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ON NONSTANDARD QUASIVARIETIES OF DIFFERENTIAL GROUPOIDS AND UNARY ALGEBRAS

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Abstract. We prove that there are continuum many nonstandard quasivarieties of differential groupoids and unary algebras.

Keywords: quasivariety, nonstandard quasivariety, differential groupoid, unary algebra.

1. Introduction

A prevariety $\mathbf{K}$ is *profinite* if each Boolean topological structure with its algebraic reduct belonging to $\mathbf{K}$ is *profinite with respect to* $\mathbf{K}$; that is, isomorphic to an inverse limit of finite structures (equipped with the discrete topology) belonging to $\mathbf{K}$. A universal Horn class $\mathbf{K}$ is *standard* if each Boolean topological structure with its algebraic reduct in $\mathbf{K}$ is isomorphic to a closed substructure of the Cartesian product of a nonempty family of finite structures from $\mathbf{K}$, where all the finite structures are equipped with the discrete topology and the Cartesian product is equipped with the product topology.

The notion of a standard class is closely related with natural dualities and profinite structures, see [3, 22]. A detailed study of standard and profinite classes of general algebraic structures as well as particular classes of algebraic structures was carried out in [4, 5, 6, 7]. It was established in [21] that there is no algorithm which would decide if the variety generated by a given finite structure is standard. This result solved in the negative a problem from [4, 6] in the case of varieties. A wide

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spectrum of examples of standard and nonstandard quasivarieties can be found in [1, 6, 7, 27].

In a recent article of the authors and their co-author, the notion of a B-class was introduced which allows us to treat certain complexity problems for quasivarieties in a uniform way [17]. Namely, the existence of such a class with respect to a quasivariety implies extremely high level of complexity from many points of view. In addition to known properties of Q-universality and the undecidability of certain decision problems, the existence of a B-class leads to continuum many subquasivarieties lacking finitely partitionable quasi-equational bases, continuum many subquasivarieties whose quasi-equational theory is undecidable, and continuum many nonstandard subquasivarieties [18, 20].

Although the variety $\mathbf{Dm}$ of differential groupoids and a certain quasivariety $\mathbf{V}$ of unary algebras with two unary operations do not admit B-classes, a series of complexity results holds for them. For example, the representation, undecidability, and independent axiomatization results can be proven for these two quasivarieties, see [15, 16]. Notice that the proof of the facts that the quasivarieties $\mathbf{Dm}$ and $\mathbf{V}$ are Q-universal in [12] and [11] required an individual approach and adjustment of known sufficient conditions for Q-universality. In the present article, we prove that, despite of the lack of B-classes in $\mathbf{Dm}$ and $\mathbf{V}$, these quasivarieties also contain continuum many nonstandard subquasivarieties, see Theorems 10, 12, 13, and 15.

2. Basic definitions and auxiliary results

For all definitions and notation concerning (algebraic) structures and their quasivarieties, we refer to the monograph [8, Ch. 1] and the articles [15, 16, 17, 18, 19, 20].

2.1. **Quasivarieties and class operators.** *Quasi-identities* are universal Horn sentences of the form

$$\forall \overline{x}\ [\varphi_1(\overline{x})\ \&\ldots\&\ \varphi_k(\overline{x}) \longrightarrow \varphi_0(\overline{x})],$$

where $\varphi_i(\overline{x})$ is an atomic formula for each $i \leqslant k$. A class $\mathbf{K}$ of structures is a *quasivariety* if it coincides with the class of all models of some set Φ of quasi-identities. Then the set Φ is called a *quasi-equational basis* of $\mathbf{K}$.

We denote structures by calligraphic letters. The universe of a structure is denoted by the corresponding italic letter. For classes of structures, we use boldface letters. We assume that all classes are *abstract*, i.e., closed under isomorphism.

Let $\mathbf{K}(\sigma)$ denote the class of all structures of type σ. For a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, let $\mathbf{K}_{\mathrm{fin}}$ denote the class of finite structures from $\mathbf{K}$ and let $\mathbf{Q}(\mathbf{K})$ denote the least quasivariety extending $\mathbf{K}$. Let $\mathbf{H}(\mathbf{K})$ denote the class of homomorphic images of structures from $\mathbf{K}$; let $\mathbf{P}(\mathbf{K})$ denote the class of structures that are isomorphic to the Cartesian product of a family of structures from $\mathbf{K}$; let $\mathbf{P}^+(\mathbf{K})$ denote the class of structures that are isomorphic to the Cartesian product of a nonempty family of structures from $\mathbf{K}$; let $\mathbf{S}(\mathbf{K})$ denote the class of structures that are embeddable into structures from $\mathbf{K}$; see, for example, [8, Sec. 1.2.7]. Finally, let $\mathbf{T}$ denote the trivial (quasi)variety.

We recall the notion of the inverse (projective) limit, see, for example, [6, Sec. 1].

Definition 1. A triple $\Lambda = \langle I, \mathcal{A}_i, \pi_{ij}\rangle$ is an *inverse spectrum* if $\langle I; \leqslant\rangle$ is an up-directed partially ordered set, $\{\mathcal{A}_i \mid i \in I\}$ is a set of structures of the same similarity type σ, the mapping $\pi_{ij}\colon \mathcal{A}_j \to \mathcal{A}_i$ is a homomorphism for all $i, j \in I$ with $i \leqslant j$, and the following holds:

(i) the mapping π_{ii} is the identity automorphism on $\mathcal{A}_i$ for each $i \in I$;
(ii) we have $\pi_{ik} = \pi_{ij}\pi_{jk}$ for all $i, j, k \in I$ with $i \leqslant j \leqslant k$.

If each homomorphism π_{ij} is onto then the inverse spectrum is said to be *surjective.* A structure $\mathcal{A} \in \mathbf{K}(\sigma)$ is the *inverse limit* of Λ if its universe

$$A = \Big\{(a_i \mid i \in I) \in \prod_{i\in I} A_i \mid \pi_{ij}(a_j) = a_i \text{ for all } i \leqslant j \text{ in } I\Big\}$$

is not empty.

We denote the inverse limit of Λ whenever it exists by $\varprojlim \Lambda$. It is clear that $\varprojlim \Lambda \in \mathbf{SP}(\mathcal{A}_i \mid i \in I)$ if Λ is as in Definition 1. For $i \in I$, we denote by π_i the canonical projection from $\varprojlim \Lambda$ to $\mathcal{A}_i$. Notice that π_i is surjective if so is Λ.

A subclass $\mathbf{K}'$ of $\mathbf{K}$ is a $\mathbf{K}$-*quasivariety* if $\mathbf{K}'$ is defined within $\mathbf{K}$ by a set of quasi-identities or, equivalently, $\mathbf{K}' = \mathbf{Q}(\mathbf{K}') \cap \mathbf{K}$.

2.2. **Finite B-classes.** As usual, we denote the least infinite cardinal by ω. For a set X, we denote by $\mathcal{P}_{\text{fin}}(X)$ the set of all finite subsets of X and by $\mathcal{P}_{\text{inf}}(X)$ the set of all infinite subsets $I \subseteq X$ such that the complement $X \backslash I$ is infinite too. Notice that $|\mathcal{P}_{\text{inf}}(X)| = 2^\omega$ for each countable infinite set X.

The following notion is introduced in [17].

Definition 2. Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety of a finite type σ. A class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\text{fin}}(\omega)\} \subseteq \mathbf{M}$ of finite structures is a *finite* B-*class* with respect to $\mathbf{M}$ if $\mathbf{A}$ satisfies the following conditions:

(B_0) $\mathcal{A}_\varnothing$ is a trivial structure;
(B_1) if $X = Y \cup Z$ in $\mathcal{P}_{\text{fin}}(\omega)$ then $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y, \mathcal{A}_Z)$;
(B_2) if $\varnothing \neq X \in \mathcal{P}_{\text{fin}}(\omega)$ and $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y)$ then $X = Y$;
(B_3) if $F \in \mathcal{P}_{\text{fin}}(\omega)$, $i \in \omega$, and $f\colon \mathcal{A}_F \to \mathcal{A}_{\{i\}}$ is a homomorphism then either $f(\mathcal{A}_F) \cong \mathcal{A}_\varnothing$ or $i \in F$;
(B_4) if $F \in \mathcal{P}_{\text{fin}}(\omega)$ then $\mathbf{H}(\mathcal{A}_F) \cap \mathbf{M} \subseteq \mathbf{A}$.

As is mentioned in the introduction, the existence of a (finite) B-class with respect to a quasivariety $\mathbf{K}$ witnesses a very complicated structure of $\mathbf{K}$ as well as of its quasivariety lattice. However, there exists natural classes of structures that do not admit B-classes by obvious reasons (say, the existence of homomorphic images in "small" subvarieties, which spoils B_3 or B_4). In the following two subsections, we recall necessary information about two such classes.

2.3. **Differential groupoids.** A *differential groupoid* is an algebra endowed with one binary operation $\cdot$ that satisfies the following identities:

$$\forall x\ [x \cdot x = x], \quad \forall x\, \forall y\ [x \cdot (x \cdot y) = x],$$
$$\forall x\, \forall y\, \forall z\, \forall t\ [(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)].$$

Let $\mathbf{Dm}$ denote the variety of all differential groupoids.

For brevity, we write $x_1 x_2 \ldots x_n$ for $(\ldots(x_1 \cdot x_2) \cdot \ldots) \cdot x_n$ and xy^n for $x\underbrace{y \ldots y}_{n}$.

We use the following representation of differential groupoids from [23]. A groupoid $\mathcal{G}$ is an $\mathbf{Lz}$-$\mathbf{Lz}$-*sum* (of orbits $\mathcal{G}_i$ over a groupoid $\mathcal{I}$) *satisfying the left normal law* if there is a partition $G = \bigcup_{i\in I} G_i$ such that, for every pair $(i, j) \in I^2$, there is a mapping $h_i^j\colon G_i \to G_i$ satisfying the following conditions:

(i) for every $i \in I$, h_i^i is the identity mapping;
(ii) we have $h_i^j(h_i^k(x)) = h_i^k(h_i^j(x))$ for all $i, j, k \in I$ and $x \in G_i$;
(iii) we have $a_i \cdot a_j = h_i^j(a_i)$ for all $i, j \in I$, $a_i \in G_i$ and $a_j \in G_j$.

According to [23, Theorem 2.2], a groupoid is differential if and only if it can be represented as an **Lz**-**Lz**-sum satisfying the left normal law. For more detailed information on differential groupoids, we refer to the monograph [24, Secs. 5.6 and 8.4].

Let $n > 0$. The structure defined in **Dm** by the generators $\{x, y\}$ and the defining relations $\{yx = y, xy^n = x\}$ is called the *cycle* of length n and is denoted by $\mathcal{D}_n$. It is convenient to regard D_n as $G_0 \cup G_1$, where G_1 is the singleton orbit $\{b\}$ and $G_0 = \{a, ab, ab^2, \ldots, ab^{n-1}\}$. We denote the trivial groupoid by $\mathcal{D}_0$.

Let $\mathbb{P}$ denote the set of all primes; we assume that $\mathbb{P} = \{p_i \mid i < \omega\}$, where $p_i \leqslant p_j$ if and only if $i \leqslant j$ for all $i, j < \omega$. For an arbitrary set $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, we put $[F] = \prod_{i \in F} p_i$ if $F \neq \varnothing$ and $[F] = 1$ if $F = \varnothing$.

We will need the following basic properties of the cycles, see [12, 15] as well as [26].

Lemma 3. *Let $n > 0$.*

(i) *The class $\{\mathcal{D}_m \mid m$ divides $n\}$ coincides with the class of nontrivial homomorphic images of $\mathcal{D}_n$.*
(ii) *If $m \in \omega$ and $\varphi\colon \mathcal{D}_n \to \mathcal{D}_m$ is a homomorphism then either $\varphi(\mathcal{D}_n) \cong \mathcal{D}_0$ or m divides n and φ is onto.*
(iii) *If $n > 0$ and $X, X_1, \ldots, X_n \in \mathcal{P}_{\mathrm{fin}}(\omega)$ are such that the set $\{1, \ldots, n\}$ is minimal with respect to the property that $\mathcal{D}_{[X]} \in \mathbf{SP}(\mathcal{D}_{[X_1]}, \ldots, \mathcal{D}_{[X_n]})$, then $X = X_1 \cup \ldots \cup X_n$.*
Conversely, if $X = X_1 \cup \ldots \cup X_n$ then $\mathcal{D}_{[X]} \in \mathbf{SP}(\mathcal{D}_{[X_1]}, \ldots, \mathcal{D}_{[X_n]})$.

The structure of the variety lattice of differential groupoids is explicitly described in [23], see also [24, Theorem 8.4.14]. In particular, each proper subvariety of **Dm** is defined by a single identity within **Dm** and is locally finite. In contrast to that description, the structure of the quasivariety lattice Lq(**Dm**) is much more complicated. Namely, the variety **Dm** is Q-universal [12], there exist 2^ω classes **K** of differential groupoids such that the set of (isomorphism types of) finite sublattices of Lq(**K**) is not computable [25, 26], and there exist continuum many quasivarieties of differential groupoids with no independent quasi-equational basis [2].

The following assertions are proven in [2, 12, 15, 16].

Theorem 4. *For each of the following properties, there exists continuum many quasivarieties of differential groupoids possessing this property*:

- *Q-universality*;
- *the undecidability of the set of* (*isomorphism types*) *of finite sublattices of the lattice of* **K**-*subquasivarieties for a suitable subclass* **K**;
- *the existence of an ω-independent quasi-equational basis and the lack of an independent quasi-equational basis within* **Dm**;
- *the existence of an independent quasi-equational basis*;
- *the undecidability of the quasi-equational theory*;
- *the undecidability of the finite membership problem and the membership problem for finitely presented differential groupoids.*

2.4. **Unary algebras.** As is proven in [11], the variety $\mathbf{K}_3$ of unary algebras of the type $\sigma = \{f, g\}$ defined by the identities

$$\forall x\,\forall y\;[f(f(x)) = f(f(y)) = f(g(y))],$$
$$\forall x\,\forall y\;[g(g(x)) = g(g(y)) = g(f(y))],$$
$$\forall x\;[f(f(x)) = g(g(x))]$$

is a minimal Q-universal variety. It follows from the proof that the proper subquasivariety $\mathbf{V} \subseteq \mathbf{K}_3$ defined by the quasi-identities

$$\forall x\;[f(x) = f(f(x)) \longrightarrow f(x) = g(x)],$$
$$\forall x\;[g(x) = g(g(x)) \longrightarrow f(x) = g(x)],$$
$$\forall x\;[f(x) = g(x) \longrightarrow f(x) = f(f(x))],$$
$$\forall x\,\forall y\;[f(x) = f(y) \longrightarrow g(x) = g(y)],$$
$$\forall x\,\forall y\;[g(x) = g(y) \longrightarrow f(x) = f(y)]$$

is Q-universal; moreover, so is the lattice of $\mathbf{W}$-quasivarieties, where $\mathbf{W}$ denotes the subclass of $\mathbf{V}$ defined by the sentences

$$\forall x\,\forall y\;[g(x) = g(y)\,\&\,x \neq y \longrightarrow g(x) = g(g(x))], \tag{1}$$
$$\forall x\;[g(x) = g(g(x))] \longrightarrow \forall x\,\forall y\;[x = y]. \tag{2}$$

We recall the notation for certain unary algebras, see [11, 13].

For $n > 1$, let $\mathcal{C}_n$ denote the algebra whose universe is

$$C_n = \{0\} \cup A_n \cup B_n \text{ with } A_n = \{a_0^n, \ldots, a_{n-1}^n\},\; B_n = \{b_0^n, \ldots, b_{n-1}^n\}$$

and the operations are defined as follows: $f(0) = g(0) = f(a_i^n) = g(a_i^n) = 0$ and $g(b_i^n) = a_i^n$ for $0 \leqslant i \leqslant n-1$, $f(b_i^n) = a_{i+1}^n$ for $0 \leqslant i \leqslant n-2$, and $f(b_{n-1}^n) = a_0^n$. Let $\mathcal{C}_1$ denote the 2-element algebra with the universe $\{0, a\}$, where $f(0) = g(0) = f(a) = g(a) = 0$. Let $\mathcal{C}_0$ denote the trivial algebra.

It is clear that, for $n \geqslant 0$, we have $\mathcal{C}_n \in \mathbf{W}$.

We recall necessary properties of the algebras $\mathcal{C}_n$ from [11, 13, 15].

Lemma 5. *If $n > 1$ then the following assertions hold.*

(i) *If m divides n then there exists a homomorphism φ from $\mathcal{C}_n$ onto $\mathcal{C}_m$; moreover, the kernels of all such homomorphisms coincide and we have*

$$\ker\varphi = \{(x, x) : x \in C_n\} \cup \{(a_i, a_j), (b_i, b_j) \mid i \equiv j \pmod{m}\}.$$

(ii) *If $\mathcal{A} \in \mathbf{W}$ is nontrivial and there exists a homomorphism from $\mathcal{C}_n$ onto $\mathcal{A}$ then $\mathcal{A}$ is isomorphic to $\mathcal{C}_m$ for a suitable divisor m of n.*

(iii) *If $\mathcal{A} \in \mathbf{V}$ is nontrivial and there exists a homomorphism φ from $\mathcal{C}_n$ onto $\mathcal{A}$ then one of the following conditions holds:*

 (a) *$\mathcal{A} \in \mathbf{W}$ and assertion* (ii) *is valid;*

 (b) *the kernel* $\ker\varphi$ *contains a pair of one of the forms $(0, a_i^n)$, $(0, b_i^n)$, (a_i^n, b_j^n), (a_i^n, a_j^n), where the difference $i - j$ is a multiple of no divisor of n, and the structure $\mathcal{A}$ satisfies the premise of sentence* (2);

 (c) *the kernel* $\ker\varphi$ *contains a pair of the form (a_i^n, a_{i+k}^n) but $(b_i^n, b_{i+k}^n) \notin \ker\varphi$, where k is the least positive number with this property, we have $k > 1$, and k divides n (in this case, the algebra $\mathcal{A}$ satisfies the premise of sentence* (1) *but violates its conclusion, i.e., $\mathcal{C}_k$ is a substructure of $\mathcal{A}$ and $|g^{-1}(\varphi(a_i^n))| > 1$).*

If $X, Y, Z \in \mathcal{P}_{\mathrm{fin}}(\omega)$ and $n > 1$ then the following assertions hold.

(iv) *There exists a homomorphism from $\mathcal{C}_{[X]}$ onto $\mathcal{C}_{[Y]}$ if and only if $Y \subseteq X$.*

(v) *We have $\mathcal{C}_{[X]} \leq \mathcal{C}_{[Y]} \times \mathcal{C}_{[Z]}$ if and only if $X = Y \cup Z$.*

If $n > 1$ and $m > 0$ then there exists a quasi-identity $q(n, m)$ such that, for every $k > 1$, the structure $\mathcal{C}_k$ satisfies $q(n, m)$ if and only if either k is not a divisor of mn or k divides n.

The following assertions are proven in [11, 13, 15, 16].

Theorem 6. *For each of the following properties, there exists continuum many* **W***-subquasivarieties possessing this property*:

- *Q-universality*;
- *the undecidability of the set of* (*isomorphism types*) *of finite sublattices of the lattice of* **K***-subquasivarieties for a suitable subclass* **K**;
- *the existence of an ω-independent quasi-equational basis and lack of an independent quasi-equational basis within* **V**;
- *the existence of an independent quasi-equational basis*;
- *the undecidability of the quasi-equational theory*;
- *the undecidability of the finite membership problem and the membership problem for finitely presented unary algebras of the type $\{f, g\}$.*

2.5. **Standard and nonstandard quasivarieties.** For a structure $\mathcal{A} \in \mathbf{K}(\sigma)$, we say that $\mathbb{A} = \langle \mathcal{A}, \tau \rangle$ is a *topological structure* if τ is a topology on A and all the basic operations of $\mathcal{A}$ are continuous and all the basic relations on $\mathcal{A}$ are closed with respect to τ. For a topological structure $\mathbb{A}$, we denote its algebraic reduct by $\mathcal{A}$ and its topology by $\tau_{\mathbb{A}}$. A topology τ on a set A is *Boolean* if the topological space $\langle A, \tau \rangle$ is compact Hausdorff and has a base of clopen sets. A topological structure $\mathbb{A}$ is *Boolean* if $\tau_{\mathbb{A}}$ is Boolean. For a class **K** of topological structures of a fixed type, let $\mathbf{S}_c(\mathbf{K})$ denote the class of structures that are isomorphic to closed substructures of structures from **K**.

A structure is *profinite* (*with respect to* **K**) if it is isomorphic to the inverse limit of a family of finite structures (from **K**). Each profinite structure is naturally equipped with a Boolean topology (in this case, this is the product topology). A prevariety **K** is *profinite* if each Boolean topological structure with its algebraic reduct belonging to **K** is profinite with respect to **K**. A universal Horn class **K** is *standard* if each Boolean topological structure with its algebraic reduct in **K** belongs to $\mathbf{S}_c\mathbf{P}^+(\mathbf{K}_{\mathrm{fin}})$, where all the finite structures are equipped with the discrete topology and the Cartesian product is equipped with the product topology.

It is immediate from [6, Theorem 2.6 and Lemma 1.4(ii)] that a universal Horn class **K** is standard if and only if it is profinite.

For more information and results on natural dualities, topological quasivarieties, Boolean topological structures, and topology, the reader is referred to [3, 6, 10].

We will need the following property of the inverse limit, see [6, Lemma 3.2].

Lemma 7. *Let σ contain no relation symbols, let $\mathcal{A} = \varprojlim_{i \in I} \mathcal{A}_i$, where Λ is a surjective inverse spectrum and $\mathcal{A}_i$ is a finite structure for every $i \in I$, let $\mathcal{B}$ be a finite structure, and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a* (*continuous*) *homomorphism. Then there exist $i \in I$ and a* (*continuous*) *homomorphism $\psi\colon \mathcal{A}_i \to \mathcal{B}$ such that $\varphi = \psi\pi_i$.*

The following notion from [6] and its connection with standard quasivarieties is an essential tool in the proof of our main results.

Definition 8. Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$, where σ contains no relation symbols. A structure $\mathcal{A}$ is *pointwise non-separable* with respect to $\mathbf{M}$ if there exist $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $\varphi(a_1) = \varphi(a_2)$ for every finite structure $\mathcal{B} \in \mathbf{M}$ and every homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$.

The following assertion is immediate from [6, Lemmas 3.3 and 3.4].

Lemma 9. *Let $\mathcal{A} = \varprojlim_{i \in I} \mathcal{A}_i$, where Λ is a surjective inverse spectrum and $\mathcal{A}_i$ is a finite structure for every $i \in I$, and let $\mathbf{K}$ be a quasivariety. If $\mathcal{A}$ is pointwise non-separable with respect to $\mathbf{K}$ then $\mathcal{A}$ is not profinite with respect to $\mathbf{K}$.*

In particular, if $\mathcal{A} \in \mathbf{K}$ then $\mathbf{K}$ is not standard.

3. Main results

We formulate and prove similar results for **Dm** and **V**. Namely, we find

(a) continuum many nonstandard subquasivarieties with no independent quasi-equational bases;

(b) and continuum many nonstandard subquasivarieties having an independent quasi-equational basis.

3.1. Differential groupoids.

Theorem 10. *There exist continuum many subquasivarieties of* **Dm** *that are not standard and have no independent quasi-equational basis.*

Proof. Let $I \subseteq \omega$ and let $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$. We denote by φ_F^I the quasi-identity

$$\forall x \, \forall y \; xy^{[F]} = x \;\&\; yx = y \;\rightarrow\; xy^{[F \cap I]} = x.$$

Let $\Phi_I = \{\varphi_F^I \mid F \in \mathcal{P}_{\mathrm{fin}}(\omega)\}$ and let $\mathbf{K}_I$ be the subquasivariety of **Dm** defined by the set of quasi-identities Φ_I.

Let $I \in \mathcal{P}_{\mathrm{inf}}(\omega)$, $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, and $p \in \omega \backslash F$. We denote by ψ_F^p the quasi-identity

$$\forall x \, \forall y \; xy^{[F \cup \{p\}]} = x \;\&\; yx = y \;\rightarrow\; xy^{[F]} = x.$$

We put $\Psi_p = \{\psi_F^p \mid F \in P_{\mathrm{fin}}(\omega),\ p \in \omega \backslash F\}$ and $\Psi_I = \bigcup_{p \in \omega \backslash I} \Psi_p$. We denote by $\mathbf{K}_I'$ the subquasivariety of **Dm** defined by the set of quasi-identities Ψ_I.

The following assertion is proven in [16, Theorem 4].

Proposition 11. *We have $\mathbf{K}_I = \mathbf{K}_I'$ for every $I \in \mathcal{P}_{\mathrm{inf}}(\omega)$. The set Ψ_I is an ω-independent quasi-equational basis of $\mathbf{K}$. There is no independent quasi-equational basis of $\mathbf{K}$.*

We fix $I \in \mathcal{P}_{\mathrm{inf}}(\omega)$ and $k \in \omega \backslash I$. Assume that the numbers in I are ordered in the natural way, i.e., we have $I = \{i_n \mid n < \omega\}$, where $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. For every $n < \omega$, we put

$$F_n = \{i_0, \ldots, i_n\},\ \mathcal{A} = \mathcal{D}_{p_k},\ \mathcal{A}_n = \mathcal{D}_{[F_n]},\ \mathcal{B}_n = \mathcal{D}_{[\{k\} \cup F_n]}.$$

By the definition of $\mathbf{K}_I$, we immediately obtain the following assertion.

Claim 1. *For every $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, we have $\mathcal{D}_{[F]} \in \mathbf{K}_I$ if and only if $F \subseteq I$.*

In particular, we have $\mathcal{A} \notin \mathbf{K}_I$ and $\mathcal{A}_n \in \mathbf{K}_I$ but $\mathcal{B}_n \notin \mathbf{K}_I$ for every $n < \omega$.

We construct an inverse spectrum Λ. By Lemma 3, if $i < j < \omega$ then there is a surjective homomorphism $\pi_{ij}\colon \mathcal{B}_j \to \mathcal{B}_i$; moreover, we may assume that $\pi_{ij}(b) = b$ and $\pi_{ij}(a) = a$, where $\{b\}$ is the singleton orbit and a is the second generator of $\mathcal{D}_n$

for every $n < \omega$. According to the definition of the inverse limit, we denote by π_{ii} the identity automorphism on $\mathcal{B}_i$ for every $i < \omega$.

We immediately obtain the following assertion.

Claim 2. *The triple $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ is a surjective inverse spectrum.*

We put $\mathcal{B} = \varprojlim \Lambda$. Since each π_{ij} fixes a and b, we conclude that $B \neq \varnothing$. Moreover, we have $\mathcal{B} \in \mathbf{SP}(\mathbf{Dm}) \subseteq \mathbf{Dm}$. For every $n < \omega$, we denote by π_n the restriction of the nth projection $\prod_{n<\omega} B_n \to B_n$ to B.

Claim 3. *The differential groupoid $\mathcal{B}$ is infinite and belongs to $\mathbf{K}_I$.*

Proof of Claim. We show first that $\mathcal{B} \models \Phi_I$. Indeed, consider an arbitrary quasi-identity $\varphi_F^I \in \Phi_I$. Suppose that the premise of φ_F^I holds in $\mathcal{B}$ under an interpretation $\gamma \colon \{x, y\} \to B$. Then there exists a homomorphism $f \colon \mathcal{D}_{[F]} \to \mathcal{B}$ such that $f(a) = \gamma(x)$ and $f(b) = \gamma(y)$. For all $n < \omega$ and $u \in D_{[F]}$, we put $f_n(u) = \pi_n(f(u))$. We find that f_n is a homomorphism from $\mathcal{D}_{[F]}$ to $\mathcal{B}_n$. By Lemma 3, we have either $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_{[G]} \leq \mathcal{B}_n$ for some nonempty $G \subseteq F$ or $f_n(\mathcal{D}_{[F]}) \leq \mathcal{D}_1$. Since all proper subgroupoids of $\mathcal{B}_n$ are left-zero modes, we conclude that either $f_n(\mathcal{D}_{[F]}) \cong \mathcal{B}_n$, or $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_1$, or $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_0$. In the first case, we have $|B_n| \leqslant [F] + 1$. Since $\mathcal{D}_{[F]}$ is a finite groupoid and $|B_n| < |B_{n+1}|$ for each $n < \omega$, there is $s < \omega$ such that $[F] + 1 < |B_n|$ for all $n \geqslant s$. This implies that $f_n(\mathcal{D}_{[F]})$ is isomorphic to a subgroupoid of $\mathcal{D}_1$ if $n \geqslant s$. We denote $J_s = \{n < \omega \mid n \geqslant s\}$.

We summarize the above conclusions as follows. The differential groupoid $\mathcal{B}$ is isomorphic to the inverse limit $\varprojlim \langle J_s, \mathcal{B}_j, \pi_{ij} \rangle$ and is a subdirect product of the family of differential groupoids $\{\mathcal{B}_n \mid n \in J_s\}$. The differential groupoid $f(\mathcal{D}_{[F]})$ is a subdirect product of the family $\{f_n(\mathcal{D}_{[F]}) \mid n \in J_s\}$ of subgroupoids of $\mathcal{D}_1$; hence, it is a left-zero mode.

We conclude that the subgroupoid $f(\mathcal{D}_{[F]})$ of $\mathcal{B}$ generated by the set $\{a, b\}$ is a left-zero mode. But then the conclusion of φ_F^I holds in $\mathcal{B}$.

Finally, if the universe B were finite then, by Lemma 7, we would obtain $|B_{n+1}| \leqslant |B| \leqslant |B_n|$ for some $n < \omega$. Since $|B_{n+1}| > |B_n|$, we arrive at a contradiction. $\square$

We remind that $\mathcal{A} \notin \mathbf{K}_I$, $\mathcal{A}_n \in \mathbf{K}_I$, and $\mathcal{B}_n \notin \mathbf{K}_I$ for every $n < \omega$. By Lemma 3, the differential groupoid $\mathcal{B}_n$ is a subdirect product of the differential groupoids $\mathcal{A}$ and $\mathcal{A}_n$. We denote the corresponding projections (which are onto homomorphisms) by $\alpha_n \colon \mathcal{B}_n \to \mathcal{A}$ and $\beta_n \colon \mathcal{B}_n \to \mathcal{A}_n$. As $\mathcal{B}_n \notin \mathbf{K}_I$ and $\mathcal{A}_n \in \mathbf{K}_I$, the kernel $\ker \beta_n$ is the least congruence on $\mathcal{B}_n$ for no $n < \omega$. For each $n < \omega$, we can find distinct elements $b_1^n, b_2^n \in B_n$ such that $\beta_n(b_1^n) = \beta_n(b_2^n)$. We conclude that $\alpha_n(b_1^n) \neq \alpha_n(b_2^n)$.

By Lemma 3, if $i < j < \omega$ then there is a surjective homomorphism $\varphi_{ij} \colon \mathcal{A}_j \to \mathcal{A}_i$; moreover, we may assume that $\varphi_{ij}(b) = b$ and $\varphi_{ij}(a) = a$, where $\{b\}$ is the singleton orbit and a is the second generator of $\mathcal{D}_n$ for every $n < \omega$. It is easy to see that, for all $i, j < \omega$ with $i < j$ and all $u \in B_j$, we have $\beta_i(\pi_{ij}(u)) = \varphi_{ij}(\beta_j(u))$.

The proof of the following assertion repeats the proof of Claim 4 from the proof of [20, Theorem 4].

Claim 4. *If $(b_1, b_2) \in \ker \beta_j \setminus \ker \alpha_j$ for some $j < \omega$ then, for $i \leqslant j$, we have*

$$\big(\pi_{ij}(b_1), \pi_{ij}(b_2)\big) \in \ker \beta_i \setminus \ker \alpha_i.$$

Claim 5. *The differential groupoid $\mathcal{B}$ is pointwise non-separable with respect to $\mathbf{K}_I$.*

Proof of Claim. We consider the set

$$H_{-1} = \bigcup_{n<\omega} \bigl\{(h_1^n, h_2^n) \in B_n^2 \mid (h_1^n, h_2^n) \in \ker \beta_n \setminus \ker \alpha_n\bigr\}.$$

Since $(b_1^n, b_2^n) \in H_{-1} \cap B_n^2 \neq \varnothing$ for every $n < \omega$, the set H_{-1} is infinite. Since the set B_0 is finite, there is a pair $\overline{c}_0$ of elements of B_0 such that the set

$$H_0 = \bigcup_{n<\omega} \bigl\{(h_1^n, h_2^n) \in H_{-1} \mid \bigl(\pi_{0n}(h_1^n), \pi_{0n}(h_2^n)\bigr) = \overline{c}_0\bigr\}$$

is infinite. By Claim 4, we have $\overline{c}_0 \in \ker \beta_0 \setminus \ker \alpha_0$.

We use induction on i and construct a chain of infinite sets H_i, $i < \omega$. Assume that, for some $i < \omega$, we have already found pairs $\overline{c}_j \in \ker \beta_j \setminus \ker \alpha_j$ for $0 \leqslant j \leqslant i$ such that the set

$$H_j = \bigcup_{n \geqslant j} \bigl\{(h_1^n, h_2^n) \in H_{j-1} \mid \bigl(\pi_{jn}(h_1^n), \pi_{jn}(h_2^n)\bigr) = \overline{c}_j\bigr\}$$

is infinite and $\pi_{jj'}$ takes the pair $\overline{c}_{j'}$ into the pair $\overline{c}_j$ whenever $0 \leqslant j \leqslant j' \leqslant i$. Since H_i is an infinite set and B_i and B_{i+1} are finite sets, there is a pair $\overline{c}_{i+1}$ of elements of B_{i+1}^m such that the set

$$H_{i+1} = \bigcup_{n \geqslant i+1} \bigl\{(h_1^n, h_2^n) \in H_i \mid \bigl(\pi_{i+1,n}(h_1^n), \pi_{i+1,n}(h_2^n)\bigr) = \overline{c}_{i+1}\bigr\}$$

is infinite. By Claim 4, we have $\overline{c}_{i+1} \in \ker \beta_{i+1} \setminus \ker \alpha_{i+1}$. Moreover, $\overline{c}_{i+1} \in H_{i+1} \subseteq H_i$. We conclude therefore that $\pi_{i,i+1}$ takes the pair $\overline{c}_{i+1}$ into the pair $\overline{c}_i$. By the definition of an inverse spectrum and the induction hypothesis, we deduce that $\pi_{j,i+1}(\overline{c}_{i+1}) = \pi_{ji}\bigl(\pi_{i,i+1}(\overline{c}_{i+1})\bigr) = \pi_{ji}(\overline{c}_i) = \overline{c}_j$ whenever $j < i+1$.

We introduce a pair $\overline{c} = (c_1, c_2)$ of elements of B such that for all $n < \omega$ and $i \in \{1, 2\}$, $\pi_i(c_i)$ is the ith component of $\overline{c}_n$.

We consider a homomorphism $f\colon \mathcal{B} \to \mathcal{D}$, where $\mathcal{D} \in \mathbf{K}_I$ and $\mathcal{D}$ is a finite differential groupoid. By Lemma 7, there exist $n < \omega$ and a homomorphism $g\colon \mathcal{B}_n \to \mathcal{D}$ such that $f = g\pi_n$. By Lemma 3 and Claim 1, we obtain $g(\mathcal{B}_n) \cong \mathcal{D}_{[G]} \in \mathbf{K}_I$ for some $G \subseteq (\{k\} \cup F_n) \cap I = F_n$ or $g(\mathcal{B}_n) \cong \mathcal{D}_0$. We can find therefore a homomorphism $h\colon \mathcal{A}_n \to \mathcal{D}$ with $g = h\beta_n$. We conclude that $f = g\pi_n = h\beta_n\pi_n$ whence $\ker \beta_n\pi_n \subseteq \ker f$. As $\overline{c}_n \in \ker \beta_n$, we obtain $(c_1, c_2) \in \ker \beta_n\pi_n \subseteq \ker f$. □

By Claims 3 and 5 and Lemma 9, the quasivariety $\mathbf{K}_I$ is not standard.

It remains to notice that, in view of Claim 1, the quasivarieties of the form $\mathbf{K}_I$ are pairwise distinct. □

Following [9], a set Φ of first-order sentences is said to be *directed* if, for arbitrary $\varphi_0, \varphi_1 \in \Phi$, there is $\varphi \in \Phi$ such that both φ_0 and φ_1 are consequences of φ. Directed sets of first-order sentences are, in a sense, antagonists of independent sets of sentences.

Theorem 12. *There exist continuum many subquasivarieties in* **Dm** *that are not standard and have an independent quasi-equational basis* [*a directed quasi-equational basis, respectively*].

Proof. This proof is similar to the one of Theorem 10 but less complicated.

We consider an infinite set $I = \{i_n \mid n < \omega\}$ and assume that its members are ordered in the natural way. That is, we have $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. We put $F_{-1} = \varnothing$, $F_n = \{i_0, \ldots, i_n\}$, and $\mathcal{B}_n = \mathcal{D}_{[F_n]}$ for every $n < \omega$.

For each $m < \omega$, let ξ_m denote the quasi-identity

$$\forall x \, \forall y \; xy^{[F_m]} = x \;\&\; yx = y \;\rightarrow\; xy^{[F_{m-1}]} = x.$$

We denote

$$\Xi_I = \{\xi_m \mid m < \omega\} \quad \text{and} \quad \mathbf{M}_I = \operatorname{Mod} \Xi_I \cap \mathbf{Dm}.$$

Claim 1. *The set Ξ_I is an independent quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{Dm}$. The quasivariety $\mathbf{M}_I$ consists of all structures $\mathcal{A} \in \mathbf{Dm}$ with the following property:*

$$\text{if } F \in \mathcal{P}_{\text{fin}}(\omega) \text{ is nonempty and } \mathcal{D}_{[F]} \in \mathbf{S}(\mathcal{A}) \text{ then } F \nsubseteq I.$$

Proof of Claim. It is not difficult to see that $\mathcal{B}_n$ satisfies ξ_m if $m \neq n$ and violates ξ_n. Hence, Ξ_I is an independent set of quasi-identities.

If $\mathcal{D}_{[F]}$ is embeddable into $\mathcal{A}$ for no $F \in \mathcal{P}_{\text{fin}}(\omega)$ with $\varnothing \neq F \subseteq I$ then, for every $m < \omega$, the premise of the quasi-identity ξ_m can hold under no interpretation $\gamma \colon \{x, y\} \to A$ in view of Lemma 3. Therefore, $\mathcal{A} \models \xi_m$ for all $m < \omega$ and $\mathcal{A} \in \mathbf{M}_I$. Conversely, let $\mathcal{A} \in \mathbf{M}_I$ and let $\mathcal{D}_{[F]}$ embed into $\mathcal{A}$ for some nonempty set $F \in \mathcal{P}_{\text{fin}}(\omega)$ with $F \subseteq I$. Let i_n be the greatest element of F; then $F \subseteq F_n$. Therefore, the differential groupoid $\mathcal{D}_{[F]}$ satisfies the premise of ξ_n under the interpretation γ with $\gamma(x) = a$, $\gamma(y) = b$ but violates the conclusion of ξ_n under the same interpretation γ in view of Lemma 3. Hence, $\mathcal{A} \not\models \xi_n$ which contradicts our assumption $\mathcal{A} \in \mathbf{M}_I$. $\square$

Notice that Claim 1 also provides us with a directed quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{Dm}$. Namely, for each nonempty set $F \in \mathcal{P}_{\text{fin}}(\omega)$ with $F \subseteq I$, we introduce the following quasi-identity χ_F:

$$\forall x \, \forall y \; xy^{[F]} = x \;\&\; yx = y \;\rightarrow\; xy = x.$$

We consider the set $X = \{\chi_F \mid F \in \mathcal{P}_{\text{fin}}(\omega) \text{ and } \varnothing \neq F \subseteq I\}$.

Claim 2. *The set X forms a directed quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{Dm}$.*

Proof of Claim. It is clear that X is a quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{Dm}$. Moreover, the quasi-identities $\chi_F, \chi_G \in X$ are consequences of the quasi-identity $\chi_{F \cup G}$ for all nonempty sets $F, G \in \mathcal{P}_{\text{fin}}(\omega)$. $\square$

We construct an inverse spectrum $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ as follows. According to Lemma 3, for all $i, j < \omega$ with $i < j$, there is a surjective homomorphism $\pi_{ij} \colon \mathcal{B}_j \to \mathcal{B}_i$; moreover, we may assume that $\pi_{ij}(b) = b$ and $\pi_{ij}(a) = a$, where $\{b\}$ is the singleton orbit and a is the second generator of $\mathcal{D}_n$ for each $n < \omega$. According to the definition of an inverse spectrum, let π_{ii} be the identity automorphism of $\mathcal{B}_i$ for each $i < \omega$.

We immediately obtain the following assertion.

Claim 3. *The triple $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ is a surjective inverse spectrum.*

We put $\mathcal{B} = \varprojlim \Lambda$. Since each π_{ij} fixes a and b, we conclude that $B \neq \varnothing$. Moreover, we have $\mathcal{B} \in \mathbf{SP}(\mathbf{Dm}) \subseteq \mathbf{Dm}$. For every $n < \omega$, we denote by π_n the restriction of the nth projection $\prod_{n<\omega} B_n \to B_n$ to B.

Claim 4. *The differential groupoid $\mathcal{B}$ is infinite and belongs to $\mathbf{M}_I$.*

Proof of Claim. The same arguments as in the proof of Theorem 10 (see Claim 3 there) show that $\mathcal{B}$ is infinite.

We prove that $\mathcal{B} \in \mathbf{M}_I$. In view of Claim 2, it suffices to prove that $\mathcal{B} \models X$. Assume that $F \in \mathcal{P}_{\text{fin}}(\omega)$ and $\varnothing \neq F \subseteq I$. If the premise of χ_F holds in $\mathcal{B}$ under an interpretation $\gamma\colon \{x, y\} \to B$ then there exists a homomorphism $f\colon \mathcal{D}_{[F]} \to \mathcal{B}$ such that $f(a) = \gamma(x)$ and $f(b) = \gamma(y)$.

For all $n < \omega$ and $u \in D_{[F]}$, we put $f_n(u) = \pi_n f(u)$. We find that f_n is a homomorphism from $\mathcal{D}_{[F]}$ to $\mathcal{B}_n$. By Lemma 3, we have either $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_{[G]} \leq \mathcal{B}_n$ for some $G \subseteq F$ or $f_n(\mathcal{D}_{[F]}) \cong \mathcal{D}_0$. This means that $f_n(\mathcal{D}_{[F]})$ is either a subgroupoid of $\mathcal{D}_1$ or isomorphic to $\mathcal{B}_n$. In the second case, we have $|B_n| \leqslant [F]+1$. Since $\mathcal{D}_{[F]}$ is a finite structure and $|B_n| < |B_{n+1}|$ for each $n < \omega$, there is $s < \omega$ such that $[F]+1 < |B_n|$ whenever $n \geqslant s$. Therefore, the differential groupoid $f_n(\mathcal{D}_{[F]})$ with $n \geqslant s$ is a subgroupoid of $\mathcal{D}_1$.

The same arguments as in the proof of Theorem 10 (see Claim 3 there) show that the subgroupoid $f(\mathcal{D}_{[F]})$ of $\mathcal{B}$ generated by the set $\{a, b\}$ is a left-zero mode, i.e., the conclusion of χ_F holds in $\mathcal{B}$. □

Claim 5. *The differential groupoid $\mathcal{B}$ is pointwise non-separable with respect to $\mathbf{M}_I$.*

Proof of Claim. As π_n maps $\mathcal{B}$ onto $\mathcal{B}_n$ for all $n < \omega$, we have $\mathcal{B} \notin \mathbf{Q}(\mathcal{D}_1)$.

Consider an arbitrary homomorphism $f\colon \mathcal{B} \to \mathcal{D}$, where $\mathcal{D} \in \mathbf{M}_I$ is a finite differential groupoid. By Lemma 7, there are $n < \omega$ and $g\colon \mathcal{B}_n \to \mathcal{D}$ such that $f = g\pi_n$ and g is a homomorphism. By Lemma 3, we have either $g(\mathcal{B}_n) \cong \mathcal{D}_{[G]} \in \mathbf{M}_I$ for some $G \subseteq F_n \subseteq I$ or $g(\mathcal{B}_n) \cong \mathcal{D}_0$. Since $\mathcal{D}_{[G]} \in \mathbf{M}_I$, we have $G = \varnothing$ by Claim 2 in the first case. This means that $f(\mathcal{B}) \cong g(\mathcal{B}_n) \leq \mathcal{D}_1$ in any case, which contradicts the fact that $\mathcal{B} \notin \mathbf{Q}(\mathcal{D}_1)$. □

By Claims 4 and 5 and Lemma 9, the quasivariety $\mathbf{M}_I$ is not standard.

It remains to notice that, in view of Claim 1, the quasivarieties of the form $\mathbf{M}_I$ are pairwise distinct. □

3.2. **Unary algebras.**

Theorem 13. *There exist continuum many subquasivarieties of $\mathbf{V}$ that are not standard and have no independent quasi-equational basis.*

Proof. Let $I \subseteq \omega$ and let $F \in \mathcal{P}_{\text{fin}}(\omega)$. We denote by φ_F^I the quasi-identity

$$\begin{aligned}\forall x\, \forall y_0 \ldots \forall y_{[F]-1}\, \forall z_0 \ldots \forall z_{[F]-1}\, \big[&f(x) = x \wedge g(x) = x \wedge \\ &\wedge \bigwedge_{0\leqslant i<[F]} f(y_i) = x \wedge \bigwedge_{0\leqslant i<[F]} g(y_i) = x \wedge \bigwedge_{0\leqslant i<[F]} g(z_i) = y_i \wedge \\ &\wedge \bigwedge_{0\leqslant i<[F]-1} f(z_i) = y_{i+1} \wedge f(z_{[F]-1}) = y_0\big] \longrightarrow g(z_0) = y_{[F\cap I]}.\end{aligned}$$

This is the quasi-identity $q(n,m)$ from Lemma 5 with $n = [F \cap I]$, $m = [F \setminus I]$.

Let $\Phi_I = \{\varphi_F^I \mid \varnothing \neq F \in \mathcal{P}_{\text{fin}}(\omega)\}$ and let $\mathbf{K}_I$ be the subquasivariety of $\mathbf{V}$ defined by the set of quasi-identities Φ_I.

The following assertion is proven in [16, Theorem 8].

Proposition 14. *Each of the quasivarieties $\mathbf{K}_I$ admits an ω-independent quasi-equational basis but lacks an independent quasi-equational basis.*

We fix $I \in \mathcal{P}_{\text{inf}}(\omega)$ and $k \in \omega \setminus I$. Assume again that $I = \{i_n \mid n < \omega\}$ and $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. For every $n < \omega$, we put

$$F_n = \{i_0, \ldots, i_n\},\ \ \mathcal{A} = \mathcal{C}_{p_k},\ \ \mathcal{A}_n = \mathcal{C}_{[F_n]},\ \ \mathcal{B}_n = \mathcal{C}_{[\{k\}\cup F_n]} = \mathcal{C}_{p_k\cdot[F_n]}.$$

By the definition of $\mathbf{K}_I$ and Lemma 5, we immediately obtain the following assertion.

Claim 1. *For every $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, we have $\mathcal{C}_{[F]} \in \mathbf{K}_I$ if and only if $F \subseteq I$.*

In particular, we have $\mathcal{A} \notin \mathbf{K}_I$ and $\mathcal{A}_n \in \mathbf{K}_I$ but $\mathcal{B}_n \notin \mathbf{K}_I$ for every $n < \omega$.

We construct an inverse spectrum Λ. By Lemma 5, if $i < j < \omega$ then there is a surjective homomorphism $\pi_{ij} \colon \mathcal{B}_j \to \mathcal{B}_i$; moreover, we may assume that $\pi_{ij}(0) = 0$, $\pi_{ij}(a^p_u) = a^q_v$, and $\pi_{ij}(b^p_u) = b^q_v$, where $p = p_k \cdot [F_j]$, $q = p_k \cdot [F_i]$, and $u \equiv v \pmod q$. According to the definition of an inverse spectrum, let π_{ii} be the identity automorphism on $\mathcal{B}_i$ for every $i < \omega$.

The following assertion holds.

Claim 2. *The triple $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ is a surjective inverse spectrum.*

We put $\mathcal{B} = \varprojlim \Lambda$. Since each homomorphism φ_{ij} fixes the element 0, we conclude that $B \neq \varnothing$. Moreover, we have $\mathcal{B} \in \mathbf{SP}(\mathbf{V}) \subseteq \mathbf{V}$. For each $n < \omega$, we denote by π_n the restriction of the nth projection $\prod_{n<\omega} \mathcal{B}_n \to \mathcal{B}_n$ to $\mathcal{B}$. As there π_n is onto for all $n < \omega$, the following statement is true.

Claim 3. *We have $\mathcal{B} \not\models \forall x\, g(g(x)) = g(x)$.*

Claim 4. *The algebra $\mathcal{B}$ is infinite and belongs to $\mathbf{K}_I$.*

Proof of Claim. The same arguments as in the proof of Theorem 10 (see the proof of Claim 3 there) show that $\mathcal{B}$ is infinite. We prove that $\mathcal{B} \models \Phi_I$.

We consider a quasi-identity $\varphi^I_F \in \Phi_I$ with $F \neq \varnothing$ and assume that the premise of φ^I_F holds in $\mathcal{B}$ under an interpretation $\gamma \colon \{x, y_0, \dots, y_{[F]-1}, z_0, \dots, z_{[F]-1}\} \to B$. Then there exists a homomorphism $\varphi \colon \mathcal{C}_{[F]} \to \mathcal{B}$ such that $\varphi(0) = \gamma(x)$, $\varphi(a^{[F]}_i) = \gamma(y_i)$, and $\varphi(b^{[F]}_i) = \gamma(z_i)$ for $i < [F]$. For all $n < \omega$ and $u \in C_{[F]}$, we put $f_n(u) = \pi_n\varphi(u)$, i.e., we again consider the composition of the projection π_n and the homomorphism φ. Then f_n is a homomorphism from $\mathcal{C}_{[F]}$ to $\mathcal{B}_n$. By Lemma 5, one of the following cases is possible:

(a) we have $\mathcal{C}_{[G]} \leq f_n(\mathcal{C}_{[F]}) \leq \mathcal{B}_n$ for some nonempty set $G \subseteq F_n \cup \{k\}$,

(b) we have $f_n(\mathcal{C}_{[F]}) \models \forall x\, g(g(x)) = g(x)$.

In case (a), we have $G = F_n \cup \{k\}$ whence $|B_n| = 2[G] + 1 \leqslant 2[F] + 1$. Since $\mathcal{C}_{[F]}$ is a finite algebra and $|B_n| < |B_{n+1}|$ for each $n < \omega$, there is $s < \omega$ such that $2[F] + 1 < |B_n|$ for all $n \geqslant s$. Therefore, $f_n(\mathcal{C}_{[F]}) \models \forall x\, g(g(x)) = g(x)$ for each $n \geqslant s$. As $\mathcal{B} \cong \varprojlim_{j \geqslant s} \mathcal{B}_j$, we conclude that $\varphi(\mathcal{C}_F) \models \forall x\, g(g(x)) = g(x)$. Therefore, the conclusion of φ^I_F also holds in $\mathcal{B}$ under γ and $\mathcal{B} \models \varphi^I_F$. □

Claim 5. *There are elements $c_1, c_2 \in B$ such that $\big(\pi_n(c_1), \pi_n(c_2)\big) \in \ker \beta_n \setminus \ker \alpha_n$ and $\pi_n(c_1), \pi_n(c_2) \in A_n$ for all $n < \omega$.*

Proof of Claim. As in the proof of Claim 5 (see the proof of Theorem 10), one can establish the existence of elements $c_1, c_2 \in B$ such that $\big(\pi_n(c_1), \pi_n(c_2)\big) \in \ker \beta_n \setminus \ker \alpha_n$ for all $n < \omega$. It follows from Lemma 5(i) that one can choose $c_1, c_2 \in B$ so that $\pi_n(c_1), \pi_n(c_2) \in A_n$ for all $n < \omega$. □

Claim 6. *The algebra $\mathcal{B}$ is not profinite with respect to $\mathbf{K}_I$.*

Proof of Claim. By Lemma 5, the algebra $\mathcal{B}_n$ is a subdirect product of the algebras $\mathcal{A}$ and $\mathcal{A}_n$. As above, we denote the corresponding projections (which are onto homomorphisms) by $\alpha_n \colon \mathcal{B}_n \to \mathcal{A}$ and $\beta_n \colon \mathcal{B}_n \to \mathcal{A}_n$. According to Claim 5, there is a pair $\bar{c} = (c_1, c_2) \in B^2$ such that $\big(\pi_n(c_1), \pi_n(c_2)\big) \in \ker \beta_n \setminus \ker \alpha_n$ and $\pi_n(c_1), \pi_n(c_2) \in A_n$ for all $n < \omega$.

Suppose that $\mathcal{B} \cong \varprojlim_{t\in T} \mathcal{U}_t$ and $\mathcal{U}_t \in \mathbf{K}_I$ is a finite unary algebra for every $t \in T$. Then $\mathcal{B} \leq_s \prod_{t\in T} \mathcal{U}_t$. Let $\pi'_t\colon \mathcal{B} \to \mathcal{U}_t$ denote the canonical projection for every $t \in T$. It follows from Lemma 7 that, for each $t \in T$, there exist $n(t) < \omega$ and a homomorphism $\psi_t\colon \mathcal{B}_{n(t)} \to \mathcal{U}_t$ such that $\pi'_t = \psi_t\pi_{n(t)}$. Since π'_t is an onto homomorphism, we conclude that ψ_t is also onto. It follows from Lemma 5 that the following two cases are possible:

(a) there is a nonempty set $G_t \subseteq \{k\}\cup F_{n(t)}$ such that $\mathcal{C}_{[G_t]} \leq \psi_t(\mathcal{B}_{n(t)}) = \mathcal{U}_t \in \mathbf{K}_I$ and $U_t = \psi_t(B_{n(t)}) = \{0\} \cup A_{[G_t]} \cup \psi_t(B_{n(t)})$, where $A_{[G_t]} = \psi_t(A_{n(t)})$;
(b) we have $\psi_t(\mathcal{B}_{n(t)}) \models \forall x\, g(g(x)) = g(x)$.

Since $\mathcal{C}_{[G_t]} \in \mathbf{K}_I$ in case (a), we conclude that $G_t \subseteq (\{k\} \cup F_{n(t)}) \cap I = F_{n(t)}$ by Claim 1. Since $\mathcal{B} \not\models \forall x\, g(g(x)) = g(x)$ by Claim 3, the set

$$T_0 = \big\{t \in T \mid \mathcal{C}_{[G_t]} \leq \psi_t(\mathcal{B}_{n(t)}) \text{ for some nonempty } G_t \subseteq F_{n(t)}\big\}$$

is coinitial in T. This implies that $\mathcal{B} \cong \varprojlim_{t\in T_0} \mathcal{U}_t$ whence $\mathcal{B} \leq_s \prod_{t\in T_0} \mathcal{U}_t$.

For each $t \in T_0$, we can find a homomorphism $\vartheta_t\colon \mathcal{A}_{n(t)} \to \mathcal{C}_{[G_t]} \leq \mathcal{U}_t$ such that $\vartheta_t\beta_{n(t)}(a) = \psi_t(a)$ for every $a \in A_{n(t)}$. This yields for each $i \in \{1, 2\}$:

$$\pi'_t(c_i) = \psi_t\pi_{n(t)}(c_i) = \vartheta_t\beta_{n(t)}\pi_{n(t)}(c_i).$$

Inclusion $\big(\pi_{n(t)}(c_1), \pi_{n(t)}(c_2)\big) \in \ker\beta_{n(t)}$ implies that $(c_1, c_2) \in \ker\pi'_t$ for every $t \in T_0$. As $\mathcal{B} \leq_s \prod_{t\in T_0} \mathcal{U}_t$, we have $c_1 = c_2$ in $\mathcal{B}$, which contradicts Claim 5. □

By Claims 4 and 6, the quasivariety $\mathbf{K}_I$ is not standard.

In view of Claim 1, the quasivarieties of the form $\mathbf{K}_I$ are pairwise distinct. It remains to refer to Proposition 14. □

Theorem 15. *There exist continuum many subquasivarieties of* $\mathbf{V}$ *that are not standard and have an independent quasi-equational basis* [*a directed quasi-equational basis, respectively*].

Proof. These arguments are similar to the proofs of Theorems 12 and 13.

We consider an infinite set $I = \{i_n \mid n < \omega\}$ and assume that its members are ordered in the natural way, i.e., $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. We put $F_{-1} = \varnothing$, $F_n = \{i_0, \dots, i_n\}$, and $\mathcal{B}_n = \mathcal{C}_{[F_n]}$ for every $n < \omega$.

For each $m < \omega$, let ξ_m denote the quasi-identity $q([F_m], [F_{m-1}])$, i.e.,

$$\begin{aligned}\forall x\,\forall y_0 \dots \forall y_{[F_m]-1}\,\forall z_0 \dots \forall z_{[F_m]-1}\,\big[f(x) = x \wedge g(x) = x \wedge{}\\ \wedge \bigwedge_{0\leqslant i<[F_m]} f(y_i) = x \wedge \bigwedge_{0\leqslant i<[F_m]} g(y_i) = x \wedge \bigwedge_{0\leqslant i<[F_m]} g(z_i) = y_i \wedge{}\\ \wedge \bigwedge_{0\leqslant i<[F_m]-1} f(z_i) = y_{i+1} \wedge f(z_{[F_m]-1}) = y_0\big] \longrightarrow g(z_0) = y_{[F_{m-1}]}.\end{aligned}$$

We denote

$$\Xi_I = \{\xi_m \mid m < \omega\} \quad \text{and} \quad \mathbf{M}_I = \operatorname{Mod}\Xi_I \cap \mathbf{V}.$$

Claim 1. *The set* Ξ_I *forms an independent quasi-equational basis of* $\mathbf{M}_I$ *within* $\mathbf{V}$. *The quasivariety* $\mathbf{M}_I$ *consists of all structures* $\mathcal{A} \in \mathbf{V}$ *with the following property:*

if $F \in \mathcal{P}_{\rm fin}(\omega)$ *is nonempty and* $\mathcal{C}_{[F]} \in \mathbf{S}(\mathcal{A})$ *then* $F \nsubseteq I$.

Proof of Claim. It is not difficult to see that $\mathcal{B}_n$ satisfies ξ_m if $m \neq n$ and violates ξ_n. Hence, Ξ_I is an independent set of quasi-identities.

If $\mathcal{C}_{[F]}$ is embeddable into $\mathcal{A}$ for no $F \in \mathcal{P}_{\text{fin}}(\omega)$ with $\varnothing \neq F \subseteq I$ then, for every $m < \omega$, the premise of the quasi-identity ξ_m holds in $\mathcal{A}$ under no interpretation $\gamma\colon \{x, y_0, \ldots, y_{[F]-1}, z_0, \ldots, z_{[F]-1}\} \to A$ in view of Lemma 5. Therefore, $\mathcal{A} \models \xi_m$ for all $m < \omega$ and $\mathcal{A} \in \mathbf{M}_I$. Conversely, let $\mathcal{A} \in \mathbf{M}_I$ and let $\mathcal{C}_{[F]}$ embed into $\mathcal{A}$ for some nonempty set $F \in \mathcal{P}_{\text{fin}}(\omega)$ with $F \subseteq I$. Let i_n be the greatest element of F; then $F \subseteq F_n$. Therefore, the algebra $\mathcal{C}_{[F]}$ satisfies the premise of ξ_n under the interpretation γ with

$$\gamma(x) = 0, \quad \gamma(y_i) = a^{[F]}_{i(\operatorname{mod}[F])}, \quad \gamma(z_i) = b^{[F]}_{i(\operatorname{mod}[F])}, \quad 0 \leqslant i < [F_n],$$

but violates the conclusion of ξ_n under the same interpretation γ, see Lemma 5. Hence, $\mathcal{A} \not\models \xi_n$ which contradicts our assumption $\mathcal{A} \in \mathbf{M}_I$. □

As above, Claim 1 also provides us with a directed quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{V}$. Namely, for each nonempty set $F \in \mathcal{P}_{\text{fin}}(\omega)$ with $F \subseteq I$, we introduce the following quasi-identity χ_F:

$$\begin{aligned} &\forall x\, \forall y_0 \ldots \forall y_{[F]-1}\, \forall z_0 \ldots \forall z_{[F]-1}\, \forall y\, \big[f(x) = x \wedge g(x) = x \wedge \\ &\qquad \wedge \bigwedge_{0 \leqslant i < [F]} f(y_i) = x \wedge \bigwedge_{0 \leqslant i < [F]} g(y_i) = x \wedge \bigwedge_{0 \leqslant i < [F]} g(z_i) = y_i \wedge \\ &\qquad\qquad \wedge \bigwedge_{0 \leqslant i < [F]-1} f(z_i) = y_{i+1} \wedge f(z_{[F]-1}) = y_0\big] \longrightarrow x = y. \end{aligned}$$

We consider the set $X = \{\chi_F \mid F \in \mathcal{P}_{\text{fin}}(\omega) \text{ and } \varnothing \neq F \subseteq I\}$.

Claim 2. *The set X forms a directed quasi-equational basis of $\mathbf{M}_I$ within $\mathbf{V}$.*

We construct an inverse spectrum $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ as follows. According to Lemma 5, for all $i, j < \omega$ with $i < j$, there is a surjective homomorphism $\pi_{ij}\colon \mathcal{B}_j \to \mathcal{B}_i$; moreover, we may assume that $\pi_{ij}(0) = 0$, $\pi_{ij}(a^{[F_j]}_k) = a^{[F_i]}_l$, and $\pi_{ij}(b^{[F_j]}_k) = b^{[F_i]}_l$, where $k \equiv l \pmod{[F_i]}$. According to the definition of an inverse spectrum, let π_{ii} be the identity automorphism of $\mathcal{B}_i$ for each $i < \omega$.

We immediately obtain the following assertion.

Claim 3. *The triple $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ is a surjective inverse spectrum.*

We put $\mathcal{B} = \varprojlim \Lambda$. Since each π_{ij} fixes 0, we conclude that $B \neq \varnothing$. Moreover, we have $\mathcal{B} \in \mathbf{SP}(\mathbf{V}) \subseteq \mathbf{V}$. For every $n < \omega$, we denote by π_n the restriction of the nth projection $\prod_{n<\omega} \mathcal{B}_n \to \mathcal{B}_n$ to $\mathcal{B}$. As π_n is onto for every $n < \omega$, the following statement is true.

Claim 4. $\mathcal{B} \not\models \forall x\, g(g(x)) = g(x)$.

Claim 5. *The algebra $\mathcal{B}$ is infinite and belongs to $\mathbf{M}_I$.*

Proof of Claim. The same arguments as in the proof of Theorem 12 (see Claim 4 there) show that B is infinite.

We prove that $\mathcal{B} \in \mathbf{M}_I$. In view of Claim 2, it suffices to prove that $\mathcal{B} \models X$. Assume that $F \in \mathcal{P}_{\text{fin}}(\omega)$ and $\varnothing \neq F \subseteq I$. If the premise of χ_F holds in $\mathcal{B}$ under an interpretation $\gamma\colon \{x, y_0, \ldots, y_{[F]-1}, z_0, \ldots, z_{[F]-1}\} \to B$ then there exists a homomorphism $\varphi\colon \mathcal{C}_{[F]} \to \mathcal{B}$ such that $\varphi(0) = 0$, $\varphi(a_k^{[F]-1}) = \gamma(y_k^{[F]-1})$, and $\varphi(b_k^{[F]-1}) = \gamma(z_k^{[F]-1})$ for $0 \leqslant k < [F]$.

For all $n < \omega$ and $u \in C_{[F]}$, we put $f_n(u) = \pi_n\varphi(u)$. We find that f_n is a homomorphism from $\mathcal{C}_{[F]}$ to $\mathcal{B}_n$. By Lemma 5, one of the following cases occurs:

(a) we have $\mathcal{C}_{[G]} \leq f_n(\mathcal{C}_{[F]}) \leq \mathcal{B}_n$ for some nonempty set $G \subseteq F$,

(b) we have $f_n(\mathcal{C}_{[F]}) \models \forall x\, g(g(x)) = g(x)$.

In case (a), we have $F_n = G \subseteq F$ whence $|B_n| = 2[G] + 1 \leqslant 2[F] + 1$. Since $\mathcal{C}_{[F]}$ is a finite algebra and $|B_n| < |B_{n+1}|$ for each $n < \omega$, there is $s < \omega$ such that $2[F] + 1 < |B_n|$ for all $n \geqslant s$. Therefore, $f_n(\mathcal{C}_{[F]}) \models \forall x\, g(g(x)) = g(x)$ for all $n \geqslant s$. As $\mathcal{B} \cong \varprojlim_{j\geqslant s} \mathcal{B}_j$, we conclude that $\mathcal{B} \models \forall x\, g(g(x)) = g(x)$, which contradicts Claim 4. Therefore, the premise of χ_F holds in $\mathcal{B}$ under no interpretation γ and $\mathcal{B} \models \chi_F$. □

Claim 6. *The algebra $\mathcal{B}$ is not profinite with respect to $\mathbf{M}_I$.*

Proof of Claim. Consider an arbitrary homomorphism $\varphi\colon \mathcal{B} \to \mathcal{D}$, where $\mathcal{D} \in \mathbf{M}_I$ is a finite structure. By Lemma 7, there is $n < \omega$ and a homomorphism $\psi\colon \mathcal{B}_n \to \mathcal{D}$ such that $\varphi = \psi\pi_n$. By Lemma 5, one of the following cases occurs:

(a) we have $\mathcal{C}_{[G]} \leq \psi(\mathcal{B}_n) \in \mathbf{M}_I$ for some nonempty set $G \subseteq F_n$;

(b) we have $\psi(\mathcal{B}_n) \models \forall x\, g(g(x)) = g(x)$.

Since $\mathcal{C}_{[G]} \in \mathbf{M}_I$ in case (a) and $\varnothing \neq G \subseteq F_n \subseteq I$, we arrive at a contradiction with Claim 2. Therefore, case (a) is impossible. Thus, we conclude that case (b) takes place and $\varphi(\mathcal{B}) = \psi(\mathcal{B}_n) \models \forall x\, g(g(x)) = g(x)$. Hence if the unary algebra $\mathcal{B}$ were profinite with respect to $\mathbf{M}_I$, it would satisfy the identity $\forall x\, g(g(x)) = g(x)$ which contradicts Claim 4. □

By Claims 5 and 6, the quasivariety $\mathbf{M}_I$ is not standard.

It remains to notice that, in view of Claim 1, the quasivarieties of the form $\mathbf{M}_I$ are pairwise distinct. □

For quasivarieties possessing finite B-classes, similar results on the existence of directed quasi-equational bases (cf. Claims 2 in the proofs of Theorems 12 and 15) were established in [14].

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