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LOGARITHMIC ASYMPTOTICS OF THE NUMBER OF CENTRAL VERTICES OF ALMOST ALL n-VERTEX GRAPHS OF DIAMETER k

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Abstract. The asymptotic behavior of the number of central vertices and F. Buckley's central ratio $\mathbb{R}_c(G) = |\mathbb{C}(G)|/|V(G)|$ for almost all n-vertex graphs G of fixed diameter k is investigated.

The logarithmic asymptotics of the number of central vertices for almost all such n-vertex graphs is established: 0 or $\log_2 n$ (1 or $\log_2 n$), respectively, for arising here subclasses of graphs of the even (odd) diameter.

It is proved that for almost all n-vertex graphs of diameter k, $\mathbb{R}_c(G) = 1$ for $k = 1, 2$, and $\mathbb{R}_c(G) = 1 - 2/n$ for graphs of diameter $k = 3$, while for $k \geq 4$ the value of the central ratio $\mathbb{R}_c(G)$ is bounded by the interval $(\frac{\Delta}{6} + r_1(n), 1 - \frac{\Delta}{6} - r_1(n))$ except no more than one value (two values) outside the interval for even diameter k (for odd diameter k) depending on k. Here $\Delta \in (0, 1)$ is arbitrary predetermined constant and $r_1(n), r_2(n)$ are positive infinitesimal functions.

Keywords: graph, diameter, radius, central vertices, number of central vertices, central ratio, center, spectrum of center, typical graphs, almost all graphs.

Introduction

We study finite labeled ordinary graphs. For a connected graph G, the *distance* $\rho_G(u, v)$ between its vertices $u, v \in V(G)$ is defined as the length of the shortest path connecting these vertices. In this case, $e_G(v) = \max_{u \in V} \rho_G(v, u)$ is the

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eccentricity of the vertex v of the graph G, $d(G) = \max_{v\in V} e_G(v)$ is the *diameter* of the graph G, and $r(G) = \min_{v\in V} e_G(v)$ is the *radius* of the graph G. A vertex is called *central* if its eccentricity is equal to the radius of the graph. The *graph center* $\mathbb{C}(G)$ is the set of all central vertices of the graph G.

The concept of the graph center is related to its numerous practical problems arising in varied fields and it is stipulated by the measurement of proximity centrality in the analysis of various kinds of networks and connections. Often in graphs corresponding to communication networks, the diameter is interpreted as the time of data transfer or the path length between the most distant nodes, the radius is as the reachability time from the most distant nodes to the central node, serving as the main distribution center. In this case, the presence of several such centers is allowed (see, for example, [17]). Finding of the graph center turns out to be useful for problems of optimal placement of publicly important institutions and enterprises (hospitals, fire stations, post offices and other emergency points), when it is required to minimize the farthest distances to these institutions [3]. So, the location of the hospital at the central vertices of the graph that arises here reduces the maximum distance that ambulances have to cover. The concept of centrality is also applied in social sciences and is actively used in the analysis of social networks [18], for example, when the most influential persons of considered network are identified. In biology, it is relevant when building models of the spread of diseases, in chemistry when analyzing molecular bonds, etc.

A number of classical results about the graph center are well known [1–4, 14, 16]. So, the realizability of an arbitrary graph as a subgraph generated by the center of a suitable graph is established. Namely, it is proved that for any graph H there exists a connected graph G such that its subgraph generated by the center $\mathbb{C}(G)$ is isomorphic to H. This fact was established by G.N. Kopylov and E.A. Timofeev [16], its simple justification was also given by S.T. Hedetniemi (see [1]). F. Buckley investigated the so-called *central ratio* $\mathbb{R}_c(G) = |\mathbb{C}(G)|/|V(G)|$ of connected graph G, for which it is obvious that the inequality $0 < \mathbb{R}_c(G) \leq 1$ holds. For any rational q in the interval $(0, 1]$, he proved the existence of a graph G such that $\mathbb{R}_c(G) = q$ [2].

In [9] the *center spectrum* $\mathbb{S}p_c(\mathcal{K})$ of an arbitrary class of graphs $\mathcal{K}$ is defined as the set of cardinalities of graphs centers from this class $\mathcal{K}$. The center spectrum of n-vertex graphs of radius r is established by J. Hu and S. Zhan [15]. The center spectrum of all and almost all n-vertex connected graphs is found in [9]. In this article [9], the author obtained a number of structural results on the center and spectrum of its cardinalities for almost all n-vertex graphs of fixed diameter k. For $k = 1, 2$ any vertex is central, while for $k \geq 3$ two types of central vertices are identified, which are necessary and sufficient to obtain the centers of almost all such graphs. While studying the possible center spectrum of almost all n-vertex graphs of fixed diameter k, an unexpected result was obtained in [9]. It turned out that the center of almost all such graphs has cardinality n for $k = 1, 2$, and $n - 2$ for $k = 3$, while for $k \geq 4$ in the distribution of values of center cardinality for the probability space of n-vertex graphs of diameter k (for almost all graphs) there is no unique value of the center cardinality (although the radius is uniquely defined [8]). Moreover, there are jumps of such cardinality values depending on the diameter value. Namely, the center spectrum is bounded by an interval of consecutive integers and additionally contains at most one value (two values) outside this interval for an even diameter k (for an odd diameter k) depending on the value of k (for more details see Theorem 2 in preliminary information). Note that the boundaries of the interval depend on a predetermined arbitrary integer p and shrink when choosing a greater value p. This leads to the need to consider the case when the parameter p depends on n.

Using this approach, in this paper we asymptotically study the behavior of the number of central vertices and the central ratio for almost all n-vertex graphs of fixed diameter k.

The necessary preliminary information is contained in Section 1. There general properties of classes of typical graphs are also given (Proposition 1). In this section it also defined a family of embedded classes $\mathcal{F}_{n,k,p}$, $p \geq 1$ (p is an integer constant independent of n) of typical n-vertex graphs of fixed diameter $k \geq 3$, which have a number of metric properties and were constructed by the author in [8].

Section 2 is of a technical nature, the purpose of which is to obtain estimates for some expressions with classical binomial coefficients (Lemma 3 and Corollary 2). Wherein properties of binomial coefficients $\binom{r}{\alpha}$ of real arguments r, $\alpha \in \mathbb{R}$, previously proved by the author in [10], are used. Namely, the analogue of the unimodality property for classical binomial coefficients (Theorem 3) and the asymptotics of such generalized binomial coefficients of a special form (Proposition 2).

In Section 3 we consider a more general case of the class of n-vertex graphs $\mathcal{F}_{n,k,p}$ of diameter $k \geq 3$, when $p = p(n)$ is a function depending on n and taking positive integer values. For selected class of functions $p(n)$ it is established asymptotically exact value of the number $|\mathcal{F}_{n,k,p(n)}|$ (Theorem 4) and it is proved that $\mathcal{F}_{n,k,p(n)}$ is the class of typical n-vertex graphs of diameter k (Corollary 3). In particular, this implies that for almost all n-vertex graphs of diameter k, neighbourhoods of any two vertices that do not belong to a fixed diametral path contain at least $p(n)$ common vertices (Corollary 4).

In Section 4, for almost all n-vertex graphs of diameter k, we find lower and upper estimates for the number of central vertices, which depend on n and refine the previously obtained interval boundaries from the center spectrum of such graphs (Theorem 6).

As a corollary, the asymptotics of the number $\log_2 |\mathbb{C}(G)|$ is established for almost all n-vertex graphs G of a fixed diameter. It is proved that the logarithmic asymptotics of the number of central vertices is 0 or $\log_2 n$ (1 or $\log_2 n$) for the corresponding subclasses of graphs of the even (odd) diameter (see Corollary 5).

Also, as a corollary, estimates of the central ratio $\mathbb{R}_c(G)$ are found for almost all n-vertex graphs G of diameter k. It is proved that for almost all such graphs, $\mathbb{R}_c(G) = 1$ for $k = 1, 2$, and $\mathbb{R}_c(G) = 1 - 2/n$ for graphs of diameter $k = 3$, while for $k \geq 4$ the value of the central ratio $\mathbb{R}_c(G)$ is bounded by the interval $(\frac{\Delta}{6} + r_1(n), 1 - \frac{\Delta}{6} - r_1(n))$ except no more than one value (two values) outside the interval for even diameter k (for odd diameter k) depending on k. Here $\Delta \in (0, 1)$ is arbitrary predetermined constant and $r_i(n) = o(1)$ are positive functions, $i = 1, 2$ (see Corollary 6 for more details).

All obtained typical properties of the center of n-vertex graphs of fixed diameter $k \geq 2$ remain typical for connected graphs of diameter at least k, as well as for graphs (not necessarily connected) containing a shortest path of length at least k (see, in particular, Corollary 7).

1. Preliminary information

The article uses the generally accepted concepts and notations of graph theory [4,14], as well as the standard concepts of combinatorial analysis [13] and the theory of functions of real variable [11]. We consider only finite ordinary (i.e., without loops and multiple edges) graphs $G = (V, E)$ with set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Denote by $B_i^G(v) = \{u \in V \mid \rho_G(v, u) \le i\}$ a *ball of radius i centered at a vertex* $v \in V$ in the metric space of the graph G with the metric ρ_G, $S_i^G(v) = \{u \in V \mid \rho_G(v, u) = i\}$ is a *sphere of radius i centered at a vertex* $v \in V$, $\lfloor x \rfloor$ is the largest integer less or equal to real nonnegative number x, $\big[\big[\, x,\, y \,\big]\big]$ is the integer interval $[x,\, y] \cap \mathbb{Z}$ between the real numbers $x, y \in \mathbb{R}$, $(n)_k = n(n-1)\cdots(n-k+1)$, and wherein we define $(n)_k = 0$ for $n < k$ and $(n)_0 = (0)_0 = 1$. A vertex of degree 1 is called *pendant*, sphere $S_1^G(v)$ of radius 1 centered at v is called the *neighbourhood of the vertex* v, a shortest path of length $d(G)$ is the *diametral path* of the graph G, and under by a *pair of diametral vertices* we mean an unordered sample of two vertices from the set V, the distance between which is equal to the diameter.

To estimate the measure of the number of graphs with a certain property, the concept of *almost all* is often used; in this approach, the studied property is considered for graphs with a large number of vertices. Let $\mathcal{J}_n$ be the class of labeled n-vertex graphs with the fixed set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Consider some property $\mathcal{P}$, by which each graph may or may not possess. Through $\mathcal{J}_n^{\mathcal{P}}$ denote the set of all graphs from $\mathcal{J}_n$ that possess the property $\mathcal{P}$. *Almost all graphs possess the property* $\mathcal{P}$ if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 1$, i.e. $|\mathcal{J}_n^{\mathcal{P}}| \sim |\mathcal{J}_n|$, and *there are almost no graphs with the property* $\mathcal{P}$, if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 0$. In a similar way, we can talk about the property $\mathcal{P}$, which is possessed by almost all n-vertex graphs of the class $\mathcal{K}$ under study, or when there are almost no graphs in the class $\mathcal{K}$ that have property $\mathcal{P}$.

In the study and selection of almost all graphs in the class of graphs under consideration it is often useful to define not characteristic properties themselves for the notion of almost all, but directly select a subclass of typical graphs itself (in [5,6] a more general concept of a class of typical combinatorial objects and an abstract typical combinatorial object for a given class of objects admitting the concept of dimension is formulated). Further we will also use this formal concept for graphs (when the dimension of a graph is understood as the number of its vertices). Let Ω be an arbitrary class of graphs such that $\Omega_n \neq \varnothing$ for all large enough n, where $\Omega_n = \Omega \cap \mathcal{J}_n$. A subclass $\Omega^* \subseteq \Omega$ is the *class of typical graphs of the class* Ω if $\lim_{n\to\infty} \frac{|\Omega_n^*|}{|\Omega_n|} = 1$. A property of graphs of the class under consideration is *typical* if almost all graphs of this class have this property. Let us formulate simple properties of classes of typical graphs related to set-theoretic operations.

Proposition 1 (properties of typical graphs)**.** *Let $\mathcal{X}$, $\mathcal{Y}$ be subclasses of a graph class $\mathcal{K}$. Then*

(i) *if $\mathcal{X}$ is the class of typical graphs of class $\mathcal{K}$ and $\mathcal{X} \subseteq \mathcal{Y}$, then $\mathcal{Y}$ is also the class of typical graphs of class $\mathcal{K}$;*

(ii) *if $\mathcal{X}$ is the class of typical graphs of class $\mathcal{K}$ and $\mathcal{Y} \subseteq \mathcal{X}$, in addition, $|\mathcal{Y}_n| = o(|\mathcal{K}_n|)$ as $n \to \infty$, then $\mathcal{X} \setminus \mathcal{Y}$ is also the class of typical graphs of class $\mathcal{K}$;*

(iii) *if $\mathcal{X}$ is the class of typical graphs of class $\mathcal{K}$ and $\mathcal{X} \subseteq \mathcal{Y}$, then $\mathcal{X}$ is the class of typical graphs of class $\mathcal{Y}$;*

(iv) *if $\mathcal{X}$, $\mathcal{Y}$ are the classes of typical graphs of class $\mathcal{K}$, then $\mathcal{X} \cap \mathcal{Y}$ is also the class of typical graphs of class $\mathcal{K}$, moreover, subclasses $\mathcal{X} \setminus \mathcal{Y}$ and $\mathcal{Y} \setminus \mathcal{X}$ have asymptotically zero fraction in $\mathcal{K}$, i.e. $|\mathcal{X}_n \setminus \mathcal{Y}_n| = o(|\mathcal{K}_n|)$ and $|\mathcal{Y}_n \setminus \mathcal{X}_n| = o(|\mathcal{K}_n|)$ as $n \to \infty$;*

(v) *if $\mathcal{X}$, $\mathcal{X}'$ are disjoint classes of typical graphs of classes $\mathcal{K}$ and $\mathcal{K}'$ respectively, then $\mathcal{X} \cup \mathcal{X}'$ is the class of typical graphs of class $\mathcal{K} \cup \mathcal{K}'$;*

(vi) *if $\mathcal{X} \cup \mathcal{Y}$ is the class of typical graphs of class $\mathcal{K}$ and $\mathcal{X} \cap \mathcal{Y} = \varnothing$, in addition, subclass $\mathcal{X}$ has an asymptotic fraction in $\mathcal{K}$ not equal to 1, then $\mathcal{Y}$ is the class of typical graphs of class $\mathcal{K} \setminus \mathcal{X}$.*

Proof. Statements (i)-(iii) follow directly from the definitions, statement (iv) is obtained from property (ii) and equality $\mathcal{X} \cap \mathcal{Y} = \mathcal{X} \setminus (\mathcal{X} \setminus \mathcal{Y})$.

Prove assertion (v). Since $\frac{|\mathcal{K}_n|}{|\mathcal{K}_n|+|\mathcal{K}'_n|}$ is a bounded sequence, as $n \to \infty$ we obtain

$$\frac{|\mathcal{K}_n|}{|\mathcal{K}_n| + |\mathcal{K}'_n|}\Big(\frac{|\mathcal{X}_n|}{|\mathcal{K}_n|} - \frac{|\mathcal{X}'_n|}{|\mathcal{K}'_n|}\Big) = o(1).$$

It remains to note that $|\mathcal{X}_n \cup \mathcal{X}'_n| = |\mathcal{X}_n| + |\mathcal{X}'_n|$ and as $n \to \infty$

$$\frac{|\mathcal{X}_n \cup \mathcal{X}'_n|}{|\mathcal{K}_n \cup \mathcal{K}'_n|} \geq \frac{|\mathcal{X}_n| + |\mathcal{X}'_n|}{|\mathcal{K}_n| + |\mathcal{K}'_n|} = \frac{|\mathcal{K}_n|}{|\mathcal{K}_n| + |\mathcal{K}'_n|}\Big(\frac{|\mathcal{X}_n|}{|\mathcal{K}_n|} - \frac{|\mathcal{X}'_n|}{|\mathcal{K}'_n|}\Big) + \frac{|\mathcal{X}'_n|}{|\mathcal{K}'_n|} \longrightarrow 1.$$

Prove statement (vi). Since $\mathcal{X} \cup \mathcal{Y}$ is the class of typical graphs of class $\mathcal{K}$, we have $|\mathcal{K}_n \setminus (\mathcal{X}_n \cup \mathcal{Y}_n)| = o(|\mathcal{K}_n|)$. By virtue of the condition on the fraction of the subclass $\mathcal{X}$, there is the following limit

$$\lim_{n\to\infty}(1 - \frac{|\mathcal{X}_n|}{|\mathcal{K}_n|}) \neq 0.$$

Besides, $\mathcal{Y}_n = \mathcal{K}_n \setminus (\mathcal{X}_n \cup \mathcal{K}_n \setminus (\mathcal{X}_n \cup \mathcal{Y}_n))$ and $|\mathcal{K}_n \setminus \mathcal{X}_n| = |\mathcal{K}_n| - |\mathcal{X}_n|$. Thus, as $n \to \infty$ we conclude

$$\frac{|\mathcal{Y}_n|}{|\mathcal{K}_n \setminus \mathcal{X}_n|} = \Big(1 - \frac{|\mathcal{X}_n|}{|\mathcal{K}_n|} - \frac{|\mathcal{K}_n \setminus (\mathcal{X}_n \cup \mathcal{Y}_n)|}{|\mathcal{K}_n|}\Big)\Big(1 - \frac{|\mathcal{X}_n|}{|\mathcal{K}_n|}\Big)^{-1} \longrightarrow 1.$$

□

Note that the condition on the fraction of the subclass $\mathcal{X}$ in assertion (vi) of Proposition 1 is essential. For example, let $\mathcal{K}_n = \mathcal{X}_n \cup \{H_n, G_n\}$, $\lim_{n\to\infty} |\mathcal{X}_n| = \infty$ and $\mathcal{Y}_n = \{G_n\}$, where $H_n, G_n \in \mathcal{J}_n \setminus \mathcal{X}_n$ are different graphs. Then $|\mathcal{K}_n| \sim |\mathcal{X}_n \cup \mathcal{Y}_n|$, i.e. $\mathcal{X} \cup \mathcal{Y}$ is the class of typical graphs of class $\mathcal{K}$. Meanwhile, it is obvious that

$$\lim_{n\to\infty} \frac{|\mathcal{Y}_n|}{|\mathcal{K}_n \setminus \mathcal{X}_n|} = \frac{1}{2}.$$

Therefore $\mathcal{Y}$ is not the class of typical graphs of class $\mathcal{K} \setminus \mathcal{X}$.

Let $\mathcal{J}_{n,\,d=k}$, $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}^*_{n,\,d\geq k}$ be the following classes of labeled n-vertex graphs: graphs of diameter k; connected graphs of diameter at least k and graphs (not necessarily connected) having a shortest path of length at least k, respectively. Obviously, the following inclusions hold $\mathcal{J}_{n,\,d=k} \subseteq \mathcal{J}_{n,\,d\geq k} \subseteq \mathcal{J}^*_{n,\,d\geq k}$. In [8] for every $k \geq 3$ a family of nested classes $\mathcal{F}_{n,k,p}$, $p \geq 1$ of typical n-vertex graphs of fixed diameter k is constructed, which are also typical for classes $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}^*_{n,\,d\geq k}$. Give the definition of the class of graphs $\mathcal{F}_{n,k,p}$. To do this, we first consider special graphs of diameter 3 and their properties. Let $x, y \in V$ and $\mathcal{F}_{n,3,p}(x, y)$ be the class of all graphs $F \in \mathcal{J}_n$, having the following properties:

a) vertices x, y are not pendant in F;

b) $\rho_F(z,x) = \rho_F(z,y) = 2$ for some vertex $z \in V$;

c) $d(F) = 3$, the graph F has the unique pair of diametral vertices x, y and does not contain coinciding balls of radius 1 with centers in different vertices;

d) the following properties of spheres hold:

$$|S_1^F(u) \cap S_1^F(v)| \geq p \ \ \forall u, v \in V \setminus \{x,y\} \text{ and } u \neq v,$$
$$|S_1^F(u) \cap S_1^F(v)| \geq p \ \ \forall v \in V \setminus \{x,y\} \ \ \forall u \in \{x,y\}.$$

Now, we define graphs of the class $\mathcal{F}_{n,k,p}$ as follows. Let $u = (u_0, u_1, \ldots, u_{k-2})$ be an arbitrary ordered sequence of different vertices from the set V. Fix a pair of neighboring elements u_s and u_{s+1}, $0 \leq s \leq k-3$. On the set $V \setminus \{u_0, \ldots, u_{s-1}, u_{s+2}, \ldots, u_{k-2}\}$ of $n-k+3$ vertices, define an arbitrary graph F from the class $\mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1})$. Finally, join by edges the vertices u_i, u_{i+1} for $i \neq s$ and $0 \leq i < k-2$. Denote the resulting graph by $G(u,s,F)$. Let $\mathcal{F}_{n,k,p}$ be the class of all constructed graphs $G(u,s,F)$ under the condition $0 \leq s \leq \lfloor \frac{k-3}{2} \rfloor$. Further, we need the following estimates of the number of graphs in this class.

Lemma 1 [8]. *Let $k \geq 3$, $p \geq 1$ and $x, y \in V$ be different vertices. Then*

$$|\mathcal{F}_{n,k,p}| = \frac{1}{2}(k-2)(n)_{k-1}\, |\mathcal{F}_{n-k+3,3,p}(x,y)|.$$

The following theorem contains estimates that give the asymptotically exact value $2^{\binom{n}{2}}\, \xi_{n,k}$ of the number of graphs in each of classes $\mathcal{F}_{n,k,p}$, $\mathcal{J}_{n,\,d=k}$, $\mathcal{J}_{n,\,d\geq k}$ and $\mathcal{J}^*_{n,\,d\geq k}$ for any fixed $k \geq 3$ and $p \geq 1$ [8].

Theorem 1 [8]. *Let $k \geq 3$, $0 < \varepsilon < 1$ and $p \geq 1$ be independent of n. Then there exists a constant $c > 0$ independent of n such that for every $n \in \mathbb{N}$ the following inequalities hold*

$$2^{\binom{n}{2}}\xi_{n,k}\Big(1 - c\Big(\tfrac{5+\varepsilon}{6}\Big)^{n-k+1}\Big) \leq |\mathcal{F}_{n,k,p}| \leq |\mathcal{J}_{n,\,d=k}|$$
$$\leq |\mathcal{J}_{n,\,d\geq k}| \leq |\mathcal{J}^*_{n,\,d\geq k}| \leq 2^{\binom{n}{2}}\xi_{n,k}\Big(1 + c\Big(\tfrac{5+\varepsilon}{6}\Big)^{n-k+1}\Big),$$

$$\textit{where } \xi_{n,k} = q_k\ (n)_{k-1}\left(\frac{3}{2^{k-1}}\right)^{n-k+1}, \quad q_k = \frac{1}{2}\,(k-2)\,2^{-\binom{k-1}{2}}.$$

In [9] the center spectrum $\mathbb{S}p_c(\mathcal{J}_{n,\,d=k})$ is investigated asymptotically, the following statement is proved.

Theorem 2 [9]. *Let $k \geq 1$ and $p \geq 1$ be fixed integer constants. Then*

(i) $|\mathbb{C}(G)| = n$ *for almost all n-vertex graphs G of diameter $k = 1, 2$;*

(ii) $|\mathbb{C}(G)| = n - 2$ *for almost all n-vertex graphs G of diameter* 3;

(iii) $|\mathbb{C}(G)| \in \big[\big[\,1+p,\ n-5-p\big]\big]$ *for almost all n-vertex graphs G of diameter* 4;

(iv) $|\mathbb{C}(G)| \in \big[\big[\,2+p,\ n-5-p\big]\big] \cup \{n-4\}$ *for almost all n-vertex graphs G of diameter* 5; *moreover, the fraction of such graphs with $(n-4)$-vertex center is asymptotically equal to $\frac{1}{3}$;*

(v) $|\mathbb{C}(G)| \in \{1\} \cup \big[\big[\,1+p,\ n-k-1-p\big]\big]$ *for almost all n-vertex graphs G of even fixed diameter $k \geq 6$; moreover, the fraction of such graphs with trivial center is asymptotically equal to $\frac{k-4}{k-2}$;*

(vi) $|\mathbb{C}(G)| \in \{2\} \cup \big[\big[\,2+p,\ n-k-p\big]\big] \cup \{n-k+1\}$ *for almost all n-vertex graphs G of odd fixed diameter $k \geq 7$; moreover, the fraction of such graphs with 2-vertex and $(n-k+1)$-vertex center is asymptotically equal to $\frac{k-5}{k-2}$ and $\frac{1}{k-2}$ respectively.*

Introduce some asymptotic notations. For numerical functions $f(n)$, $g(n) : \mathbb{N} \to \mathbb{R}$ we write $f(n) \lesssim g(n)$ (respectively $f(n) \gtrsim g(n)$) if there exists $N \in \mathbb{N}$ such that for every $n \geq N$ the inequality $f(n) \leq g(n)$ $(f(n) \geq g(n))$ holds. To denote the *asymptotic equality* of the numerical functions $f(n)$ and $g(n)$ as $n \to \infty$, we use the notation $f(n) \sim g(n)$, which by definition means that $f(n) = g(n)(1 + r(n))$ for all large enough n, where $r(n) = o(1)$, or, equivalently, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ (for functions positive in some neighborhood of ∞). Let Ω be the class of graphs under consideration and an numerical characteristic $\mathcal{X} : \Omega \to \mathbb{R}$ assigns to each graph $G \in \Omega$ some real number $\mathcal{X}(G) \in \mathbb{R}$. For almost all n-vertex graphs G of class Ω, the numerical characteristic $\mathcal{X}(G)$ *is asymptotically equal to* the numerical function $f(n)$ (symbolically $\mathcal{X}(G) \sim f(n)$), if for some functions f_1, $f_2 : \mathbb{N} \to \mathbb{R}$ almost all n-vertex graphs G of class Ω satisfy the relation $f_1(n) \leq \mathcal{X}(G) \leq f_2(n)$, wherein the numerical functions $f_1(n)$, $f_2(n)$, $f(n)$ asymptotically coincide as $n \to \infty$.

2. Estimates of some expressions with binomial coefficients

In this section, we obtain estimates for some expressions with usual binomial coefficients. Wherein, we need the properties of binomial coefficients $\binom{r}{\alpha}$ of real arguments $r, \alpha \in \mathbb{R}$, proved by the author in [10]. Recall that the binomial coefficient of real arguments (as a generalization of the classical binomial coefficient $\binom{n}{m}$ for non-negative integers n and m) is defined as follows (see, for example, [12]):

$$\binom{r}{\alpha} = \frac{\Gamma(1+r)}{\Gamma(1+\alpha)\Gamma(1+r-\alpha)}, \tag{1}$$

here $\Gamma(\alpha)$ is the Euler gamma function, which can be defined in the *Euler-Gauss form* (see, for example, [11]) as

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{(n-1)!\, n^{\alpha}}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}, \ \alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\}.$$

Note that if $r \in (-1, +\infty)$ and $\alpha \in (-1, r+1)$, the binomial coefficient $\binom{r}{\alpha}$ is defined by (1) correctly.

In [10] a number of properties of real binomial coefficients is established, in particular, the following statements are proved.

Theorem 3 [10]**.** *Let $r \in (-1, +\infty)$ and $\alpha \in (-1, r+1)$. Then $\binom{r}{\alpha} > 0$ and binomial coefficient $\phi(\alpha) = \binom{r}{\alpha}$ is strictly increasing on the interval $(-1, \frac{r}{2}]$ and strictly decreasing on the interval $[\frac{r}{2}, r+1)$.*

Proposition 2 [10]**.** *Let r takes real values, $\alpha \in \mathbb{R}$ does not depend on r and $0 < \alpha < 1$. Then the following asymptotic equality is valid as r tends to infinity*

$$\binom{r}{r\alpha} \sim \sqrt{\frac{1}{2\pi\alpha(1-\alpha)r}} \left(\frac{1}{\alpha}\right)^{\alpha r} \left(\frac{1}{1-\alpha}\right)^{(1-\alpha)r}. \tag{2}$$

Corollary 1 [10]**.** *Let r takes non-negative integer values, $\alpha \in \mathbb{R}$ does not depend on r and $0 < \alpha < 1$. Then the asymptotic equality* (2) *is valid as r tends to infinity.*

Recall that the sequence of numbers $a_1, a_2, \dots, a_n$ is called *unimodal* if there exists s such that $a_1 < a_2 < \dots < a_s \geq a_{s+1} > a_{s+2} > \dots > a_n$ (see, for example, [19]).

Lemma 2. *Let $n \geq q \geq 1$. Then the sequence $\binom{n}{s}q^{-s}$, $s = 0, 1, \ldots, n$ is unimodal, moreover, for $s = \lfloor \frac{n+1}{q+1} \rfloor$ its largest value is reached and such s is the largest.*

Proof. Introduce sequences $\alpha_s = \binom{n}{s}q^{-s}$, $0 \leq s \leq n$, and $\beta_s = \frac{\alpha_{s+1}}{\alpha_s}$ if $0 \leq s \leq n-1$. It is directly established that

$$\beta_s = \frac{n-s}{(s+1)q}\,,$$

$$\frac{\beta_{s+1}}{\beta_s} = \frac{n-(s+1)}{n-s} \cdot \frac{s+1}{s+2} < 1 \text{ if } 0 \leq s \leq n-2,$$

i.e. β_s is a strictly decreasing sequence. Moreover, $\beta_0 \geq 1$ due to the condition $n \geq q$. Therefore, for some s^* the sequence α_s strictly increases for $0 \leq s \leq s^*$, $\alpha_{s^*} \leq \alpha_{s^*+1}$ and then strictly decreases for $s^*+1 \leq s \leq n$. Note that s^* is the largest s satisfying the inequality $\beta_s \geq 1$. Hence $s^* = \lfloor \frac{n-q}{q+1} \rfloor$ and $s^*+1 = \lfloor \frac{n+1}{q+1} \rfloor$. $\square$

Lemma 3. *Let $p(n) = \lfloor \frac{n}{q+1}\Delta \rfloor + 1$, where $q \geq 1$ and $0 < \Delta < 1$ do not depend on n. Then there is ε_Δ such that ε_Δ does not depend on n, $0 < \varepsilon_\Delta < 1$ and the following relation is fulfilled as $n \to \infty$*

$$\Big(\frac{q}{q+1}\Big)^n \sum_{s=0}^{p(n)-1} \binom{n}{s} q^{-s} = \varepsilon_\Delta^n\, O(\sqrt{n}).$$

Proof. By Proposition 2 we obtain

$$\binom{n}{\frac{n}{q+1}\Delta} \sim \sqrt{\frac{(q+1)^2}{2\pi\Delta(q+1-\Delta)n}}\,(q+1)^n \Delta^{-\frac{\Delta}{q+1}n}\,(q+1-\Delta)^{-\frac{q+1-\Delta}{q+1}n}. \qquad (3)$$

Therefore

$$\binom{n}{\frac{n}{q+1}\Delta} = (q+1)^n \Delta^{-\frac{\Delta}{q+1}n}\,(q+1-\Delta)^{-\frac{q+1-\Delta}{q+1}n}\, O\Big(\frac{1}{\sqrt{n}}\Big).$$

Hence,

$$n\Big(\frac{q}{q+1}\Big)^n \binom{n}{\frac{n}{q+1}\Delta} q^{-\frac{n}{q+1}\Delta} = \varepsilon_\Delta^n\, O(\sqrt{n}), \text{ where} \qquad (4)$$

$$\varepsilon_\Delta = q\,\Delta^{-\frac{\Delta}{q+1}}\,(q+1-\Delta)^{-\frac{q+1-\Delta}{q+1}}\; q^{-\frac{\Delta}{q+1}}. \qquad (5)$$

Since $0 < \Delta < 1$ and $q \geq 1$, we have $\varepsilon_\Delta > 0$. Consider the functions

$$f(x) = x^{\frac{x}{q+1}}\,(q+1-x)^{\frac{q+1-x}{q+1}}\; q^{\frac{x}{q+1}} \text{ and } g(x) = \ln f(x)$$

on the interval $(0, q+1)$. Let us study intervals of monotonicity of the function $f(x)$. It is directly calculated

$$g'(x) = \frac{1}{q+1} \ln \frac{qx}{q+1-x}\,.$$

Note that $g'(x) < 0$ on $(0, 1)$ and $g'(1) = 0$. Therefore, the function $g(x)$ strictly decreases on $(0, 1]$. The function $f(x)$ behaves similarly on the interval $(0, 1]$. Therefore, the relation $f(\Delta) > f(1) = q$ is valid. Now, using (5), we obtain $\varepsilon_\Delta = \frac{q}{f(\Delta)} < 1$.

Further, due to the inequality $p(n) - 1 \leq \frac{n}{q+1}\Delta \leq \frac{n}{2}$ by Theorem 3 we have

$$\binom{n}{p(n)-1} \leq \binom{n}{\frac{n}{q+1}\Delta}. \qquad (6)$$

Now, using the relations $p(n)-1 \le \lfloor \frac{n+1}{q+1} \rfloor$, $\lfloor x \rfloor \ge x-1$, (4), (6) and Lemma 2, as $n \to \infty$ we conclude

$$\begin{aligned}
\Big(\frac{q}{q+1}\Big)^n \sum_{s=0}^{p(n)-1} \binom{n}{s} q^{-s} &\le p(n)\Big(\frac{q}{q+1}\Big)^n \binom{n}{p(n)-1} q^{-(p(n)-1)} \\
&= \Big(\frac{q}{q+1}\Big)^n \binom{n}{\frac{n}{q+1}\Delta} q^{-\frac{n}{q+1}\Delta}\, O(n) \\
&= \varepsilon_\Delta^n\, O(\sqrt{n}).
\end{aligned}$$

□

Remark 1. *The statement of Lemma* 3 *is not valid for* $\Delta = 1$.

Indeed, let $\Delta = 1$. First consider the case $q > 1$. Using Theorem 3 and the inequality $\frac{n}{q+1} \le p(n) \lesssim \frac{n}{2}$, we obtain

$$\begin{aligned}
\binom{n}{p(n)-1} &= \binom{n}{p(n)} \frac{p(n)}{n-p(n)+1} \\
&= \binom{n}{\frac{n}{q+1}} \frac{\lfloor \frac{n}{q+1} \rfloor + 1}{n - \lfloor \frac{n}{q+1} \rfloor}\, \Omega(1) \qquad (7) \\
&= \binom{n}{\frac{n}{q+1}} \frac{\Omega(n)}{nq+q+1}\,.
\end{aligned}$$

From the asymptotic equality (3) (which is also valid for $\Delta = 1$) we have

$$\binom{n}{\frac{n}{q+1}} = \frac{(q+1)^n}{\sqrt{n}}\; q^{-\frac{q}{q+1}n}\, \Omega(1). \qquad (8)$$

Now, from (7),(8) it is easy to see that

$$\begin{aligned}
\Big(\frac{q}{q+1}\Big)^n \sum_{s=0}^{p(n)-1} \binom{n}{s} q^{-s} &= \Big(\frac{q}{q+1}\Big)^n \binom{n}{p(n)-1} q^{-(p(n)-1)}\, \Omega(1) \\
&= \Big(\frac{q}{q+1}\Big)^n \frac{(q+1)^n}{\sqrt{n}}\; q^{-\frac{q}{q+1}n}\, q^{-\frac{n}{q+1}} \frac{\Omega(n)}{nq+q+1} \\
&= \frac{\Omega(\sqrt{n})}{nq+q+1}.
\end{aligned}$$

Therefore, if the assertion of Lemma 3 holds, then $(nq+q+1)\,\varepsilon_\Delta^n = \Omega(1)$ and we get a contradiction.

Now consider the case $q = 1$. It is known that as $n \to \infty$

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \sim \frac{2^n}{\sqrt{\pi n/2}}$$

(see, for example, [19]). Therefore

$$2^{-n} \sum_{s=0}^{p(n)-1} \binom{n}{s} = 2^{-n} \binom{n}{\lfloor \frac{n}{2} \rfloor}\, \Omega(1) = \frac{\Omega(1)}{\sqrt{n}}$$

and similarly, as in the case of $q > 1$, we obtain a contradiction.

Corollary 2. *Let $p(n) = \lfloor \frac{n}{q+1}\Delta \rfloor + 1$, where is an integer $q \geq 1$ and $0 < \Delta < 1$ do not depend on n. Then there is ε_Δ such that $0 < \varepsilon_\Delta < 1$, ε_Δ does not depend on n and as $n \to \infty$*

$$\sum_{i=1}^{q}\Big(\frac{i}{i+1}\Big)^n \sum_{s=0}^{p(n)-1}\binom{n}{s} i^{-s} = \Big(\frac{q+\varepsilon_\Delta}{q+1}\Big)^n O(\sqrt{n}).$$

Proof. Since $p(n) \leq \lfloor \frac{n}{i+1}\Delta \rfloor + 1$ for every $i \leq q$, by Lemma 3 there is ε such that $0 < \varepsilon < 1$, ε does not depend on n and as $n \to \infty$

$$\sum_{i=1}^{q}\Big(\frac{i}{i+1}\Big)^n \sum_{s=0}^{p(n)-1}\binom{n}{s} i^{-s} \leq \sum_{i=1}^{q}\Big(\frac{i}{i+1}\Big)^n \sum_{s=0}^{\lfloor \frac{n}{i+1}\Delta \rfloor}\binom{n}{s} i^{-s} = \varepsilon^n O(\sqrt{n}).$$

We can assume that $\frac{q}{q+1} < \varepsilon < 1$. Remains to note that $\varepsilon_\Delta = \varepsilon(q+1) - q$ will be required. $\square$

3. Class of graphs $\mathcal{F}_{n,k,p(n)}$

This section deals with the more general case of the graphs class $\mathcal{F}_{n,k,p}$ (constructed for every $k \geq 3$ and $p \geq 1$ in [8]) when $p = p(n)$ is a function depending on n and taking positive integer values. Let us estimate the number of graphs of the class $\mathcal{F}_{n,k,p(n)}$ for a class of functions $p(n)$ distinguished in this section. Further, we will use the estimates obtained in [5, 8] for the number of graphs of the following subclasses of labeled n-vertex graphs. Let x, y, u, v be different elements of V, $p \geq 1$, $0 \leq s < p$ and

$$a_n = |\mathcal{A}_n(x,y)|, \text{ where } \mathcal{A}_n(x,y) = \{G \in \mathcal{J}_n \mid B_1^G(x) \cap B_1^G(y) = \varnothing\},$$
$$\mathcal{B}_n(x,y,u,v;s) = \{G \in \mathcal{J}_n \mid B_1^G(x) \cap B_1^G(y) = \varnothing \text{ and } |S_1^G(u) \cap S_1^G(v)| = s\},$$
$$\mathcal{C}_n(x,y,u;s) = \{G \in \mathcal{J}_n \mid B_1^G(x) \cap B_1^G(y) = \varnothing \text{ and } |S_1^G(x) \cap S_1^G(u)| = s\},$$
$$\beta_{n,p} = |\mathcal{B}_{n,p}(x,y,u,v)|, \text{ where } \mathcal{B}_{n,p}(x,y,u,v) = \bigcup_{0 \leq s < p} \mathcal{B}_n(x,y,u,v;s),$$
$$\gamma_{n,p} = |\mathcal{C}_{n,p}(x,y,u)|, \text{ where } \mathcal{C}_{n,p}(x,y,u) = \bigcup_{0 \leq s < p} \mathcal{C}_n(x,y,u;s).$$

Lemma 4 [5,8]**.** *Let x, y, u, v be different elements in V and $p \geq 1$. Then for every $n \geq p$ the following relations hold*

(i) $a_n = 2^{\binom{n}{2}}\, \frac{8}{9} \left(\frac{3}{4}\right)^n$;

(ii) $\beta_{n,p} \leq a_n b_p(n) \Big(\frac{3}{4}\Big)^n$, *where* $b_p(n) = 128 \sum_{s=0}^{p-1} \binom{n}{s}\, 3^{-(3+s)}$;

(iii) $\gamma_{n,p} \leq a_n c_p(n) \Big(\frac{5}{6}\Big)^n$, *where* $c_p(n) = 72 \sum_{s=0}^{p-1} \binom{n}{s}\, 5^{-(1+s)}$.

Lemma 5 [8]**.** *Let x, y be different vertices in V and $p \geq 1$, $\lambda > 0$, $0 < \varepsilon < 1$ be arbitrary constants independent of n. Then*

$$|\mathcal{F}_{n,3,p}(x,y)| \gtrsim a_n \Big(1 - \lambda \Big(\frac{5+\varepsilon}{6}\Big)^{n-2}\Big).$$

The next lemma extends Lemma 5 on a lower estimate of the number of graphs of class $\mathcal{F}_{n,3,p}(x,y)$, where p is a fixed integer, to the case $p = p(n)$ for the considered class of functions.

Lemma 6. *Let x, y be different vertices in V, $p(n) = \lfloor \frac{n}{6}\Delta \rfloor + 1$ and $0 < \Delta < 1$, $\lambda > 0$ are arbitrary constants independent of n. Then there exists ε_Δ such that $0 < \varepsilon_\Delta < 1$, ε_Δ does not depend on n, and for every ε satisfying the inequality $\varepsilon_\Delta < \varepsilon < 1$, the following relation is fulfilled*

$$|\mathcal{F}_{n,3,p(n)}(x, y)| \gtrsim a_n\Big(1 - \lambda \Big(\frac{5+\varepsilon}{6}\Big)^{n-2}\Big).$$

Proof. Using Lemma 4, Corollary 2 and the inequality $n \gtrsim p(n)$, we obtain the existence of ε_Δ such that $0 < \varepsilon_\Delta < 1$, ε_Δ does not depend on n, and for every ε satisfying the inequality $\varepsilon_\Delta < \varepsilon < 1$, as $n \to \infty$ we have

$$\begin{aligned} n^2\beta_{n,p(n)} + 2n\gamma_{n,p(n)} &= a_n O(n^2)\Big(\Big(\frac{3}{4}\Big)^n b_{p(n)}(n) + \Big(\frac{5}{6}\Big)^n c_{p(n)}(n)\Big) \\ &= a_n O(n^2) \sum_{i=1}^{5} \Big(\frac{i}{i+1}\Big)^n \sum_{s=0}^{p(n)-1} \binom{n}{s} i^{-s} \\ &= a_n \Big(\frac{5+\varepsilon_\Delta}{6}\Big)^n O(n^{\frac{5}{2}}) \\ &\lesssim a_n \frac{\lambda}{2} \Big(\frac{5+\varepsilon}{6}\Big)^{n-2}. \end{aligned}$$

It is not difficult to see that for every n the following inclusion holds (see the proof of Lemma 4 and relation (2) in [8] similarly):

$$\mathcal{F}_{n,3,1}(x, y) \setminus (\mathcal{B}_{n,p(n)}(x, y) \cup \mathcal{C}_{n,p(n)}(x, y) \cup \mathcal{C}_{n,p(n)}(y, x)) \subseteq \mathcal{F}_{n,3,p(n)}(x, y), \text{ where}$$

$$\mathcal{B}_{n,p(n)}(x, y) = \bigcup_{\substack{u,v \in V \setminus \{x,y\} \\ u \neq v}} \mathcal{B}_{n,p(n)}(x, y, u, v),$$

$$\mathcal{C}_{n,p(n)}(x, y) = \bigcup_{u \in V \setminus \{x,y\}} \mathcal{C}_{n,p(n)}(x, y, u).$$

Thus, reckoning Lemma 5, conclude

$$\begin{aligned} |\mathcal{F}_{n,3,p(n)}| &\gtrsim |\mathcal{F}_{n,3,1}(x, y)| - n^2|\mathcal{B}_{n,p(n)}(x, y, u, v)| - 2n|\mathcal{C}_{n,p(n)}(x, y, u)| \\ &\gtrsim |\mathcal{F}_{n,3,1}(x, y)| - a_n \frac{\lambda}{2} \Big(\frac{5+\varepsilon}{6}\Big)^{n-2} \\ &\gtrsim a_n\Big(1 - \frac{\lambda}{2} \Big(\frac{5+\varepsilon}{6}\Big)^{n-2}\Big) - a_n \frac{\lambda}{2} \Big(\frac{5+\varepsilon}{6}\Big)^{n-2}. \end{aligned}$$

□

Remark 2. *In Lemma 6, the value ε_Δ depends only on Δ.*

Further, for an arbitrary Δ, $0 < \Delta < 1$, the notation ε_Δ will be used for the constant found in Lemma 6.

Lemma 7 (lower estimate of $|\mathcal{F}_{n,k,p(n)}|$)**.** *Let $k \geq 3$, $0 < \Delta < 1$, $\varepsilon_\Delta < \varepsilon < 1$, $p(n) = \left\lfloor \frac{n}{6}\Delta \right\rfloor + 1$ and k, Δ, ε do not depend on n. Then there exists a constant $c > 0$ independent of n and k such that for every $n \in \mathbb{N}$ the following inequality holds*

$$|\mathcal{F}_{n,k,p(n)}| \geq 2^{\binom{n}{2}} \xi_{n,k}\Big(1 - c\Big(\frac{5+\varepsilon}{6}\Big)^{n-k+1}\Big).$$

Proof. Similarly, taking into account Lemmas 1, 4(i) and Lemma 6 for $\lambda = 1$, to the proof of the lower estimate $|\mathcal{F}_{n,k,p}|$ for fixed p (see Lemma 8 in [**?**]). □

Theorem 4 (asymptotics of $|\mathcal{F}_{n,k,p(n)}|$). *Let $k \geq 3$, $0 < \Delta < 1$, $\varepsilon_\Delta < \varepsilon < 1$, $p(n) = \left\lfloor \frac{n}{6}\Delta \right\rfloor + 1$ and k, Δ, ε do not depend on n. Then there exists a constant $c > 0$ independent of n such that for every $n \in \mathbb{N}$ the following inequalities are valid*

$$2^{\binom{n}{2}}\xi_{n,k}\Big(1 - c\Big(\tfrac{5+\varepsilon}{6}\Big)^{n-k+1}\Big) \leq |\mathcal{F}_{n,k,p(n)}| \leq |\mathcal{J}_{n,\, d=k}|$$
$$\leq |\mathcal{J}_{n,\, d\geq k}| \leq |\mathcal{J}^*_{n,\, d\geq k}| \leq 2^{\binom{n}{2}}\xi_{n,k}\Big(1 + c\Big(\tfrac{5+\varepsilon}{6}\Big)^{n-k+1}\Big).$$

Proof. It follows directly from Lemma 7, Theorem 1 and the inclusion $\mathcal{F}_{n,k,p(n)} \subseteq \mathcal{F}_{n,k,1}$. □

Corollary 3. *Let $k \geq 3$, $0 < \Delta < 1$ do not depend on n and $p(n) = \left\lfloor \frac{n}{6}\Delta \right\rfloor + 1$. Then as $n \to \infty$*

$$|\mathcal{F}_{n,k,p(n)}| \sim |\mathcal{J}_{n,\, d=k}| \sim |\mathcal{J}_{n,\, d\geq k}| \sim |\mathcal{J}^*_{n,\, d\geq k}| \sim 2^{\binom{n}{2}}\xi_{n,k}.$$

The following statement follows directly from the definition of the class $\mathcal{F}_{n,k,p(n)}$ and Corollary 3.

Corollary 4. *Let $k \geq 3$, $0 < \Delta < 1$ be independent of n. Then almost all n-vertex graphs of diameter k contain a diametral path such that neighbourhoods of any two vertices from the set outside this path have at least $\left\lfloor \frac{n}{6}\Delta \right\rfloor + 1$ common vertices.*

4. The spectrum of the center of almost all graphs from $\mathcal{J}_{n,\, d=k}$

Let us establish estimates for the number of central vertices of almost all n-vertex graphs G of fixed diameter k, which depend on n and give the logarithmic asymptotics of this number $|\mathbb{C}(G)|$. First, turn to the center spectrum of the class $\mathcal{F}_{n,k,p(n)}$.

Theorem 5 (spectrum $\mathbb{S}p_c(\mathcal{F}_{n,k,p(n)})$). *Let $k \geq 3$ and $p = p(n)$ be a function depending on n and taking positive integer values. Then for every $n \geq 2p(n) + k + 4$ the following equalities hold*

(i) $\mathbb{S}p_c(\mathcal{F}_{n,k=3,p(n)}) = \{n-2\}$;

(ii) $\mathbb{S}p_c(\mathcal{F}_{n,k=4,p(n)}) = [[\, 1 + p(n),\, n - 5 - p(n)\,]]$;

(iii) $\mathbb{S}p_c(\mathcal{F}_{n,k=5,p(n)}) = [[\, 2 + p(n),\, n - 5 - p(n)\,]] \cup \{n-4\}$;

(iv) $\mathbb{S}p_c(\mathcal{F}_{n,k,p(n)}) = \{1\} \cup [[\, 1 + p(n),\, n - k - 1 - p(n)\,]]$ *for even* $k \geq 6$;

(v) $\mathbb{S}p_c(\mathcal{F}_{n,k,p(n)}) = \{2\} \cup [[\, 2 + p(n),\, n - k - p(n)\,]] \cup \{n-k+1\}$ *for odd* $k \geq 7$.

Proof. It follows from the description of the center spectrum $\mathbb{S}p_c(\mathcal{F}_{n,k,p})$ for fixed integer values $k \geq 3$ and $p \geq 1$ (Theorem 5 [9]). □

Theorem 6 (spectrum of the center of almost all graphs from $\mathcal{J}_{n,\, d=k}$). *Let $k \geq 1$ and $0 < \Delta < 1$ be independent of n. Then*

(i) $|\mathbb{C}(G)| = n$ *for almost all n-vertex graphs G of diameter $k = 1, 2$;*

(ii) $|\mathbb{C}(G)| = n - 2$ *for almost all n-vertex graphs G of diameter* 3;

(iii) $|\mathbb{C}(G)| \in [[\, 2 + \lfloor \frac{n}{6}\Delta \rfloor,\, n - \lfloor \frac{n}{6}\Delta \rfloor - 6\,]]$ *for almost all n-vertex graphs G of diameter* 4;

(iv) $|\mathbb{C}(G)| \in [[\, 3 + \lfloor \frac{n}{6}\Delta \rfloor,\, n - \lfloor \frac{n}{6}\Delta \rfloor - 6\,]] \cup \{n-4\}$ *for almost all n-vertex graphs G of diameter* 5; *moreover, the fraction of such graphs with $(n-4)$-vertex center asymptotically equals $\frac{1}{3}$;*

(v) $|\mathbb{C}(G)| \in \{1\} \cup \big[\big[\, 2 + \lfloor \frac{n}{6}\Delta \rfloor,\, n - \lfloor \frac{n}{6}\Delta \rfloor - k - 2 \,\big]\big]$ *for almost all n-vertex graphs G of even fixed diameter $k \geq 6$; moreover, the fraction of such graphs with trivial center asymptotically equals $\frac{k-4}{k-2}$;*

(vi) $|\mathbb{C}(G)| \in \{2\} \cup \big[\big[\, 3 + \lfloor \frac{n}{6}\Delta \rfloor,\, n - \lfloor \frac{n}{6}\Delta \rfloor - k - 1 \,\big]\big] \cup \{n-k+1\}$ *for almost all n-vertex graphs G of odd fixed diameter $k \geq 7$; moreover, the fraction of such graphs with 2-vertex and $(n-k+1)$-vertex center asymptotically equals $\frac{k-5}{k-2}$ and $\frac{1}{k-2}$ respectively.*

Proof. The cases $k = 1, 2, 3$ are considered in Theorem 2. Let now $k \geq 4$ and $p(n) = \left\lfloor \frac{n}{6}\Delta \right\rfloor + 1$. Then $n \gtrsim 2p(n) + k + 4$.

Consider, for example, the case $k = 4$. By Theorem 5, for all large enough n we have $\mathbb{S}p_c(\mathcal{F}_{n,k=4,p(n)}) = \big[\big[\, 1 + p(n),\, n - 5 - p(n) \,\big]\big]$. Therefore, taking into account Corollary 3, as $n \to \infty$ we obtain

$$\frac{|\{G \in \mathcal{J}_{n,\, d=4} \mid |\mathbb{C}(G)| \in \big[\big[\, 1 + p(n),\, n - 5 - p(n) \,\big]\big]\}|}{|\mathcal{J}_{n,\, d=4}|} \geq \frac{|\mathcal{F}_{n,k=4,p(n)}|}{|\mathcal{J}_{n,\, d=4}|} \longrightarrow 1.$$

Similarly, from Corollary 3 and Theorem 5 we obtain the possible cardinalities of the center of almost all n-vertex graphs of fixed diameter $k \geq 5$ for all other indicated cases (iv)-(vi). Asymptotic value of the corresponding fractions of the graphs is established in Theorem 2. □

Theorem 6 implies a number of properties of the centers of almost all graphs of fixed diameter k. For example, there are almost no graphs of diameter $k = 2, 4$ and odd diameter k with a trivial center, while for any even $k \geq 6$ this is not true. Similarly, there are almost no graphs with a 2-vertex center of diameter $k = 1, 3, 5$ and even diameter k, however, for every odd $k \geq 7$ this does not hold. Unexpected is the jump of the center cardinality outside the interval of consecutive integer values both from above from $n - \lfloor \frac{n}{6}\Delta \rfloor - k - 1$ to $n - k + 1$ for odd $k \geq 5$, and from below from $2 + \lfloor \frac{n}{6}\Delta \rfloor$ to 1 for even $k \geq 6$ and from $3 + \lfloor \frac{n}{6}\Delta \rfloor$ to 2 for odd $k \geq 7$.

Theorem 6 also allows to find the logarithmic asymptotics of the number of central vertices of almost all n-vertex graphs of fixed diameter.

Corollary 5 (logarithmic asymptotics)**.** *Let $k \geq 1$ be an integer. The following asymptotic equality is valid*

$$\log_2 |\mathbb{C}(G)| \sim \log_2(n)$$

for almost all n-vertex graphs G of each of the following classes: graphs of fixed diameter $k \leq 5$; graphs of even fixed diameter $k \geq 6$ with nontrivial center; graphs of odd fixed diameter $k \geq 7$ whose the center cardinality is not equal to 2.

Proof. Let $k \geq 6$ and $0 < \Delta < 1$ be a fixed constant. By statements (v),(vi) of Theorem 6 and Proposition 1(v) for almost all n-vertex graphs G of even fixed diameter $k \geq 6$ with a nontrivial center we have $|\mathbb{C}(G)| \in \big[\big[\, 2 + \lfloor \frac{n}{6}\Delta \rfloor,\, n - \lfloor \frac{n}{6}\Delta \rfloor - k - 2 \,\big]\big]$, and for almost all n-vertex graphs G of odd fixed diameter $k \geq 7$ whose the center cardinality is not equal to 2, we obtain $|\mathbb{C}(G)| \in \big[\big[\, 3 + \lfloor \frac{n}{6}\Delta \rfloor,\, n - \lfloor \frac{n}{6}\Delta \rfloor - k - 1 \,\big]\big] \cup \{n - k + 1\}$. It remains to note that if $P(n)$ is a polynomial over $\mathbb{R}$ of non-zero degree m with positive leading coefficient, then $\log_2 P(n) \sim m \log_2 n$ as $n \to \infty$.

The case $k \leq 5$ is considered similarly using statements (i)-(iv) of Theorem 6. □

Remark 3. *The fraction of n-vertex graphs of fixed even diameter $k \geq 6$ with trivial center (of odd diameter $k \geq 7$ with 2-vertex center) is asymptotically equal to $\frac{k-4}{k-2}$ ($\frac{k-5}{k-2}$). Moreover, typical graphs of these classes are constructed in* [9].

Thus, for almost all n-vertex graphs of a fixed diameter, the logarithmic asymptotics of the number of central vertices is 0 or $\log_2 n$ (1 or $\log_2 n$) for the corresponding subclasses of graphs of the even (odd) diameter.

Theorem 6 also implies estimates of the central ratio $\mathbb{R}_c(G)$ for almost all n-vertex graphs G of fixed diameter k.

Corollary 6 (estimates of $\mathbb{R}_c(G)$). *Let $k \geq 1$ and $0 < \Delta < 1$ do not depend on n. Then*

(i) $\mathbb{R}_c(G) = 1$ *for almost all n-vertex graphs G of diameter $k = 1, 2$;*

(ii) $\mathbb{R}_c(G) = 1 - 2/n$ *for almost all n-vertex graphs G of diameter* 3;

(iii) $\frac{\Delta}{6} + r_1(n) \leq \mathbb{R}_c(G) \leq 1 - \frac{\Delta}{6} - r_2(n)$ *for almost all n-vertex graphs G of each of the following classes: graphs of diameter 4; graphs of diameter 5 whose center cardinality is not equal to $n - 4$; graphs of even fixed diameter $k \geq 6$ with nontrivial center; graphs of odd fixed diameter $k \geq 7$ whose center cardinality is not equal to 2 and $n - k + 1$. Here $r_1(n)$, $r_1(n)$ are positive functions and $r_1(n) = o(n)$, $r_2(n) = o(n)$ as $n \to \infty$.*

Proof. Similar to the proof of Corollary 5 using Theorem 6, Proposition 1 and the inequality $\lfloor x \rfloor \geq x - 1$. $\square$

Remark 4. *The fraction of n-vertex graphs of fixed even diameter $k \geq 6$ with trivial center (of odd diameter $k \geq 5$ with 2-vertex and $(n - k + 1)$-vertex center) asymptotically equals $\frac{k-4}{k-2}$ ($\frac{k-5}{k-2}$ and $\frac{1}{k-2}$ respectively). Moreover, typical graphs of these classes are constructed in* [9].

Note that from Corollary 6, by virtue of Proposition 1(v), we also find possible values of the central ratio $\mathbb{R}_c(G)$ for almost all graphs G of the whole class $\mathcal{J}_{n,\, d=k}$.

The obtained properties of the centers are also valid, by virtue of Corollary 3, for graphs of classes $\mathcal{J}_{n,\, d \geq k}$ and $\mathcal{J}^*_{n,\, d \geq k}$.

Corollary 7. *For every fixed $k \geq 2$, almost all n-vertex graphs of each of the following classes $\mathcal{J}_{n,\, d \geq k}$, $\mathcal{J}^*_{n,\, d \geq k}$ are connected, have diameter k, and the properties stated in Theorem 6 and its corollaries hold for the center.*

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