2, 14] and are also applied for regularization of forward-backward parabolic equations [9, 30, 31, 39]. The presence of $\partial^3_{xxt} u_\varepsilon$ is essential in our study. The purely parabolic equation of the form (2a), i.e., the equation where the third order derivative $\partial^3_{xxt} u_\varepsilon$ is discarded, requires an additional research and this question lays beyond the present article. As well, an interesting direction of further research may be devoted to inclusion of a non-local term of the Fredholm type into (2a). Specifically, inserting the term

$$K^\tau_\varepsilon(t)\int_0^T K^\tau_\varepsilon(s)\partial^2_{xx}u_\varepsilon(x,s)\,ds$$

on the place of $K^\tau_\varepsilon(t)\partial^2_{xx}u_\varepsilon(x,t)$, we encounter the generalized Fredholm type integro-differential pseudoparabolic equation, which may have a significant place in the theory of non-local in time partial differential equations [41, 42, 47] and in the theory of impulsive integro-differential equations [23].

2. Well-posedness of problem (2). The first and the second energy estimates

We start our study of problem (2) by establishing the following:

Proposition 1. *Whenever $g \in W^{2,2}(0,1)\cap W^{1,2}_0(0,1)$, there is a unique strong generalized solution $u_\varepsilon = u_\varepsilon(x,t)$ to problem (2). Moreover, the solution satisfies the uniform in ε first and second energy estimates*

$$\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(0,1))} + \|\partial_x u_\varepsilon\|^2_{L^\infty(0,T;L^2(0,1))} \leqslant C\big(\|g\|^2_{L^2(0,1)} + \|\partial_x g\|^2_{L^2(0,1)}\big) \overset{\text{def}}{=} M_1,\quad \forall\,\varepsilon\in(0,\varepsilon_0], \tag{9}$$

$$\|\partial_x u_\varepsilon\|^2_{L^\infty(0,T;L^2(0,1))} + \|\partial^2_{xx} u_\varepsilon\|^2_{L^\infty(0,T;L^2(0,1))} \leqslant M_2, \tag{10}$$

and the bound

$$\|\partial_x u_\varepsilon\|_{L^\infty(\Pi_T)} \leqslant M_3, \tag{11}$$

where $C, M_1, M_2, M_3 = \mathrm{const} > 0$ do not depend on ε.

The notion of strong generalized solution is quite standard:

Definition 1. *A function $u_\varepsilon\colon \Pi_T \mapsto \mathbb{R}$ is called a* ***strong generalized solution*** *of problem (2) if*

- **(i)** $u_\varepsilon,\ \partial_t u_\varepsilon \in L^\infty(0,T;W^{2,2}(0,1)\cap W^{1,2}_0(0,1))$,
- **(ii)** u_ε *satisfies* (2a) *a.e. in* Π_T *and* (2c) *in the strong sense, namely,*

$$\|u_\varepsilon(\cdot,t) - g(\cdot)\|_{W^{2,2}(0,1)} \underset{t\to 0+}{\longrightarrow} 0.$$

The proof of Proposition 1 can be divided into four stages and is given in Sections 2.1–2.4 further.

2.1. **Existence.** Since we deal with the linear case, we apply the Fourier series method. Taking into account homogeneous conditions, a solution is sought as a Fourier series:

$$u_\varepsilon(x,t) = \sum_{n=1}^\infty c_{\varepsilon,n}(t)\varphi_n(x), \tag{12}$$

where $\varphi_n(x) = \sin(\lambda_n x)$, $\lambda_n = n\pi$. Therefore, the coefficients satisfy the set of Cauchy problems ($n\in\mathbb{N}$)

$$c'_{\varepsilon,n}(t) = \frac{-\lambda_n^2}{1+\lambda_n^2}(1+\alpha K^\tau_\varepsilon(t))c_{\varepsilon,n}(t),\quad t\in(0,T), \tag{13a}$$

$$c_{\varepsilon,n}(0) = g_n, \tag{13b}$$

where, in turn, g_n are the Fourier coefficients of initial data $g \in W^{2,2}(0,1) \cap W_0^{1,2}(0,1)$.

Now, let us apply the results from [36, 37] to (13). Namely, for every $t \in [0, +\infty)$ and $\varepsilon > 0$ we introduce the linear operator $E_\varepsilon(t)$: $L^2(0,1) \mapsto L^2(0,1)$ by the rule:

$$E_\varepsilon(t)v = \sum_{n=1}^{\infty} \exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t + \alpha \int_0^t K_\varepsilon^\tau(s)\, ds\right)\right) v_n \varphi_n(x), \quad t > 0,$$

for an arbitrary function v with $\|v\|^2_{L^2(0,1)} = \sum\limits_{n=1}^{\infty} v_n^2 < \infty$.

Operator $E_\varepsilon(t)$ depends on t as on a parameter and has the following properties.

(i) $E_\varepsilon(0) = I$,

(ii) $\|E_\varepsilon\| \leqslant \exp(|\alpha|)$.

Proof of assertion (ii). By Parseval's identity, we have

$$\begin{aligned}\|E_\varepsilon(t)v\|^2_{L^2(0,1)} &= \sum_{n=1}^{\infty} v_n^2 \exp\left(-\frac{2\lambda_n^2}{1+\lambda_n^2}\left(t + \alpha \int_0^t K_\varepsilon^\tau(s)\, ds\right)\right) \\ &\leqslant \exp(2|\alpha|) \sum_{n=1}^{\infty} v_n^2 = \exp(2|\alpha|)\|v\|^2_{L^2(0,1)},\end{aligned}$$

and the estimate follows.

(iii) $E_\varepsilon(\cdot)v \in C([0, +\infty); L^2(0,1))$ for any $v \in L^2(0,1)$.

Proof of assertion (iii). On the strength of Parseval's identity followed by simple estimation, we have

$$\begin{aligned}&\int_0^1 (E_\varepsilon(t_2)v - E_\varepsilon(t_1)v)^2\, dx \\ &\quad \leqslant \frac{1}{2}\sum_{n=1}^{\infty}\left[\exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t_2 + \alpha\int_0^{t_2} K_\varepsilon^\tau(s)\, ds\right)\right)\right. \\ &\qquad \left. - \exp\left(-\frac{\lambda_n^2}{1+\lambda_n^2}\left(t_1 + \alpha\int_0^{t_1} K_\varepsilon^\tau(s)\, ds\right)\right)\right]^2 v_n^2 \to 0, \quad \text{as } t_2 - t_1 \to 0,\end{aligned}$$

since the series in the right hand side of the inequality are uniformly continuous in $t := t_1, t_2$ with respect to n (but not with respect to ε, though).

Thus assertion (iii) follows.

(iv) For any fixed $\varepsilon > 0$, function $u_\varepsilon = E_\varepsilon(\cdot)g$ is a strong generalized solution of problem (2) in the sense of Definition 1.

The proof of assertion (iv) is analogous to the one in [36, Secs. 4, 5], where a more general linear case is treated.

2.2. **The first energy estimate uniform in ε. Uniqueness.** We multiply (2a) by the solution u_ε, integrate over $(0,1) \times (0,t)$, where $t \in (0,T)$, integrate by parts in x, taking into account (2c), and then use (2b) to get the first energy identity

$$(14) \qquad \frac{1}{2}\int_0^1 \left(|u_\varepsilon(x,t)|^2 + |\partial_x u_\varepsilon(x,t)|^2\right) dx - \frac{1}{2}\int_0^1 \left(|g(x)|^2 + |\partial_x g(x)|^2\right) dx$$

$$
\begin{aligned}
&= -\int_0^t \int_0^1 (1+\alpha K_\varepsilon^\tau(t'))\, |\partial_x u_\varepsilon(x,t')|^2 \, dxdt' \\
&\leqslant |\alpha| \int_0^t \int_0^1 K_\varepsilon^\tau(t')\, |\partial_x u_\varepsilon(x,t')|^2 \, dxdt', \quad \forall\, t \in [0,T].
\end{aligned}
$$

Applying the Grönwall–Bellman lemma to this inequality, we immediately establish estimate (9). Uniqueness of solution u_ε follows immediately from (9) due to linearity of the problem.

2.3. **The second energy estimate uniform in ε.** Since $u_\varepsilon = E_\varepsilon(t)g$, $\partial^2_{xx} g \in L^2(0,1)$ and $\partial^2_{xx} u_\varepsilon(\cdot,t) \in L^2(0,1)$, we are eligible to multiply (2a) by $-\partial^2_{xx} u_\varepsilon$, integrate over $(0,1)\times(0,t)$, where $t \in (0,T)$, integrate by parts in x in the first two terms, taking into account (2c), and then use (2b) to get *the second energy identity*

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 \left(|\partial_x u_\varepsilon(x,t)|^2 + |\partial^2_{xx} u_\varepsilon(x,t)|^2\right) dx - \frac{1}{2}\int_0^1 \left(|\partial_x g(x)|^2 + |\partial^2_{xx} g(x)|^2\right) dx \\
&\quad = -\int_0^t \int_0^1 (1+\alpha K_\varepsilon^\tau(t'))\, |\partial^2_{xx} u_\varepsilon(x,t')|^2 \, dxdt' \\
&\quad \leqslant \int_0^t \int_0^1 |\alpha| K_\varepsilon^\tau(t') |\partial^2_{xx} u_\varepsilon(x,t')|^2 \, dxdt', \quad \forall\, t \in [0,T].
\end{aligned}
$$

Applying the Grönwall–Bellman lemma to this inequality, we arrive at the second energy estimate (10).

2.4. **The bound on $\partial_x u_\varepsilon$ uniform in ε.** The bound (11) directly follows from the two energy estimates and the Newton–Leibnitz formula.

Proposition 1 is fully proved.

3. The main results

The main results of the article are the following Theorems 1 and 2. In Theorem 1 we deal with the limit $u(x,t) = \lim\limits_{\varepsilon\to 0+} u_\varepsilon$ apart from the section $\{t=\tau\}$. In Theorem 2 we formulate the initial-boundary value problem on the 'transition layer' between $t=\tau-0$ and $t=\tau+0$. Solution of this problem links $u(x,\tau-0)$ with $u(x,\tau+0)$.

Theorem 1. *The family $\{u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is relatively compact in $L^2(0,T;W_0^{1,2}(0,1))$ and relatively weakly* compact in $L^\infty(0,T;W^{2,2}(0,1))$, as $\varepsilon \to 0+$. In other terms, there exist a subsequence from $\{u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, still labeled by ε, and a limit function $u \in L^\infty(0,T;W^{2,2}(0,1)\cap W_0^{1,2}(0,1))$ such that*

(15a) $\quad u_\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} u$ *strongly in* $L^2(0,T;W_0^{1,2}(0,1))$ *and weakly* in* $L^\infty(0,T;W^{2,2}(0,1))$,

(15b) $\quad \partial_t u = \partial^3_{xxt} u + \partial^2_{xx} u$ *for* $(x,t) \in \Pi_T \setminus \{t=\tau\}$,

(15c) $\quad u(0,t) = u(1,t) = 0$ *for* $x \in (0,1)$, $t \in (0,\tau)\cup(\tau,T)$,

(15d) $\quad u(x,0) = g(x)$ *for* $x \in (0,1)$.

Equation (15b) is understood a.e. on $(0,1)\times(0,\tau)$ and a.e. on $(0,1)\times(\tau,T)$. Initial condition (15d) is understood in the sense of the strong trace, namely,

$$
\|u(\cdot,t) - g(\cdot)\|_{W^{2,2}(0,1)} \underset{t\to 0+}{\longrightarrow} 0.
$$

In Sections 4 and 5, the identities for the one-sided limits are proved:

$$u(x,\tau-0)=\lim_{\varepsilon\to0+}u_\varepsilon(x,\tau-0)=\lim_{\varepsilon\to0+}u_\varepsilon\Big(x,\tau-\frac{\varepsilon}{2}\Big),$$

$$u(x,\tau+0)=\lim_{\varepsilon\to0+}u_\varepsilon(x,\tau+0)=\lim_{\varepsilon\to0+}u_\varepsilon\Big(x,\tau+\frac{\varepsilon}{2}\Big).$$

Therefore, following the idea presented in [20, 21, 44], along with the family $\{u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, in order to link $u(x,\tau-0)$ and $u(x,\tau+0)$, we use rescaling on the transition layer $\Big[\tau-\frac{\varepsilon}{2},\tau+\frac{\varepsilon}{2}\Big]$ and deal with the rescaled solutions

$$\bar u_\varepsilon(x,\bar t)=u_\varepsilon(x,\tau+\varepsilon\bar t),$$

where $\bar u_\varepsilon\colon\Pi\mapsto\mathbb{R}$ and we denote $\Pi:=(0,1)\times\Big(-\frac{1}{2},\frac{1}{2}\Big)$.

For every fixed $\varepsilon>0$, with the help of a new variable $\bar t=\dfrac{t-\tau}{\varepsilon}\in\Big[-\frac{1}{2},\frac{1}{2}\Big]$ and multiplication by ε, from (2a) we deduce the rescaled equation

$$\partial_{\bar t}\bar u_\varepsilon=\partial^3_{xx\bar t}\bar u_\varepsilon+(\varepsilon+\alpha K(\bar t))\,\partial^2_{xx}\bar u_\varepsilon,\quad (x,\bar t)\in\Pi. \tag{16}$$

Theorem 2. *The family of rescaled solutions* $\{\bar u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ *is relatively compact in* $L^2(\Pi)$ *and relatively weakly** *compact in* $L^\infty\Big(-\frac{1}{2},\frac{1}{2};W^{2,2}(0,1)\Big)$. *In other terms, there exist a subsequence from* $\{\bar u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, *still labeled by* ε, *and a limit function* $\bar u\in L^\infty\Big(-\frac{1}{2},\frac{1}{2};W^{2,2}(0,1)\cap W^{1,2}_0(0,1)\Big)$ *such that*

$$\bar u_\varepsilon\underset{\varepsilon\to0+}{\longrightarrow}\bar u\quad\textit{strongly in } L^2(\Pi)\ \textit{and weakly}^\star\textit{ in } L^\infty\Big(-\frac{1}{2},\frac{1}{2};W^{2,2}(0,1)\Big). \tag{17a}$$

Moreover, on the transition layer we write out the pseudo-parabolic equation supplemented with the homogeneous boundary conditions and the initial condition:

$$\partial_{\bar t}\bar u=\partial^3_{xx\bar t}\bar u+\alpha K(\bar t)\partial^2_{xx}\bar u\ \textit{for } (x,\bar t)\in\Pi, \tag{17b}$$

$$\bar u(0,\bar t)=\bar u(1,\bar t)=0\ \textit{for } \bar t\in\left(-\frac{1}{2},\frac{1}{2}\right), \tag{17c}$$

$$\bar u\left(x,-\frac{1}{2}+0\right)=u(x,\tau-0)\ \textit{for } x\in(0,1). \tag{17d}$$

Finally, the matching condition for the transition layer solution $\bar u$ *on the section* $\Big\{\bar t=\frac{1}{2}-0\Big\}$ *and the* outer solution u *on the section* $\{t=\tau+0\}$ *holds true as follows:*

$$u(x,\tau+0)=\bar u\left(x,\frac{1}{2}-0\right)\ \textit{for } x\in(0,1). \tag{17e}$$

Equation (17b) is understood a.e. in Π. Initial condition (17d) and matching condition (17e) are understood in the sense of the strong traces, namely,

$$\big\|\bar u(\cdot,\bar t)-u(\cdot,\tau-0)\big\|_{W^{2,2}(0,1)}\underset{\bar t\to-\frac{1}{2}+0}{\longrightarrow}0, \tag{18}$$

$$\Big\|u(\cdot,t)-\bar u\Big(\cdot,\frac{1}{2}-0\Big)\Big\|_{W^{2,2}(0,1)}\underset{t\to\tau+0}{\longrightarrow}0. \tag{19}$$

Remark 1. *As it has already been mentioned in Sec. 1, the system* (15b)–(15d), (17b)–(17e) *is a closed limit two-scale model describing the process in which an instantaneous impulsive phenomenon manifests itself. Existence of a strong generalized solution* $(u, \bar{u})$ *to this model follows directly from Theorems 1 and 2 on the strength of the limiting passages, as* $\varepsilon \to 0+$.

Moreover, the solution $(u, \bar{u})$ *is unique. Indeed, in fact, the system* (15b)–(15d), (17b)–(17e) *is a set of the three initial-boundary value problems that should be solved successively. In the first step, we solve problem* (15b)–(15d) *on the slow time interval* $\{0 < t < \tau\}$. *In the second step, we solve problem* (17b)–(17d) *on the fast time interval* $\Big\{-\dfrac{1}{2} < \bar{t} < \dfrac{1}{2}\Big\}$, *where the initial data is taken from the first step:* $\bar{u}\Big(x, -\dfrac{1}{2}\Big) = u(x, \tau - 0)$. *Finally, we solve problem* (15b), (15c), (17e) *on the slow time interval* $\{\tau < t < T\}$, *where the initial data is taken from the second step:* $u(x, \tau + 0) = \bar{u}\Big(x, \dfrac{1}{2} - 0\Big)$. *Each of these three problems is linear and admits the first energy estimate, analogous to estimate* (14). *Therefore the solution of each of the problems is unique.*

4. Proof of Theorem 1 (beginning)

4.1. Uniform boundedness and equi-continuity of $\{u_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$.

Lemma 1. *For all* $\varepsilon \in (0, \varepsilon_0]$, *the family of strong solutions of problem* (2) *satisfies the following demands:*

- *uniform boundedness in* $L^2(0, T; W^{2,2}(0, 1))$;
- *integral equi-continuity*
$$\|\tau_h u_\varepsilon - u_\varepsilon\|_{L^2(0, T-h; W_0^{1,2}(0,1))} \leqslant Ch, \quad \forall\, h > 0 \ (h \ll 1),$$
where $\tau_h u_\varepsilon(x, t) := u_\varepsilon(x, t + h)$ *and constant* $C > 0$ *does not depend on* ε *and* h.

Proof. The uniform boundedness follows from (10). In order to prove the integral equi-continuity, we rewrite

$$\|\tau_h u_\varepsilon - u_\varepsilon\|^2_{L^2(0, T-h; W_0^{1,2}(0,1))}$$
$$= \int_0^{T-h} \big(\|u_\varepsilon(\cdot, t + h) - u_\varepsilon(\cdot, t)\|^2_{L^2(0,1)} + \|\partial_x u_\varepsilon(\cdot, t + h) - \partial_x u_\varepsilon(\cdot, t)\|^2_{L^2(0,1)}\big)\, dt.$$

Therefore,

$$\begin{aligned}
&\|u_\varepsilon(\cdot, t + h) - u_\varepsilon(\cdot, t)\|^2_{L^2(0,1)} + \|\partial_x u_\varepsilon(\cdot, t + h) - \partial_x u_\varepsilon(\cdot, t)\|^2_{L^2(0,1)} \\
&\quad = \sum_{n=1}^{\infty} (1 + \lambda_n^2)\, (c_{\varepsilon,n}(t + h) - c_{\varepsilon,n}(t))^2 \\
&\quad = \sum_{n=1}^{\infty} (1 + \lambda_n^2) \int_t^{t+h} c'_{\varepsilon,n}(s)\, ds\, (c_{\varepsilon,n}(t + h) - c_{\varepsilon,n}(t)) \\
&\quad = \sum_{n=1}^{\infty} \lambda_n^2 \int_t^{t+h} c_{\varepsilon,n}(s)(1 + \alpha K_\varepsilon^\tau(s))\, ds\, (c_{\varepsilon,n}(t + h) - c_{\varepsilon,n}(t)) \\
&\quad \leqslant 2 \int_t^{t+h} (1 + |\alpha| K_\varepsilon^\tau(s))\, ds \sum_{n=1}^{\infty} \lambda_n^2 \max_{t \in [0,T]} |c_{\varepsilon,n}(t)|^2
\end{aligned}$$

$$\leqslant 2C \int_t^{t+h} (1 + |\alpha| K_\varepsilon^\tau(s))\, ds \sum_{n=1}^{\infty} \lambda_n^2 g_n^2.$$

This provides

$$\begin{aligned}
&\|\tau_h u_\varepsilon - u_\varepsilon\|^2_{L^2(0,T-h;W_0^{1,2}(0,1))} \\
&\quad \leqslant 2C \int_0^{T-h} \left(\int_t^{t+h} (1 + |\alpha| K_\varepsilon^\tau(s))\, ds \right) dt \sum_{n=1}^{\infty} \lambda_n^2 g_n^2 \leqslant Ch,
\end{aligned}$$

since

$$\begin{aligned}
\int_0^{T-h} \left(\int_t^{t+h} K_\varepsilon^\tau(s)\, ds \right) dt &= \int_0^h K_\varepsilon^\tau(s) s\, ds + \int_h^{T-h} K_\varepsilon^\tau(s) h\, ds \\
&+ \int_{T-h}^{T} K_\varepsilon^\tau(s)(T-s)\, ds \leqslant h \int_0^T K_\varepsilon^\tau(s)\, ds = h,
\end{aligned}$$

for $0 < \tau < T$ and $h, \varepsilon \ll 1$.

Lemma 1 is proved. □

4.2. **Relative compactness of $\{u_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ in $L^2(0,T;W_0^{1,2}(0,1))$.** Lemma 1 enables to apply the Aubin–Lions–Simon lemma, see [38, Theorem 3], where $X = W^{2,2}(0,1) \cap W_0^{1,2}(0,1)$ and $B = W_0^{1,2}(0,1)$, so that X is compactly embedded in B. Due to this and Lemma 1, sequence $\{u_\varepsilon\}_{\varepsilon \to 0+}$ is relatively compact in $L^2(0,T;W_0^{1,2}(0,1))$. Bound (10) implies that $\{u_\varepsilon\}_{\varepsilon \to 0+}$ is relatively weakly* compact in $L^\infty(0,T;W^{2,2}(0,1))$. Due to these properties, there exist a subsequence from $\{u_\varepsilon\}_{\varepsilon \to 0+}$ and a limit function $u \in L^\infty(0,T;W^{2,2}(0,1) \cap W_0^{1,2}(0,1))$ satisfying the limiting relation (15a) and equation (15b) in the sense of distributions in $(0,1) \times (0,\tau)$ and in $(0,1) \times (\tau,T)$.

4.3. **Equation** (15b) **on $(0,1) \times (0,\tau)$. Initial condition** (15d)**.** Note that equation (15b) coincides with equation (2a) for $t \in \left(0, \tau - \dfrac{\varepsilon}{2}\right)$, since K_ε^τ is supported on the segment $\left\{\tau - \dfrac{\varepsilon}{2} \leqslant t \leqslant \tau + \dfrac{\varepsilon}{2}\right\}$. Therefore u coincides with u_ε for $t \in \left(0, \tau - \dfrac{\varepsilon}{2}\right)$. Hence the initial value g is attained by u in the strong trace sense, as in Definition 1 and, on $(0,1) \times (0,\tau)$, the limit function u has the same regularity properties as the solution u_ε in Definition 1. This implies that equation (15b) holds a.e. on $(0,1) \times (0,\tau)$.

In order to complete the proof of Theorem 1, it remains to show that equation (15b) holds a.e. on $(0,1) \times (\tau,T)$, as well. This is done further in Sec. 6.

5. Proof of Theorem 2

In the present section, we rescale the solution u_ε in the time variable, correspondingly, in the three domains:

- $(0,1) \times \left(0, \tau - \dfrac{\varepsilon}{2}\right)$ is mapped into $(0,1) \times (0,\tau)$ and a rescaled solution $\widehat{u}_\varepsilon$, defined by Fourier coefficients (20), satisfies the problem (21), stated further;

- $(0,1)\times\left(\tau-\frac{\varepsilon}{2},\tau+\frac{\varepsilon}{2}\right)$ is mapped into $(0,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)$ and a rescaled solution $\overline{u}_\varepsilon$, defined by the Fourier coefficients (22), satisfies problem (16), (23a), (23b);
- $(0,1)\times\left(\tau+\frac{\varepsilon}{2},T\right)$ is mapped into $(0,1)\times(\tau,T)$ and a rescaled solution $\widetilde{u}_\varepsilon$, defined by the Fourier coefficients (24), satisfies problem (25).

With the help of this rescaling, we can link $\lim\limits_{\varepsilon\to 0+} u_\varepsilon(x,\tau-0)$ with $\lim\limits_{\varepsilon\to 0+} u_\varepsilon(x,\tau+0)$ and, correspondingly, $\lim\limits_{\varepsilon\to 0+} \partial_x u_\varepsilon(x,\tau-0)$ with $\lim\limits_{\varepsilon\to 0+} \partial_x u_\varepsilon(x,\tau+0)$. In each of the three domains, we apply the original version of the Aubin–Lions lemma.

5.1. **Rescaling on $\boldsymbol{t\in\left[0,\tau-\frac{\varepsilon}{2}\right]}$.** Here we are going to rescale system (13). On $\left\{0\leqslant t\leqslant \tau-\frac{\varepsilon}{2}\right\}$ we take $\widehat{t}:=\dfrac{t\tau}{\left(\tau-\frac{\varepsilon}{2}\right)}\in[0,\tau]$ and

$$\widehat{c}_{\varepsilon,n}(\widehat{t}):=c_{\varepsilon,n}\left(\frac{\left(\tau-\frac{\varepsilon}{2}\right)}{\tau}\widehat{t}\right).$$

With $dt=\dfrac{\left(\tau-\frac{\varepsilon}{2}\right)}{\tau}\,d\widehat{t}$, system (13) is rewritten in the form:

$$\widehat{c}\,'_{\varepsilon,n}(\widehat{t})=-\frac{\left(\tau-\frac{\varepsilon}{2}\right)}{\tau}\frac{\lambda_n^2}{1+\lambda_n^2}\widehat{c}_{\varepsilon,n}(\widehat{t}),\quad \widehat{t}\in(0,\tau),\quad n\in\mathbb{N}, \tag{20a}$$

$$\widehat{c}_{\varepsilon,n}(0)=g_n. \tag{20b}$$

We introduce a function

$$\widehat{u}_\varepsilon(x,\widehat{t})=\sum_{n=1}^{\infty}\widehat{c}_{\varepsilon,n}(\widehat{t})\varphi_n(x)$$

as a solution of the rescaled equation

$$\partial_{\widehat{t}}\widehat{u}_\varepsilon=\frac{\left(\tau-\frac{\varepsilon}{2}\right)}{\tau}\partial^2_{xx}\widehat{u}_\varepsilon+\partial^3_{xx\widehat{t}}\widehat{u}_\varepsilon \tag{21a}$$

satisfying the respective initial and homogeneous boundary conditions

$$\widehat{u}_\varepsilon(x,0)=g(x), \tag{21b}$$

$$\widehat{u}_\varepsilon(0,\widehat{t})=\widehat{u}_\varepsilon(1,\widehat{t})=0. \tag{21c}$$

Lemma 2. *The following estimate holds true:*

$$\sup_{\widehat{t}\in(0,\tau)}\|\widehat{u}_\varepsilon(\cdot,\widehat{t})\|^2_{W^{2,2}(0,1)}+\sup_{\widehat{t}\in(0,\tau)}\|\partial_{\widehat{t}}\widehat{u}_\varepsilon(\cdot,\widehat{t})\|^2_{W_0^{1,2}(0,1)}\leqslant C\|g\|^2_{W^{2,2}(0,1)}.$$

Proof. The proof of this lemma is similar to the deduction of the first and the second energy estimates, see Sections 2.2 and 2.3. After rescaling, it is feasible to estimate $\partial^2_{x\widehat{t}}\widehat{u}_\varepsilon$ and $\partial^3_{xx\widehat{t}}\widehat{u}_\varepsilon$ in $L^\infty(0,\tau;L^2(0,1))$. But in the present paper, there is no need in the latter estimate. □

The family of the functions $\widehat{u}_\varepsilon$ is compact in $C([0,\tau]; W_0^{1,2}(0,1))$. This is guaranteed by Proposition 1, Lemma 2 and the original version of the Aubin–Lions lemma, since the space

$$\Big\{v \in L^\infty(0,\tau; W^{2,2}(0,1) \cap W_0^{1,2}(0,1)) : \partial_t v \in L^\infty(0,\tau; W_0^{1,2}(0,1))\Big\}$$

is compactly embedded in $C([0,\tau]; W_0^{1,2}(0,1))$.

5.2. **Rescaling on $\boldsymbol{t \in \Big[\tau - \frac{\varepsilon}{2}, \tau + \frac{\varepsilon}{2}\Big]}$.** Let $\bar{t} := \dfrac{t-\tau}{\varepsilon}$ for $t \in \Big[\tau - \dfrac{\varepsilon}{2}, \tau + \dfrac{\varepsilon}{2}\Big]$. Note that $\bar{t} \in \Big[-\dfrac{1}{2}, \dfrac{1}{2}\Big]$, $dt = \varepsilon d\bar{t}$, and $t = \tau + \varepsilon\bar{t}$. Taking into account (3), for

$$\bar{c}_{\varepsilon,n}(\bar{t}) := c_{\varepsilon,n}(\tau + \varepsilon\bar{t})$$

system (13) transforms into

$$\bar{c}'_{\varepsilon,n}(\bar{t}) = -\frac{\lambda_n^2}{1+\lambda_n^2}\big(\varepsilon + \alpha K(\bar{t})\big)\bar{c}_{\varepsilon,n}(\bar{t}), \quad \bar{t} \in \Big(-\frac{1}{2}, \frac{1}{2}\Big), \quad n \in \mathbb{N}, \tag{22a}$$

$$\bar{c}_{\varepsilon,n}\Big(-\frac{1}{2}\Big) = \widehat{c}_{\varepsilon,n}(\tau). \tag{22b}$$

As in Sec. 5.1, from (22) it follows that

$$\bar{u}_\varepsilon(x,\bar{t}) = \sum_{n=1}^{\infty} \bar{c}_{\varepsilon,n}(\bar{t})\varphi_n(x)$$

satisfies equation (16) with initial and boundary conditions

$$\bar{u}_\varepsilon\Big(x, -\frac{1}{2}\Big) = \widehat{u}_\varepsilon(x,\tau), \tag{23a}$$

$$\bar{u}_\varepsilon(0,\bar{t}) = \bar{u}_\varepsilon(1,\bar{t}) = 0. \tag{23b}$$

Lemma 3. *The following estimate holds true:*

$$\sup_{\bar{t}\in(-\frac{1}{2},\frac{1}{2})} \|\bar{u}_\varepsilon(\cdot,\bar{t})\|^2_{W^{2,2}(0,1)} + \sup_{\bar{t}\in(-\frac{1}{2},\frac{1}{2})} \|\partial_{\bar{t}}\bar{u}_\varepsilon(\cdot,\bar{t})\|^2_{W_0^{1,2}(0,1)} \leqslant \overline{C}_\alpha \|g\|^2_{W^{2,2}(0,1)}.$$

Proof. Justification of the lemma is based on Lemma 2 and is similar to the proof of the first and the second energy inequalities, see Secs. 2.2 and 2.3. □

The family of functions $\bar{u}_\varepsilon$ is compact in $C\Big(\Big[-\dfrac{1}{2}, \dfrac{1}{2}\Big]; W_0^{1,2}(0,1)\Big)$. This is guaranteed by Lemma 3 and the original version of the Aubin–Lions lemma, since the space

$$\Big\{v \in L^\infty\Big(-\frac{1}{2}, \frac{1}{2}; W^{2,2}(0,1) \cap W_0^{1,2}(0,1)\Big) : \partial_t v \in L^\infty\Big(-\frac{1}{2}, \frac{1}{2}; W_0^{1,2}(0,1)\Big)\Big\}$$

is compactly embedded in $C\Big(\Big[-\dfrac{1}{2}, \dfrac{1}{2}\Big]; W_0^{1,2}(0,1)\Big)$.

5.3. **Rescaling on $\boldsymbol{t \in \Big[\tau + \frac{\varepsilon}{2}, T\Big)}$.** For $t \in \Big[\tau + \dfrac{\varepsilon}{2}, T\Big)$ we set

$$\widetilde{t} := \frac{-\varepsilon T}{2\Big(T - \tau - \frac{\varepsilon}{2}\Big)} + \frac{(T-\tau)\,t}{\Big(T - \tau - \frac{\varepsilon}{2}\Big)} \in [\tau, T),$$

and, correspondingly,

$$\widetilde{c}_{\varepsilon,n}(\widetilde{t}) = c_{\varepsilon,n}\left(\frac{\Big(T-\tau-\dfrac{\varepsilon}{2}\Big)}{(T-\tau)}\,\widetilde{t} + \frac{-\varepsilon T}{2(T-\tau)}\right), \quad dt = \frac{(T-\tau-\frac{\varepsilon}{2})}{(T-\tau)}\,d\widetilde{t}.$$

Finally, we rewrite (13) as

$$\widetilde{c}_{\varepsilon,n}'(\widetilde{t}) = -\frac{\Big(T-\tau-\dfrac{\varepsilon}{2}\Big)}{(T-\tau)}\,\frac{\lambda_n^2}{1+\lambda_n^2}\,\widetilde{c}_{\varepsilon,n}(\widetilde{t}), \quad \widetilde{t}\in(\tau,T),\ n\in\mathbb{N}, \tag{24a}$$

$$\widetilde{c}_{\varepsilon,n}(\tau) = \overline{c}_{\varepsilon,n}\Big(\frac{1}{2}+0\Big). \tag{24b}$$

We introduce a solution

$$\widetilde{u}_\varepsilon(x,\widetilde{t}) = \sum_{n=1}^{\infty}\widetilde{c}_{\varepsilon,n}(\widetilde{t})\varphi_n(x)$$

of the rescaled problem

$$\partial_{\widetilde{t}}\widetilde{u}_\varepsilon = \frac{\Big(T-\tau-\dfrac{\varepsilon}{2}\Big)}{(T-\tau)}\,\partial^2_{xx}\widetilde{u}_\varepsilon + \partial^3_{xx\widetilde{t}}\widetilde{u}_\varepsilon, \tag{25a}$$

$$\widetilde{u}_\varepsilon(x,\tau) = \bar{u}_\varepsilon\Big(x,\frac{1}{2}+0\Big), \tag{25b}$$

$$\widetilde{u}_\varepsilon(0,\widetilde{t}) = \widetilde{u}_\varepsilon(1,\widetilde{t}) = 0. \tag{25c}$$

Lemma 4. *The following estimate holds true:*

$$\sup_{\widetilde{t}\in(\tau,T)}\|\widetilde{u}_\varepsilon(\cdot,\widetilde{t})\|^2_{W^{2,2}(0,1)} + \sup_{\widetilde{t}\in(\tau,T)}\|\partial_{\widetilde{t}}\widetilde{u}_\varepsilon(\cdot,\widetilde{t})\|^2_{W_0^{1,2}(0,1)} \leqslant \widetilde{C}_\alpha\|g\|^2_{W_0^{2,2}(0,1)}.$$

Proof. Justification of the lemma is based on Lemma 2 and is similar to the proof of the first and the second energy inequalities, see Secs. 2.2 and 2.3. □

The family of the functions $\widetilde{u}_\varepsilon$ is compact in $C([\tau,T];W_0^{1,2}(0,1))$. This is guaranteed by Lemma 4 and the original version of the Aubin–Lions lemma, since the space

$$\Big\{v\in L^\infty(\tau,T;W_0^{2,2}(0,1)) :\ \partial_t v\in L^\infty(\tau,T;W_0^{1,2}(0,1))\Big\}$$

is compactly embedded in $C([\tau,T];W_0^{1,2}(0,1))$.

5.4. **Matching conditions.** Summarizing Secs. 5.1–5.3, we state that the sequences $\{\widehat{u}_\varepsilon\}_{\varepsilon\to0+}$, $\{\bar{u}_\varepsilon\}_{\varepsilon\to0+}$, and $\{\widetilde{u}_\varepsilon\}_{\varepsilon\to0+}$ are compact in the following spaces, respectively:

$$C([0,\tau];W_0^{1,2}(0,1)), \quad C\Big(\Big[-\frac{1}{2},\frac{1}{2}\Big];W_0^{1,2}(0,1)\Big), \quad \text{and } C([\tau,T];W_0^{1,2}(0,1)).$$

Moreover, we conclude that the limit of the sequence $\{\widehat{u}_\varepsilon\}_{\varepsilon\to0+}$ satisfies the system (15b)–(15d) on $(0,1)\times(0,\tau)$ in the strong sense (see Remark 1 and Sec. 4.3) and the limit of the sequence $\{\widetilde{u}_\varepsilon\}_{\varepsilon\to0+}$ satisfies the system (15b)–(15c) on $(0,1)\times(\tau,T)$ and the limit of the sequence $\{\bar{u}_\varepsilon\}_{\varepsilon\to0+}$ satisfies the system (17b)–(17c) on Π, at least, in the weak sense (in the sense of distributions) so far. From initial conditions (23a) and (25b), the matching conditions

$$\lim_{\varepsilon\to 0+} u_\varepsilon(x,\tau-0) = \lim_{\varepsilon\to 0+} \widehat{u}_\varepsilon(x,\tau-0) = \lim_{\varepsilon\to 0+} \bar{u}_\varepsilon\Big(x,-\frac{1}{2}+0\Big)$$

and

$$\lim_{\varepsilon\to 0+} \bar{u}_\varepsilon\Big(x,\frac{1}{2}-0\Big) = \lim_{\varepsilon\to 0+} \widetilde{u}_\varepsilon(x,\tau+0) = \lim_{\varepsilon\to 0+} u_\varepsilon(x,\tau+0)$$

follow, i.e., equalities (17d) and (17e) hold true.

5.5. **Equation** (17b) **on Π.** Problem (17b)–(17d) (for $\bar{u}$) has the form similar to the formulation of problem (2) and

$$\bar{u}\Big(\cdot,-\frac{1}{2}+0\Big) = u(\cdot,\tau-0) \in W^{2,2}(0,1)\cap W_0^{1,2}(0,1).$$

Therefore $\bar{u}$ is unique and meets the same regularity requirements, as the solution u_ε of (2) in Definition 1. In particular, equation (17b) holds a.e. on Π and initial condition (17d) holds in the sense of the limiting relation (18). Theorem 2 is fully proved.

6. Proof of Theorem 1 (completion)

The initial-boundary value problem for equation (15b) on $(0,1)\times(\tau,T)$ supplemented with the boundary conditions (15c) and the initial conditions (17e) has the form similar to the formulation of problem (2) and

$$u(\cdot,\tau+0) = \bar{u}\Big(\cdot,\frac{1}{2}-0\Big) \in W^{2,2}(0,1)\cap W_0^{1,2}(0,1),$$

due to the arguments in Sec. 5.5. Therefore u is unique and meets the same regularity requirements, as the solution u_ε of (2) in Definition 1. In particular, equation (15b) holds a.e. on $(0,1)\times(0,\tau)$ and initial condition (17e) holds in the sense of the limiting relation (19).

Proof of Theorem 1 is complete.

7. Remark on the strong convergence

In the end of the article, let us make an additional remark on strong convergence leading to some generalization of the established results.

Remark 2. *Note that the strong convergence properties for* $\{u_\varepsilon\}_{\varepsilon\to 0+}$ *and* $\{\bar{u}_\varepsilon\}_{\varepsilon\to 0+}$ *(see the limiting relations* (15a) *and* (17a)*) are redundant for the derivation of the limit model* (15b)–(15d), (17b)–(17e)*, since the original model* (2) *is linear and therefore the mere properties of weak convergence would have been sufficient for the limiting transitions in* (2)*, in fact.*

On the other hand, presence of strong convergence manifests that problem (2) *can be regarded to as a 'good' approximation of the limit problem* (15b)–(15d), (17b)–(17e)*, in a sense. Besides, as an example, precisely following the arguments in the article, we can naturally generalize the research onto the case of the semilinear equation*

$$\partial_t u_\varepsilon = \partial^3_{xxt} u_\varepsilon + (1+\alpha K^\tau_\varepsilon(t))\,\partial^2_{xx} u_\varepsilon - f(u_\varepsilon),$$

where f is a smooth given sublinear function.

Acknowledgements

The authors are very grateful to Professor Stanislav N. Antontsev (Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia) for many fruitful discussions and to the anonymous referee for valuable remarks and recommendations that helped improve the earlier version of the paper.

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