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ON RECOGNITION OF $A_6 \times A_6$ BY THE SET OF CONJUGACY CLASS SIZES

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Abstract. For a finite group G denote by $N(G)$ the set of conjugacy class sizes of G. Recently the following question has been asked: Is it true that for each nonabelian finite simple group S and each $n \in \mathbb{N}$, if the set of class sizes of a finite group G with trivial center is the same as the set of class sizes of the direct power S^n, then $G \simeq S^n$? In this paper we approach an answer to this question by proving that $A_6 \times A_6$ is uniquely determined by $N(A_6 \times A_6)$ among finite groups with trivial center.

Keywords: finite groups, conjugacy classes, class sizes.

1. Introduction

For a finite group G denote by $N(G)$ the set of conjugacy class sizes of G. In 1980s J.G. Thompson posed the following conjecture: if L is a nonabelian finite simple group, G is a finite group with $Z(G) = 1$ and $N(G) = N(L)$, then $G \simeq L$. Later A.S. Kondrat'ev added this conjecture to the *Kourovka Notebook* [6, Question 12.38]. In a series of papers of different authors, it took more than twenty years to confirm the conjecture. The final step was done in [4], where a full historical overview of the proof can be found.

There are many ways to generalize Thompson's conjecture, in this paper we consider one of them.

Question. [6, Question 20.29] *Let S be a nonabelian finite simple group. Is it true that for any $n \in \mathbb{N}$, if the set of class sizes of a finite group G with $Z(G) = 1$ is the same as the set of class sizes of the direct power S^n, then $G \simeq S^n$?*

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There are no examples when the answer to this question is negative and the only studied case with $n > 1$ is when $S = A_5$ and $n = 2$ [5]. We continue studying this question by proving the following theorem.

Theorem. *If G is a group with $Z(G) = 1$ and $N(G) = N(A_6 \times A_6)$, then $G \simeq A_6 \times A_6$.*

2. Preliminaries

Let n, m be integers and p a prime. By n_p we denote the maximal power of p dividing n. The least common multiple and the greatest common divisor of n and m are denoted by $\mathrm{lcm}(n, m)$ and $\gcd(n, m)$ respectively.

By $\pi(G)$ we denote the set of all prime divisors of the order of G. Given an element x of a group G, denote by x^G the set $\{x^g \mid g \in G\}$, that is, the conjugacy class contaning x, and by $|x^G|$ its size.

Lemma 2.1. *$N(A_6) = \{1, 40, 45, 72, 90\}$, and $n \in N(A_6 \times A_6)$ if and only if $n = a \cdot b$, where $a, b \in N(A_6)$.*

Proof. The first statement can be easily verified. The second statement follows from the well-known fact that $N(G \times H) = N(G) \cdot N(H) = \{n \mid n = a \cdot b,\ a \in N(G), b \in N(H)\}$. □

Lemma 2.2. [3, Theorem 5.2.3] *Let A be a p'-group of automorphisms of an abelian p-group P. Then $P = C_P(A) \times [P, A]$.*

In the following two lemmas we list well-known facts on sizes of conjugacy classes in finite groups (see, e.g. [5, Lemma 1]).

Lemma 2.3. *Suppose that G is a finite group, $K \trianglelefteq G$, $x \in G$, and $\overline{x} \in \overline{G} = G/K$ is an image of x in G/K. Then $|x^K|$ and $|\overline{x}^{\overline{G}}|$ divide $|x^G|$.*

Lemma 2.4. *Let G be a finite group and $x, y \in G$, $xy = yx$, and $\gcd(|x|, |y|) = 1$. Then $C_G(xy) = C_G(x) \cap C_G(y)$, $|x^G| \cdot |y^G| \geq |(xy)^G|$ and $\mathrm{lcm}(|x^G|, |y^G|)$ divides $|(xy)^G|$.*

Lemma 2.5. *If $G = H \wr \langle g \rangle$ is the wreath product of a group H and involution g, then $|g^G| = |H|$.*

Proof. If $B = H \times H$ is the base of the wreath product, then $C_B(g) = \{(h, h) \mid h \in H\}$, so the statement follows. □

Lemma 2.6. *Let $x \in G$ with $p^n = |x^G|_p < |G|_p$, where $n \geq 1$. Then there is a p-element $y \in C_G(x)$ with $|y^G|_p \leq p^{n-1}$. If, in addition, x is a p'-element, then $|(xy)^G|_p = |x^G|_p$.*

Proof. Take $P \in Syl_p(G)$ such that $C_G(x) \cap P = \tilde{P} \in Syl_p(C_G(x))$, so $|P : \tilde{P}| = p^n$. Put $N = N_P(\tilde{P})$. There is a nontrivial element $y \in Z(N) \cap \tilde{P}$. Since $|P : N| \leq p^{n-1}$, it follows that $|y^G|_p \leq p^{n-1}$.

By the above paragraph, $\tilde{P} \leq C_G(x) \cap C_G(y) \leq C_G(xy)$. If x is a p'-element, then Lemma 2.4 implies that $|x^G|$ divides $|(xy)^G|$, so the second statement of the lemma follows. □

3. Proof of the theorem

Let G be a finite group with $Z(G) = 1$ such that $N(G) = N(A_6 \times A_6)$. It follows from [1, Corollary 1] that $\pi(G) = \pi(A_6 \times A_6) = \{2, 3, 5\}$.

Lemma 3.1. *There are a* 5*-element* $x \in G$ *with* $|x^G| = 72$ *and a* 3*-element* $u \in G$ *with* $|u^G| = 40$.

Proof. Lemma 2.1 yields that there is an element x of prime order with $|x^G| = 72$. If $|x| = 5$, then we are done, so we can assume that $|x| \in \{2, 3\}$.

Suppose that $|x| = 2$. Lemma 2.6 implies that there is a 3-element $y \in C_G(x)$ with $|y^G|_3 \leq 3$, so $|y^G| \in \{40, 40^2\}$ due to Lemma 2.1. If $|y^G| = 40^2$, then $\mathrm{lcm}(|x^G|, |y^G|) = \mathrm{lcm}(40^2, 72) = 120^2$. By Lemma 2.4, there must be a multiple of 120^2 in $N(G)$; a contradiction. Hence $|y^G| = 40$.

Applying Lemma 2.6 once more, we obtain a 5-element $z \in C_G(y)$ with $|z^G|_5 \leq 1$, so $|z^G| \in \{72, 72^2\}$. If $|z^G| = 72^2$, then $N(G)$ must contain a multiple of $72^2 \cdot 5$; a contradiction. Thus, z is the desired 5-element with $|z^G| = 72$.

The case $|x| = 3$ can be treated similarly: for a suitable 2-element $y \in C_G(x)$, there is a 5-element $z \in C_G(y)$ with $|z^G| = 72$.

Again, Lemma 2.1 yields that there is an element u of prime order with $|u^G| = 40$. Applying the same arguments as above we obtain a 3-element satisfying this property. □

In the next three lemmas we show that $O_p(G) = 1$ for every $p \in \pi(G)$; so the socle of G must be the direct product of nonabelian simple groups.

Lemma 3.2. $O_5(G) = 1$.

Proof. Suppose the contrary. Let V be a nontrivial normal abelian 5-subgroup of G.

Let w be a 3-element with $|w^G| = 40$. Lemma 2.2 yields $V = C_V(w) \times [V, w]$. Since w acts freely on $[V, w]$, we have that $|[V, w]| - 1$ is a multiple of 3, so $|[V, w]| = 5^{2k}$ for $k \geq 0$. In view of Lemma 2.3,

$$|w^V| = |V : C_V(w)| = |[V, w]| = 5^{2k} \text{ divides } |w^G|_5 = 5.$$

Thus, $|[V, w]| = 1$ and $V = C_V(w)$.

If $P \in Syl_5(G)$ and $y \in V \cap Z(P) \setminus \{1\}$, then $|y^G|_5 = 1$. It follows from Lemma 2.1 that $|y^G| \in \{72, 72^2\}$. Suppose that $|y^G| = 72^2$. Since $\gcd(|w|, |y|) = 1$, Lemma 2.4 implies that $\mathrm{lcm}(|w^G|, |y^G|) = 72^2 \cdot 5$ divides $|(wy)^G|$, which contradicts Lemma 2.1. Hence $|y^G| = 72$.

By Lemma 2.6, there is a 2-element $z \in C_G(y)$ with $|z^G|_2 \leq 2^2$, so $|z^G| \in \{45, 90, 45^2, 45 \cdot 90, 90^2\}$. If $|z^G| \in \{45^2, 45 \cdot 90, 90^2\}$, then Lemma 2.4 implies that $45^2 \cdot 2^3$ divides $|(yz)^G|$, a contradiction. Therefore, $|z^G| \in \{45, 90\}$. Since y is a $2'$-element, $|(yz)^G|_2 = |y^G|_2 = 2^3$ due to Lemma 2.6. It follows from Lemma 2.4 that $\mathrm{lcm}(|y^G|, |z^G|) = 2^3 \cdot 3^2 \cdot 5 = 360$ divides $|(yz)^G|$. Thus, $|(yz)^G| \in \{40 \cdot 45, 45 \cdot 72\}$.

Applying Lemma 2.6 once more, we obtain a 3-element $x \in C_G(z)$ such that $|x^G|_3 \leq 3$, whence $|x^G| \in \{40, 40^2\}$. In fact $|x^G| = 40$, otherwise Lemma 2.4 implies that $40^2 \cdot 3^2$ divides $|(xz)^G|$, which is impossible. We have $|(xz)^G|_3 = |z^G|_3 = 3^2$ by Lemma 2.6 and $|(xz)^G|$ is a multiple of $\mathrm{lcm}(|x^G|, |z^G|) = 360$. Therefore, $|(xz)^G| \in \{40 \cdot 45, 40 \cdot 72, 40 \cdot 90\}$.

By the second paragraph of the proof, x centralizes V, in particular, $x \in C_G(y)$. Lemma 2.4 implies that $|(xy)^G| \leq |x^G| \cdot |y^G| = 40 \cdot 72$ and $|(xy)^G|$ is a multiple of $\mathrm{lcm}(|x^G|, |y^G|) = 360$. Therefore, $|(xy)^G| \in \{40 \cdot 45, 40 \cdot 72\}$.

Since x, y, z are of coprime orders and centralize each other, $|(xyz)^G|$ is a multiple of $\mathrm{lcm}(|(xy)^G|, |(xz)^G|, |(yz)^G|)$. Therefore, we can assume that $|(xy)^G| = |(yz)^G| = 40 \cdot 45$ and $|(xz)^G| \in \{40 \cdot 45, 40 \cdot 90\}$, because otherwise $|(xyz)^G| \notin N(G)$.

It is clear that $|(xyz)^G|_5 = |(xy)^G|_5$ or, equivalently, $|C_G(xy) \cap C_G(z)|_5 = |C_G(xy)|_5$. The latter implies that a Sylow 5-subgroup Q of $C_G(xy) \cap C_G(z)$ is a Sylow 5-subgroup of $C_G(xy)$. Since V is a normal 5-subgroup of $C_G(xy) = C_G(x) \cap C_G(y)$, it follows that $V \leq Q \leq C_G(z)$.

Suppose that P is a Sylow 5-subgroup of G containing a Sylow 5-subgroup $\tilde{P}$ of $C_G(x)$ and $u \in V \cap Z(P) \setminus \{1\}$. Then $|u^G| = 72$ by the arguments from the third paragraph of the proof. Since u centralizes $\tilde{P}$, it follows that $|(xu)^G|_5 = |x^G|_5 = 5$. Lemma 2.4 and Lemma 2.6 yield $|(xu)^G| = 40 \cdot 72$. Note that $u \in V \leq C_G(z)$, so $|(xuz)^G|$ is a multiple of $\mathrm{lcm}(|(xu)^G|, |(xz)^G|) = 2^6 \cdot 3^2 \cdot 5^2$, a contradiction. □

The proofs of the next two lemmas are very similar to that of Lemma 3.2. However, they are different from it, so we provide them.

Lemma 3.3. $O_3(G) = 1$.

Proof. Suppose, to the contrary, that V is a nontrivial abelian normal 3-subgroup of G.

By the same arguments as in the proof of Lemma 3.2, an arbitrary 5-element $w \in G$ with $|w^G| = 72$ centralizes V and there is an element $y \in V$ with $|y^G| = 40$.

In view of Lemma 2.6, there is a 2-element $z \in C_G(y)$ with $|z^G|_2 \leq 2^2$. Following exactly the proof of the previous lemma, we obtain that $|z^G| \in \{45, 90\}$ and $|(yz)^G| \in \{40 \cdot 45, 45 \cdot 72\}$.

Applying Lemma 2.6 once more, we obtain a 5-element $x \in C_G(z)$ such that $|x^G|_5 \leq 1$, whence $|x^G| \in \{72, 72^2\}$. In fact $|x^G| = 72$, otherwise Lemma 2.4 implies that $72^2 \cdot 5$ divides $|(xz)^G|$, which is impossible. We have $|(xz)^G|_5 = |z^G|_5 = 5$ by Lemma 2.6 and $|(xz)^G|$ is a multiple of $\mathrm{lcm}(|x^G|, |z^G|) = 360$. Therefore, $|(xz)^G| \in \{40 \cdot 72, 45 \cdot 72, 72 \cdot 90\}$.

By the second paragraph of the proof, $x \in C_G(y)$. It follows that $|(xy)^G| \leq |x^G| \cdot |y^G| = 40 \cdot 72$ and $|(xy)^G|$ is a multiple of $\mathrm{lcm}(|x^G|, |y^G|) = 360$ due to Lemma 2.4. Therefore, $|(xy)^G| \in \{40 \cdot 45, 40 \cdot 72\}$.

The elements x, y, z are of coprime orders and centralize each other. Hence, $|(xyz)^G|$ is a multiple of $\mathrm{lcm}(|(xy)^G|, |(xz)^G|, |(yz)^G|)$, which is impossible due to Lemma 2.1. □

Lemma 3.4. $O_2(G) = 1$.

Proof. Suppose that V is a nontrivial normal abelian 2-subgroup of G.

Applying the same arguments as in the proof of Lemma 3.2, we see that an arbitrary 5-element $w \in G$ with $|w^G| = 72$ centralizes V and there is an element $y \in V$ with $|y^G| = 45$.

By Lemma 2.6, there is a 3-element $z \in C_G(y)$ with $|z^G|_3 \leq 3$, so $|z^G| \in \{40, 40^2\}$. If $|z^G| = 40^2$, then Lemma 2.4 implies that $40^2 \cdot 3^2$ divides $|(yz)^G|$, a contradiction. Therefore, $|z^G| = 40$. It follows from Lemma 2.4 that $\mathrm{lcm}(|y^G|, |z^G|) = 360$ divides $|(yz)^G|$ and $40 \cdot 45 \geq |(yz)^G|$. Thus, $|(yz)^G| = 40 \cdot 45$.

Applying Lemma 2.6 once more, we obtain a 5-element $x \in C_G(z)$ with $|x^G|_5 = 1$, so $|x^G| \in \{72, 72^2\}$. If $|x^G| = 72^2$, then Lemma 2.4 implies that $72^2 \cdot 5$ divides $|(xz)^G|$, a contradiction. Therefore, $|x^G| = 72$. Moreover, Lemma 2.6 yields $|(xz)^G|_5 = |z^G|_5 = 5$. We have that $\operatorname{lcm}(|x^G|, |z^G|)$ divides $|(xz)^G|$ and $40 \cdot 72 \geq |(xz)^G|$ due to Lemma 2.4. Therefore, $|(xz)^G| = 40 \cdot 72$.

By the second paragraph of the proof, $x \in C_G(y)$. It follows that x, y, z are of coprime orders and centralize each other. Hence, $|(xyz)^G|$ is a multiple of the least common multiple of $|(xz)^G|$ and $|(yz)^G|$, that is, $40 \cdot 72 \cdot 5$, a contradiction. □

Thus, $M \leq G \leq \operatorname{Aut}(M)$, where $M = S_1 \times \cdots \times S_k$ is a direct product of finite nonabelian simple groups $S_1, \ldots, S_k$. Since $\pi(S_i) \subseteq \pi(G)$, it follows that $S_i \in \{A_5, A_6, U_4(2)\}$. We will use information about these groups from [2].

Lemma 3.5. *$M \simeq A_6 \times A_6$.*

Proof. If $k = 1$, then $|G|_5 > |\operatorname{Aut}(M)|_5$, a contradiction. If $k \geq 3$, then there is an element $x \in M$ with $|x^M|_5 = 5^3$. Lemma 2.3 implies that $|x^M|$ divides $|x^G|$, a contradiction.

Therefore, $k = 2$. If one of the two direct factors is isomorphic to $U_4(2)$, then there is an element $x \in M$ with $|x^M|_2 > 2^6$, which is impossible.

Thus, we have $M \in \{A_5 \times A_5, A_5 \times A_6, A_6 \times A_6\}$. If $M \in \{A_5 \times A_5, A_5 \times A_6\}$, then $|G|_3 > |\operatorname{Aut}(M)|_3$, a contradiction. So $M \simeq A_6 \times A_6$, as claimed. □

Lemma 3.6. *$G \simeq A_6 \times A_6$.*

Proof. By Lemma 3.5, we have $S_1 \times S_2 \leq G \leq \operatorname{Aut}(S_1 \times S_2) = \operatorname{Aut}(S_1) \wr \langle a \rangle$, where a is an involution and $S_1 \simeq S_2 \simeq A_6$.

If $a \in G$, then $G \simeq H \wr \langle a \rangle$, where $A_6 \leq H \leq \operatorname{Aut}(A_6)$. Lemma 2.5 states that $|a^G| = |H|$. This is a contradiction since $|H| \notin N(G)$.

Suppose that $a \notin G$, so $S_1 \times S_2 \leq G \leq \operatorname{Aut}(S_1) \times \operatorname{Aut}(S_2)$. If $S_1 \times S_2 < G$, then G includes a subgroup G_1 of the form $(S_1 \times S_2).2 = \langle S_1 \times S_2, (\varphi_1, \varphi_2) \rangle$, where $\varphi_i \in \operatorname{Aut}(S_i)$. We may assume that $\varphi_1 \notin S_1$. Also observe that G_1 is normal in G.

There are three isomorphism types of $A_6.2$. For every possible isomorphism type of $A_6.2$ there is an element $x \in A_6$ with $|x^{A_6.2}|_2 > 2^3$, so $C_{A_6.2}(x) = C_{A_6}(x)$.

Take an element $x = (x_1, x_2) \in G_1$ such that $|{x_1}^{\langle S_1, \varphi_1 \rangle}|_2 > 2^3$ and $|{x_2}^{\langle S_2, \varphi_2 \rangle}|_2 \geq 2^3$. Since $C_{\langle S_i, \varphi_i \rangle}(x_i) = C_{S_i}(x_i)$ regardless of whether φ_i is outer or not, we have $C_{G_1}(x) = C_{S_1}(x_1) \times C_{S_2}(x_2)$. It follows that $|x^G|_2 > 2^6$. This contradiction completes the proof of the lemma and the theorem. □

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