S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
`http://semr.math.nsc.ru`

Том 19, №2, стр. 792–803 (2022) УДК 512.554.7
DOI 10.33048/semi.2022.19.066 MSC 17C70

DUAL COALGEBRA OF THE DIFFERENTIAL POLYNOMIAL ALGEBRA IN ONE VARIABLE AND RELATED COALGEBRAS

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Abstract. We show that the dual coalgebra of the polynomial algebra in one variable is the space of linearly recursive sequences over an arbitrary field. Moreover, this coalgebra is a differential one relative to the dual standard derivation and does not contain nonzero finite-dimensional differentially closed subcoalgebras if the characteristic of the ground field is zero. We construct a Novikov coalgebra which is the dual coalgebra of the left-symmetric Witt algebra of index one. Also, we construct a Jordan supercoalgebra which is dual to the Jordan superalgebra of vector type of the polynomial algebra in one variable. All these coalgebras do not contain non-zero finite-dimensional subcoalgebras if the characteristic of ground field is zero. It is shown that over a field of characteristic different from 2 the adjoint Lie coalgebra of the dual coalgebra of the left-symmetric Witt algebra of index one is isomorphic to the dual coalgebra of the Witt algebra of index one.

Key words: coalgebra, coderivation, associative commutative algebra, differential algebra, Novikov algebra, Lie algebra, Witt algebra, Jordan superalgebra, locally finite coalgebra

In general, an algebra is a vector space A over a ground field F equipped with a linear map $m : A \otimes A \to A$. The dual notion is known as a coalgebra. The theory of coalgebras has been initially developed within the framework of the theory of Hopf algebras [1]. The main result of the theory of associative coalgebras is the Fundamental Theorem on Coalgebras, which asserts that every associative coalgebra over a field is locally finite. The latter means that every finitely generated coalgebra is finite-dimensional.

Zhelyabin, V.N., Kolesnikov, P.S. Dual coalgebra of the differential polinomial algebra in one variable and related coalgebras.

Supported by RAS Fundamental Research Program, project FWNF-2022-0002 .

Поступила 24 июня 2022 г., опубликована 11 ноября 2022 г.

The interest to nonassociative coalgebras is related to the notion of a quantum group introduced by Drinfeld [2]. Initially, it was associated with the notion of a Lie bialgebra. The latter was introduced in [3] as one of the most important notions of quantum group theory. A Lie bialgebra is simultaneously a Lie algebra and a Lie coalgebra. In contrast to associative coalgebras there exist Lie coalgebras that are not locally finite [4].

It is known (see [1, 4]) that the dual algebra of an associative coalgebra or a Lie coalgebra is associative or Lie, respectively. In [5] (1994), the notion of a coalgebra related to some variety of algebras was introduced. In particular, alternative and Jordan coalgebras were defined, and their local finiteness was proved. An analogue of this result is true for structurable coalgebras [6], for Jordan copairs [7], for right alternative Malcev admissible coalgebras, and for binary $(-1,1)$-coalgebras [8]. In [9] (1995), some necessary and sufficient conditions for a Lie coalgebra to be locally finite were found.

In the papers [10, 11] the first author found the connection between Jordan and Lie (super)coalgebras, which is an analogue of the well-known Kantor–Koecher–Tits construction for usual (super)algebras. In [12, 13], it was shown that every Malcev coalgebra is embedded into a Lie coalgebra with triality.

As mentioned above, there exist non-locally finite Lie coalgebras. In [14], an example of a non-locally finite right-symmetric coalgebra was constructed. In contrast to Jordan coalgebras, non-locally finite Jordan supercoalgebras exist [11].

In [15], it was constructed an example of a non-locally finite differential coalgebra. On this differential coalgebra we can define a Lie comultiplication so that the obtained Lie coalgebra coincides with the example of Michaelis presented in [4]. The dual analogue of the Gelfand–Dorfman construction was proposed in [15], which implied the construction of Novikov coalgebras based on differential associative commutative coalgebras. Using this construction, an example of a non-locally finite Novikov coalgebra was built. A dual analogue of the Kantor construction for usual Jordan superalgebra was also presented in [15], as a corollary, a new example of a non-locally finite Jordan supercoalgebra was constructed.

In [15] and [16], examples non-locally finite right alternative coalgebras were constructed.

In [17], it was shown that the dual coalgebra W_1° of the Witt Lie algebra W_1 is a non-zero Lie coalgebra which does not contain non-zero finite-dimensional subcoalgebras. An analogue of this result for Jordan supercoalgebra was obtained in [21]. The structure of the dual Lie coalgebra of the Witt algebra over field of characteristic not 2 and zero was also described in [18, 19]. These results were generalized to the case of several derivations in [20]. Namely in [18] was shown that W_1° is the space of linearly recursive sequences, if $\operatorname{char} F = 0$ or $\operatorname{char} F = p > 2$.

It is known that both the Witt Lie algebra W_1 and the left-symmetric algebra $\mathcal{L}_1$ can be obtained from the differential polynomial algebra in one variable by means of appropriate constructions. It follows from [19, Theorem 1] that the dual coalgebra W_1° is obtained from P_1° by means of the dual construction. The purpose of this paper is to obtain the dual analogue of these results for the dual coalgebra $\mathcal{L}_1^\circ$.

In particular, we prove that the dual coalgebra P_1° of the algebra P_1 in one variable is the space of linearly recursive sequences over an arbitrary field, Moreover, if the characteristic of ground field is zero then the coalgebra P_1° does not contain finite-dimensional differentially closed subcoalgebras.

We also show that over field of characteristic different from 2 the dual coalgebra of the left-symmetric Witt algebra of index one can be obtained from the coalgebra P_1° by means of the dual Gelfand–Dorfman construction.

Finally, we describe the dual supercoalgebra of a Jordan superalgebra obtained by the Kantor construction from the differential polynomial algebra in one variable.

1. COALGEBRAS AND CODERIVATIONS

Let F be an arbitrary field. Denote by $\underbrace{V\otimes\ldots\otimes V}_{n}$ the nth tensor power of the vector space V over F. Denote by V^* the dual vector space of V, i. e., $V^* = \mathrm{Hom}_F(V,F)$.

The map

$$\rho : \underbrace{V^*\otimes\ldots\otimes V^*}_{n} \to (\underbrace{V\otimes\ldots\otimes V}_{n})^*$$

defined by

$$\rho(f_1\otimes\ldots\otimes f_n)\Big(\sum_{i_1,\ldots,i_n} v_{i_1}\otimes\ldots\otimes v_{i_n}\Big) = \sum_{i_1,\ldots,i_n} f_1(v_{i_1})\ldots f_n(v_{i_n})$$

is injective. Therefore, we can assume that

$$\underbrace{V^*\otimes\ldots\otimes V^*}_{n} \subseteq (\underbrace{V\otimes\ldots\otimes V}_{n})^*.$$

If $\phi : V \to U$ is a linear map of vector spaces then the transpose $\phi^* : U^* \to V^*$ of ϕ is defined by the rule $\phi^*(u^*)(v) = u^*(\phi(v))$, where $v\in V$, $u^*\in U^*$.

Definition 1. *A pair (C,Δ), where C is a vector space over F and $\Delta : C\to C\otimes C$ is a linear map, is called a coalgebra. The map Δ is said to be a comultiplication of C. For an element $a\in C$, we will use the Sweedler's notation (see* [1]*), namely, $\Delta(a) = \sum\limits_{(a)} a_{(1)}\otimes a_{(2)}$.*

If (C,Δ) is a coalgebra then the rule $f\otimes g\mapsto fg$, where

$$(fg)(a) = \rho(f\otimes g)(\Delta(a)) = \sum_{(a)} f(a_{(1)})g(a_{(2)}), \quad f,g\in C^*,\ a\in C,$$

defines a product on C^*, so that C^* is an algebra. The algebra C^* is called the *dual algebra* of (C,Δ).

The dual algebra C^* has natural left and right actions on the initial coalgebra C. Namely,

$$\alpha\cdot a = \sum_{(a)} a_{(1)}\alpha(a_{(2)}), \quad a\cdot\alpha = \sum_{(a)} \alpha(a_{(1)})a_{(2)},$$

for $\alpha\in C^*$, $a\in C$. Hence, C is a C^*-bimodule.

A linear map $d : C\to C$ is called a *coderivation* of a coalgebra (C,Δ) if

$$\Delta d = (d\otimes id + id\otimes d)\Delta,$$

i. e., for every $a\in C$

$$\Delta(d(a)) = \sum_{(a)} d(a_{(1)})\otimes a_{(2)} + a_{(1)}\otimes d(a_{(2)}).$$

The following statement is well-known.

Lemma 1. *Let* d *be a coderivation of a coalgebra* (C, Δ)*. Then its transpose map* d^* *is a derivation of the dual algebra* C^**, i. e.,*

$$d^*(fg) = d^*(f)g + fd^*(g)$$

holds in the algebra C^* *for all* $f, g \in C^*$*.*

A coalgebra equipped with a coderivation is called a *differential coalgebra.*

A coalgebra (C, Δ) is said to be associative (coassociative) if

$$(\Delta \otimes id - id \otimes \Delta)\Delta = 0,$$

i. e., for every $a \in C$

$$\sum_{(a)} (\Delta(a_{(1)}) \otimes a_{(2)} - a_{(1)} \otimes \Delta(a_{(2)})) = 0.$$

It is known that a coalgebra (C, Δ) is associative if and only if its dual algebra C^* is associative. By this reason, the following definition of a coalgebra related to some variety of algebras was stated in [5].

Definition 2. *Let* $\mathcal{M}$ *be a variety of algebras. Then a coalgebra* (C, Δ) *is called an* $\mathcal{M}$*-coalgebra if its dual algebra* C^* *is an algebra of* $\mathcal{M}$*.*

Let V be a vector space, and let the linear map $\tau : V \otimes V \to V \otimes V$ is defined by $\tau(x \otimes y) = y \otimes x$, $x, y \in V$.

A coalgebra (C, Δ) is commutative (cocommutative) if

$$\Delta = \tau\Delta,$$

i. e.,

$$\sum_{(a)} a_{(1)} \otimes a_{(2)} = \sum_{(a)} a_{(2)} \otimes a_{(1)}$$

for every $a \in C$.

Let (C, Δ) be an arbitrary coalgebra. A vector subspace B of C is a *subcoalgebra* of (C, Δ) if $\Delta(B) \subseteq B \otimes B$.

It is known (see [5]) that a vector space B of C is a subcoalgebra if and only if B is a submodule of the C^*-bimodule C. Therefore, the intersection of a family of subcoalgebras of (C, Δ) is again a subcoalgebra of C.

Recall that the orthogonal complement

$$B^{\perp} = \{\alpha \in C^* \mid B \subseteq \ker \alpha\}$$

of a subcoalgebra B of (C, Δ) is an ideal of the algebra C^*. Conversely, the orthogonal complement $I^{\perp}$ of an ideal I of C^* is a subcoalgebra of (C, Δ) (see [1, Proposition 1.4.9]).

Let S be a subset of C. The smallest subcoalgebra which contains S is called the *subcoalgebra generated by* S and denoted by $Coalg(S)$. In other words, $Coalg(S)$ is the submodule of the C^*-bimodule C, generated by S. If S is a finite set then $Coalg(S)$ is called finitely generated coalgebra.

Definition 3. *A coalgebra* (C, Δ) *is called locally finite if every finitely generated subcoalgebra of* C *is finite-dimensional.*

Let (C, Δ) be a coalgebra. Denote by $Loc(C)$ the sum of all locally finite subcoalgebras of C. It is clear that $Loc(C)$ is a locally finite coalgebra.

Let A be an algebra over a field F with a multiplication $m : A \otimes A \to A$, i. e., $m(a \otimes b) = ab$ for $a, b \in A$. Then $m^* : A^* \to (A \otimes A)^*$ is the transpose map of m. A vector subspace V of A^* is called *good* if $m^*(V) \subseteq \rho(V \otimes V)$. If V is a good subspace then we can define the comultiplication Δ_V on V by the rule $\Delta_V = \rho^{-1} m^*$.

Let A° be the sum of all good subspaces of A^*. Then A° is the largest good subspace of A^*, and hence the pair (A°, Δ°) is a coalgebra with the comultiplication $\Delta^\circ = \Delta_{A^\circ}$ [4, 5]. The coalgebra (A°, Δ°) is called the *finite dual coalgebra* (or simply *dual coalgebra*) of A. For every $a, b \in A$ and for every $f \in A^\circ$ we have $f(ab) = \sum_f f_{(1)}(a) f_{(2)}(b)$, where $\Delta^\circ(f) = \sum_f f_{(1)} \otimes f_{(2)}$.

If A is finite dimensional then $A^* \otimes A^* \equiv \rho(A^* \otimes A^*) = (A \otimes A)^*$ and the dual space A^* is a coalgebra with the comultiplication $\Delta = \rho^{-1} m^*$. Therefore, $(A^\circ, \Delta^\circ) = (A^*, \Delta)$.

In [5], it was shown that the dual coalgebra A° is a $\mathcal{M}$-coalgebra if A is an algebra of variety $\mathcal{M}$.

Lemma 2. *Let A be an algebra over a field F, and let (V, Δ) be a coalgebra, where $V \subseteq A^*$. If $f(ab) = \rho(\Delta(f))(a \otimes b)$ holds for every $a, b \in A$ and for every $f \in V$ then V is a good subspace of A^*.*

Proof. By the definition of m^*, we have $m^*(f)(a \otimes b) = f(ab)$ for all $a, b \in A$, $f \in A^*$. If $f(ab) = \rho(\Delta(f))(a \otimes b)$ holds for $f \in V$ then

$$m^*(f)(a \otimes b) = \rho(\Delta(f))(a \otimes b).$$

Consequently, $m^*(f) = \rho(\Delta(f))$. Thus $m^*(V) \subseteq \rho(V \otimes V)$. □

Let $\mathcal{I}$ be the set of finite-codimensional ideals of an algebra A. Let us define

$$A^*_{\mathcal{I}} = \{\alpha \in A^* \mid \text{exists } I \in \mathcal{I} \text{ such that } I \subseteq \ker \alpha\}.$$

As it was shown in [5], $A^*_{\mathcal{I}} = Loc(A^\circ)$. It is known that $A^*_{\mathcal{I}} = Loc(A^\circ) = A^\circ$ for every associative algebra A.

2. Dual coalgebra of the differential polynomial algebra $F[x]$

Let $P_1 = F[x]$ be the polynomial algebra in one variable equipped with the standard derivation $d = \frac{d}{dx}$. Then

$$P_1^\circ = \{\alpha \in P_1^* \mid \text{ exists } f(x) \in F[x], f(x) \neq 0 \text{ such that } f(x)F[x] \subseteq \ker \alpha\}.$$

Let $\Delta^\circ = \Delta_{P_1^\circ}$. For $\alpha \in P_1^*$ and $a \in P_1$ put $\langle \alpha, a \rangle = \alpha(a)$. Also, let us denote $d^\circ = d^*|_{P_1^\circ}$, where d^* is the transpose map of the derivation d. Let $(P_1, d)^\circ$ be the sum of all good subspaces of P_1^* which are differential coalgebras with the coderivation d°.

Lemma 3. *The inclusion $d^\circ(P_1^\circ) \subseteq P_1^\circ$ holds. The triple $(P_1^\circ, \Delta^\circ, d^\circ)$ is an associative and commutative differential coalgebra with the coderivation d°. Moreover, $(P_1^\circ, \Delta^\circ, d^\circ) = (P_1, d)^\circ$.*

Proof. Let us show that $d^\circ(P_1^\circ) \subseteq P_1^\circ$. Suppose $u \in P_1^\circ$, then there exists $I = f(x)P_1$, $f(x) \neq 0$, such that $\langle u, I \rangle = 0$. Consider $J = I^2$: this is an ideal of finite codimension in P_1, and $\langle d^*(u), J \rangle = \langle u, d(J) \rangle \subseteq \langle u, I \rangle = 0$ since $d(I^2) \subseteq Id(I) \subseteq I$.

Obviously, $(P_1^\circ, \Delta^\circ, d^\circ)$ is a differential coalgebra with coderivation d°. Hence, $(P_1^\circ, \Delta^\circ, d^\circ) = (P_1, d)^\circ$. $\square$

Remark 1. *The proof of Lemma 3 remains valid for an arbitrary associative commutative algebra A equipped with an arbitrary derivation D. Hence, the equality $(A^\circ, \Delta^\circ, D^\circ) = (A, D)^\circ$ holds in general.*

Theorem 1. *Let F be a field of characteristic zero. Then the coalgebra $(P_1^\circ, \Delta^\circ, d^\circ)$ does not contain non-zero finite-dimensional differentially closed subcoalgebras.*

Proof. Let B be a finite-dimensional differentially closed subcoalgebra of the differential coalgebra $(P_1^\circ, \Delta^\circ, d^\circ)$. Then the orthogonal complement $B^\perp$ is a differentially closed ideal of P_1. Moreover, the ideal $B^\perp$ has finite codimension. Since the characteristic of F is zero then P_1 is a differentially simple algebra. Therefore, either $B^\perp = 0$ or $B^\perp = P_1$. The first option is impossible since P_1 is an infinite-dimensional algebra. Consenquently, $B^\perp = P_1$ and $B = 0$. $\square$

Let W_1 be the Witt algebra of index one, i. e., $W_1 = \mathrm{Der}_F(P_1)$; P_1^* and W_1^* are isomorphic as vector spaces.

Recall that the vector space P_1 is a Lie algebra relative to the operation

$$[f, g]_d = fd(g) - d(f)g,$$

this Lie algebra is isomorphic to the Witt algebra W_1 under isomorphism $f \mapsto fd$. Thus, we can identify $(P_1, [\,,]_d)$ and W_1.

Define the new comultiplication $\Delta_{d^\circ}^{(-)}$ on the vector space P_1°:

$$\Delta_{d^\circ}^{(-)}(\alpha) = (id \otimes d^\circ - d^\circ \otimes id)\Delta^\circ(\alpha).$$

Then

$$\langle \alpha, [f, g]_d \rangle = \langle \alpha, fd(g) - d(f)g \rangle = \rho(\Delta_{d^\circ}^{(-)}(\alpha))(f \otimes g)$$

for $\alpha \in P_1^\circ$ and $f, g \in P_1$. Therefore, P_1° is a good subspace of W_1^*. Consequently, we can assume that $(P_1^\circ, \Delta_{d^\circ}^{(-)})$ is a subcoalgebra of $(W_1^\circ, \Delta_{W_1^\circ})$.

In the algebra P_1 we put $x_i = x^{i+1}$, $i = -1, 0, 1, \ldots$. Then we have $x_i \cdot x_j = x_{i+j+1}$ in the algebra P_1.

We identify elements of P_1^* with sequences of elements of F. Namely, every $\alpha \in P_1^*$ corresponds to $(\alpha(x_n))_{n \geq -1}$.

Following [18], we say that a sequence $(a_n)_{n \geq -1}$ of elements of F is $(F$-$)$*linearly recursive* if there exist $\beta_0, \beta_1, \ldots, \beta_r \in F$, not all equal to zero, and a number k such that $\sum_{i=0}^{r} \beta_i a_{n+i} = 0$ for all $n \geq k$.

In the next statement, the base field F is of arbitrary characteristic.

Theorem 2. *Let V be the space of linearly recursive sequences. Then $P_1^\circ = V$. In particular, the space V is the dual coalgebra of P_1. Therefore, the coalgebra $(V, \Delta_{d^\circ}^{(-)})$ is a Lie subcoalgebra of $(W_1)^\circ$.*

Proof. Let $\alpha \in P_1^*$ and $a_n = \alpha(x_n)$, where $n \geq -1$.

Assume $\alpha \in P_1^\circ$. We show that the sequence $(a_n)_{n \geq -1}$ is linearly recursive. There exists $f(x) \in P_1$, $f(x) \neq 0$, such that $\alpha(f(x)g(x)) = 0$ for all $g(x) \in P_1$. Let $f(x) = \beta_r x_{r-1} + \ldots + \beta_0 x_{-1}$, where $\beta_0, \beta_1, \ldots, \beta_r \in F$. Since $f(x) \neq 0$, not all $\beta_0, \beta_1, \ldots, \beta_r$ are zero. Then for all $n \geq 0$ we have

$$\sum_{i=0}^{r} \beta_i a_{n+i} = \sum_{i=0}^{r} \beta_i \alpha(x_{n+i}) = \sum_{i=0}^{r} \beta_i \alpha(x^{n+i+1}) = \alpha(f(x)x^{n+1}) = 0.$$

Hence, the sequence $(a_n)_{n\geq -1}$ is linearly recursive.

Let $(a_n)_{n\geq -1}$ be a linearly recursive sequence. Then there exist $\beta_0, \beta_1, ..., \beta_r \in F$, not all zero, and a number k, such that $\sum_{i=0}^{r}\beta_i a_{n+i} = 0$ for all $n \geq k$. Put $f(x) = \beta_r x_{k+r} + \ldots + \beta_0 x_k$. Then for all $j \geq -1$ we get

$$\alpha(f(x)x_j) = \sum_{i=0}^{r}\beta_i\alpha(x_{k+i+j+1}) = \sum_{i=0}^{r}\beta_i a_{k+i+j+1} = 0,$$

since $k+j+1 \geq k$. Consequently, $f(x)P_1 \subseteq \ker\alpha$.

Since $(P_1^\circ, \Delta_{d^\circ}^{(-)})$ is a subcoalgebra of $(W_1^\circ, \Delta_{W_1^\circ})$ then the space V is a Lie subcoalgebra of $(W_1)^\circ$. ☐

By [18, Theorem 5] , if F has characteristic 0 or characteristic $p \neq 2$, then W_1° is the space of linearly recursive sequences. Hence, $P_1^\circ = W_1^\circ$ by Theorem 2.

Therefore, the following statement holds.

Corollary 1 (see also [19]). *Define the comultiplication*

$$\Delta_{d^\circ}^{(-)}(\alpha) = (id\otimes d^\circ - d^\circ\otimes id)\Delta^\circ(\alpha).$$

on the vector space P_1°. *If* F *is a field of characteristic* $\neq 2$ *then* $(P_1^\circ, \Delta_{d^\circ}^{(-)}) = (W_1^\circ, \Delta_{W_1^\circ})$.

Following [17], we define elements y_n, of W_1^*, $n \geq -1$, by $y_n(x_i) = \delta_{n,i}$. Define $Y = span(y_n \mid n \geq -1)$, and let Δ_Y be the restriction of m^* in P_1^* on the space Y. Then

$$\Delta_Y(y_n) = \sum_{i+j=n-1} y_i \otimes y_j.$$

By Lemma 2, Y is a good subspace of P_1^*. Consequently, (Y, Δ_Y) is a subcoalgebra of $(P_1^\circ, \Delta^\circ)$.

Now following [18], define $y_{a,j} \in P_1^*$, for $0 \neq a \in F$, $j \geq -1$, by putting $y_{a,j}(x_i) = a^i\binom{i+1}{j+1}$. Define $Y_a = span(y_{a,j} \mid j \geq -1)$, and let Δ_{Y_a} be the restriction of m^* in P_1^* on the space Y_a. For the sake of uniformity, we set $Y_0 = Y$. Let us turn Y_a into a differential subcoalgebra of P_1°.

Lemma 4. *The space* Y_a *is a good subspace of* P_1^*, *and* $(Y_a, \Delta_{Y_a}, d^\circ)$ *is a differential coalgebra. Consequently,* $(Y_a, \Delta_{Y_a}, d^\circ) \subseteq (P_1, d)^\circ$. *If* F *is an algebraically closed field of characteristic zero then* $P_1^\circ = \oplus_{a\in F} Y_a$

Proof. First let $a = 0$. We prove that $d^\circ(y_k) = (k+2)y_{k+1}$ for $k \geq -1$. Indeed, if $i > -1$ then

$$d^\circ(y_k)(x_i) = y_k(d((x_i)) = (i+1)y_k(x_{i-1}) = (i+1)y_{k+1}(x_i) = (k+2)y_{k+1}(x_i).$$

If $i = -1$ then $d^\circ(y_k)(x_{-1}) = y_k(d((1)) = 0$. On the other hand $y_{k+1}(x_{-1}) = \delta_{k+1,-1} = 0$. Hence, $d^\circ(y_k)(x_{-1}) = (k+2)y_{k+1}(x_{-1})$.

It is clearly that d° is a coderivation of (Y, Δ_Y). Therefore, (Y, Δ_Y, d°) is a good subspace of $(P_1, d)^*$. Consequently, $(Y, \Delta_Y, d^\circ) \subseteq (P_1, d)^\circ$.

Let $0 \neq a \in F$. Recall that for binomial coefficients we have

$$\binom{r+s+2}{k+1} = \sum_{i+j=k-1,\, i,j\geq -1}\binom{r+1}{i+1}\binom{s+1}{j+1},$$

where $r, s, k \geq -1$.

The comultiplication Δ_{Y_a} on the space Y_a is given by

$$\Delta_{Y_a}(y_{a,n}) = \sum_{i+j=n-1,\, i,j\geq -1} a y_{a,i} \otimes y_{a,j}.$$

Indeed,

$$\rho(\Delta_{Y_a}(y_{a,k}))(x_r \otimes x_s) = \sum_{i+j=k-1,\, i,j\geq -1} a y_{a,i}(x_r) y_{a,j}(x_s) = \sum_{i+j=k-1,\, i,j\geq -1} a^{r+s+1} \binom{r+1}{i+1}\binom{s+1}{j+1} = a^{r+s+1}\binom{r+s+2}{k+1} = $$
$$= y_{a,k}(x_{r+s+1}) = y_{a,k}(x_r \cdot x_s).$$

Therefore, Y_a is a good subspace of P_1^*. Consequently, (Y_a, Δ_{Y_a}) is a subcoalgebra of $(P_1^\circ, \Delta^\circ)$.

Let us show that for the coderivation d° we have $d^\circ(y_{a,k}) = a^{-1}(k+2)y_{a,k+1}$. Indeed, if $i > -1$ then

$$d^\circ(y_{a,k})(x_i) = y_{a,k}(d(x_i)) = (i+1)y_{a,k}(x_{i-1}) = a^{i-1}(i+1)\binom{i}{k+1} = a^{i-1}(i+1)\frac{i!}{(k+1)!(i-k-1)!} = a^{-1}(k+2)a^i \frac{(i+1)!}{(k+2)!(i+1-k-2)!} =$$
$$= a^{-1}(k+2)y_{a,k+1}(x_i).$$

If $i = -1$ then $d^\circ(y_{a,k})(x_{-1}) = y_{a,k}(d(1)) = 0$. On the other hand, $y_{a,k+1}(x_{-1}) = a^{-1}\binom{0}{k+2} = 0$. Therefore, $d^\circ(y_{a,k})(x_{-1}) = a^{-1}(k+2)y_{a,k+1}(x_{-1})$.

Consequently, $(Y_a, \Delta_{Y_a}, d^\circ)$ is a good subspace of $(P_1, d)^*$, so $(Y_a, \Delta_{Y_a}, d^\circ) \subseteq (P_1, d)^\circ$.

If F is an algebraically closed field of characteristic zero then Theorems 4 and 5 from [18] imply $W_1^\circ = \oplus_{a\in F} Y_a$. Hence, $P_1^\circ = \oplus_{a\in F} Y_a$. $\square$

An algebra $(A, \circ)$ with a multiplication operation $\circ$ is called a *Novikov algebra* (see [22, 23]) if A satisfies the following identities:

$$x\circ(y\circ z) - (x\circ y)\circ z = y\circ(x\circ z) - (y\circ x)\circ z \text{ (left symmetry)},$$

$$(x\circ y)\circ z = (x\circ z)\circ y \text{ (right commutativity)}.$$

Define the operation of multiplication $\circ$ on P_1 by the rule $f\circ g = fd(g)$, $f, g \in P_1$. Then $(P_1, \circ)$ is a Novikov algebra (see [22]). Following [24], we will call it left-symmetric Witt algebra of index 1 and denote by $\mathcal{L}_1$. Let $x_i = x^{i+1}$, $i = -1, 0, 1, \ldots$. Then the equality $x_i \circ x_j = (j+1)x_{i+j}$ holds in the algebra $\mathcal{L}_1$. It is known that the algebra $\mathcal{L}_1$ is simple, if F is field of characteristic zero.

Define the comultiplication Δ_N on P_1° by the rule

$$\Delta_N(\alpha) = (id \otimes d^\circ)\Delta^\circ(\alpha).$$

As it was shown in [15], the coalgebra (P_1°, Δ_N) is a Novikov coalgebra. Obviously, P_1° is a good subspace of $\mathcal{L}_1^*$. Therefore, $(P_1^\circ, \Delta_N) \subseteq (\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$.

The following Theorem holds.

Theorem 3. *Let $\Delta_N = (id \otimes d^\circ)\Delta^\circ$. Then $(P_1^\circ, \Delta_N) \subseteq (\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$. Therefore, the dual coalgebra $(\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$ of $\mathcal{L}_1$ is a non-zero Novikov coalgebra. If F is a field of characteristic zero then the coalgebra $(\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$ does not contain non-zero proper finite-dimensional subcoalgebras.*

Let $f, g \in \mathcal{L}_1$. It is easy to see that in the algebra $\mathcal{L}_1$ we have

$$[f, g] = f \circ g - g \circ f = fd(g) - d(f)g = [f, g]_d.$$

Consequently, the algebra $(\mathcal{L}_1, [\,,\,])$ is isomorphic to the Witt algebra W_1.

Define the new comultiplication $\Delta^{(-)}_{\mathcal{L}_1^\circ} = (1-\tau)\Delta_{\mathcal{L}_1^\circ}$ on the vector space $\mathcal{L}_1^\circ$. Then for every $f, g \in P_1$ and $\alpha \in \mathcal{L}_1^\circ$ we have

$$\langle \alpha, [f, g] \rangle = \langle \alpha, f \circ g - f \circ g \rangle = \rho(\Delta^{(-)}_{\mathcal{L}_1^\circ}(\alpha))(f \otimes g).$$

Therefore, $(\mathcal{L}_1^\circ, \Delta^{(-)}_{\mathcal{L}_1^\circ})$ is a good subspace of $(\mathcal{L}_1, [\,,\,])^*$, and we can assume that $(\mathcal{L}_1^\circ, \Delta^{(-)}_{\mathcal{L}_1^\circ})$ is a good subspace of W_1^*. Consequently, $(\mathcal{L}_1^\circ, \Delta^{(-)}_{\mathcal{L}_1^\circ}) \subseteq (W_1^\circ, \Delta_{W_1^\circ})$. Theorem 3 implies $(P_1^\circ, \Delta^{(-)}_{d^\circ}) \subseteq (\mathcal{L}_1^\circ, \Delta^{(-)}_{\mathcal{L}_1^\circ})$.

The following statement holds.

Theorem 4. *Let F be a field of characteristic $\neq 2$. Then $(P_1^\circ, \Delta_N) = (\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ})$ and $(W_1^\circ, \Delta_{W_1^\circ}) = (\mathcal{L}_1^\circ, \Delta^{(-)}_{\mathcal{L}_1^\circ})$, where $\Delta^{(-)}_{\mathcal{L}_1^\circ} = (1-\tau)\Delta_{\mathcal{L}_1^\circ}$.*

Proof. By Theorem 3, we have $P_1^\circ \subseteq \mathcal{L}_1^\circ$. It was shown that $\mathcal{L}_1^\circ \subseteq W_1^\circ$. Let F be a field of characteristic $\neq 2$. Then, by Theorem 2, $P_1^\circ = W_1^\circ$. Consequently, $P_1^\circ = \mathcal{L}_1^\circ = W_1^\circ$. Therefore, $(W_1^\circ, \Delta_{W_1^\circ}) = (\mathcal{L}_1^\circ, \Delta^{(-)}_{\mathcal{L}_1^\circ})$.

Let us prove that $\Delta_{\mathcal{L}_1^\circ} = \Delta_N$. Suppose f, g are arbitrary polynomials in P_1, and $\alpha \in \mathcal{L}_1^\circ$. Then

$$\langle \alpha, fd(g) \rangle = \rho(\Delta^\circ(\alpha))(f \otimes d(g)) = \rho((id \otimes d^\circ)\Delta^\circ(\alpha))(f \otimes g) = \rho(\Delta_N(\alpha))(f \otimes g).$$

On the other hand,

$$\langle \alpha, fd(g) \rangle = \langle \alpha, f \circ g \rangle = \rho(\Delta_{\mathcal{L}_1^\circ}(\alpha))(f \otimes g).$$

Hence, $\Delta_N = \Delta_{\mathcal{L}_1^\circ}$, so $(\mathcal{L}_1^\circ, \Delta_{\mathcal{L}_1^\circ}) = (P_1^\circ, \Delta_N)$.

□

3. Dual coalgebra of the Kantor construction of polynomial algebra in one variable

Let G be the Grassmann algebra with identity. The algebra $G = G_0 + G_1$ is a $\mathbb{Z}_2$-graded algebra. Let $J = J_0 + J_1$ be a $\mathbb{Z}_2$-graded algebra. Then $G(J) = J_0 \otimes G_0 + J_1 \otimes G_1$ is a subalgebra of $J \otimes G$ and it is called the *Grassmann envelope* of J.

An algebra J is called a *Jordan superalgebra* if its Grassmann envelope is a Jordan algebra, i. e., $G(J)$ satisfies the identities

$$xy = yx, \quad (x^2 y)x = x^2(xy).$$

Recall the Kantor constuction [25]. Let A be an associative commutative algebra over a field F with a derivation D. Denote by $\overline{A}$ an isomorphic copy of the vector space A with an isomorphism $a \mapsto \overline{a}$. On the direct sum of the vector spaces $J(A, D) = A + \overline{A}$ define a product $(\cdot)$ by

$$a \cdot b = ab, \quad a \cdot \overline{b} = \overline{ab}, \quad \overline{a} \cdot b = \overline{ab}, \quad \overline{a} \cdot \overline{b} = aD(b) - D(a)b,$$

where $a, b \in A$ and ab is the product of elemets in A. Then $J(A, D)$ is an Jordan superalgebra. The superalgebra $J(A, D)$ is said to be a superalgebra *of vector type.*

Let us give the dual analogue of the Kantor costruction for coalgebras. Suppose (C, Δ, d) is an associative commutative differential coalgebra with a coderivation d. Let $\overline{C}$ be an isomorphic copy of the vector space C with an isomorphism $c \mapsto \overline{c}$. Define the comultiplication Δ_J on the direct sum of vector spaces $J^c(C, \Delta_J, d) = C + \overline{C}$ by

$$\Delta_J(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} + \overline{c_{(1)}} \otimes \overline{d(c_{(2)})} - \overline{d(c_{(1)})} \otimes \overline{c_{(2)}},$$

$$\Delta_J(\overline{c}) = \sum_{(c)} \overline{c_{(1)}} \otimes c_{(2)} + c_{(1)} \otimes \overline{c_{(2)}},$$

where $c \in C$ and $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$. It is shown in [15] that the dual algebra of the coalgebra $J^c(C, \Delta_J, d)$ is a Jordan superalgebra of vector type $J(C^*, d^*)$, where d^* is the transpose map to d. Therefore, $J^c(C, \Delta_J, d)$ is a Jordan supercoalgebra.

Lemma 5. *Let $(P_1^\circ, \Delta^\circ, d^\circ)$ be the dual coalgebra of (P_1, d). Then $J^c(P_1^\circ, \Delta_J^\circ, d^\circ) = J(P_1, d)^\circ$.*

Proof. Set $J = J(P_1, d)$, and let $(J^\circ, \Delta_{J^\circ})$ be the dual supercoalgebra of J. It was noted above that $J^c = J^c(P_1^\circ, \Delta_J^\circ, d^\circ)$ is a Jordan supercoalgebra. It is clear that J^c is a good subspace of J^*. Therefore, $J^c \subseteq J^\circ$.

Let us put $\langle \alpha, a \rangle = \alpha(a)$ for $\alpha \in J^*$, $a \in J$, as above. Since $J^c = P_1^\circ + \overline{P_1^\circ}$, where $\overline{P_1^\circ}$ is an isomorphic copy of the vector space P_1° with the isomorphism $a \mapsto \overline{a}$, we can assume $\langle \alpha, a \rangle = \langle \overline{\alpha}, \overline{a} \rangle$ for $\alpha \in P_1^\circ$ and $a \in P_1$.

The algebra J acts on J^* as on a bimodule by the rule

$$\langle \alpha \cdot a, b \rangle = \langle \alpha, ab \rangle, \quad \langle a \cdot \alpha, b \rangle = \langle \alpha, ba \rangle,$$

where $\alpha \in J^*$, $a, b \in J$.

Since $J = P_1 \oplus \overline{P_1}$, then we can assume that $J^* = P_1^* \oplus \overline{P_1}^*$ (e.g., $P_1^* \cong \overline{P_1}^\perp \subset J^*$). Moreover, P_1^* is a P_1-subbimodule of J^*. Indeed, let $\alpha \in P_1^*$ and $f \in P_1$. Then

$$\langle \alpha \cdot f, \overline{P_1} \rangle = \langle \alpha, f\overline{P_1} \rangle \subseteq \langle \alpha, \overline{P_1} \rangle = 0.$$

Consequently, $\alpha \cdot f \in P_1^*$. Similarly, $f \cdot \alpha \in P_1^*$. Hence, P_1^* is a P_1-subbimodule of J^*. In the same way, one may show that $\overline{P_1}^* \cong P_1^\perp \subset J^*$ is a P_1-subbimodule of J^*.

Denote by V and W the projections of J° on the spaces P_1^* and $\overline{P_1}^*$, respectively. Since J° is a P_1-subbimodule of J^* then V and W are also P_1-subbimodules of J^*.

Let us show that V is a good subspace of P_1^*. For $\alpha \in V$ there exist $\gamma \in J^\circ$ and $\beta \in W$ such that $\gamma = \alpha + \beta$. Let $\Delta_{J^\circ}(\gamma) = \sum_\gamma \gamma_{(1)} \otimes \gamma_{(2)}$. Then $\gamma \cdot P_1 \subseteq \sum_\gamma \langle \gamma_{(1)}, P_1 \rangle \gamma_{(2)}$. Consequently, the space $\gamma \cdot P_1$ is finite-dimensional. The space $\alpha \cdot P_1$ is the projection of $\gamma \cdot P_1$ to V, therefore, $\alpha \cdot P_1$ is finite-dimensional. Similarly the space $P_1 \cdot \alpha$ is finite-dimensional. By Corollary 2.5 from [5] we get that V a good subspace of P_1^*.

Consequently, $V \subseteq P_1^\circ$. Since $P_1^\circ \subseteq J^c \subseteq J^\circ$ then $P_1^\circ \subseteq V$. Therefore, $P_1^\circ = V$. Then $V \subseteq J^\circ$ and $W \subseteq J^\circ$. Hence, the coalgebra J° is a $\mathbb{Z}_2$-graded space, where $V = (J^\circ)_0$, $W = (J^\circ)_1$. Moreover, $\overline{V} = \overline{P_1^\circ} \subseteq W$.

Assume $w \in W$. Then $\Delta_{J^\circ}(w) = \sum_i v_i \otimes w_i + w_i \otimes v_i$, where $v_i \in V$, $w_i \in W$, and for all $a \in P_1$ we have

$$\langle w, \overline{a} \rangle = \langle w, \overline{1} \cdot a \rangle = \left\langle \sum_i \langle w_i, \overline{1} \rangle v_i, a \right\rangle = \left\langle \sum_i \langle w_i, \overline{1} \rangle \overline{v_i}, \overline{a} \right\rangle.$$

Since $W \subseteq \overline{P_1}^*$, we have $w = \sum_i \langle w_i, \overline{1} \rangle \overline{v_i}$. Consequently, $W = \overline{V}$, and thus $J^c(P_1^\circ, \Delta_J^\circ, d^\circ) = J^\circ$. $\square$

It is known (see [21]) that a Jordan superalgebra is simple if and only if it is simple as an algebra. Also, recall that the superalgebra $J(A, D)$ is simple if and only if the algebra A is a differentially simple algebra [26].

Therefore, the following Theorem is true.

Theorem 5. *Let $P_1 = F[x]$ be the polynomial algebra in the variable x, and $d = \frac{d}{dx}$ is the derivation with respect to the variable x. Consider the Jordan superalgebra $J(P_1, d)$ of vector type. Then $J(P_1, d)^\circ = J^c(P_1^\circ, \Delta_J^\circ, d^\circ)$. Moreover, if F is a field of characteristic 0 then $J(P_1, d)^\circ$ does not contain non-zero finite-dimensional subcoalgebras.*

REFERENCES

[1] M. Sweedler, *Hopf algebras*, W.A. Benjamin, Inc., New York, 1969. Zbl 0194.32901

[2] V.G. Drinfel'd, *Quantum groups*, in *Proc. Int. Congr. Math., Berkeley/Calif. 1986, Vol. 1*, 798–820, Amer. Math. Soc., Providence, 1987. Zbl 0667.16003

[3] V.G. Drinfel'd, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang-Baxter equations,* Sov. Math., Dokl., **27** (1983), 68–71. Zbl 0526.58017

[4] W. Michaelis, *Lie coalgebras*, Adv. Math., **38** (1980), 1–54. Zbl 0451.16006

[5] J. Anquella, T. Cortés, F. Montaner, *Nonassociative coalgebras*, Comm. Algebra, **22**:12 (1994), 4693–4716. Zbl 0815.16023

[6] V.N. Zhelyabin, *Structurable coalgebras*, Algebra Logic **35**:5 (1996), 296–304. Zbl 0976.16031

[7] V.N. Zhelyabin, *Embedding of Jordan copairs into Lie coalgebras*, Commun. Algebra, **35**:2 (2007), 561–576. Zbl 1144.17015

[8] G. Santos Filho, L. Murakami, I. Shestakov, *Locally finite coalgebras and the locally nilpotent radical I*, Linear Algebra Appl., **621** (2021), 235–253. Zbl 7337677

[9] A.M. Slinko, *Local finiteness of coalgebraic Lie coalgebras*, Commun. Algebra, **23**:3 (1995), 1165–1170. Zbl 0826.16039

[10] V.N. Zhelyabin, *The Kantor-Koecher-Tits construction for Jordan coalgebras*, Algebra Logic, **35**:2 (1996), 96–104. Zbl 0906.17026

[11] V.N. Zhelyabin, *Jordan (super)coalgebras and Lie (super)coalgebras*, Sib. Math. J., **44**:1 (2003), 73–92. Zbl 1049.17029

[12] M.E. Goncharov, V.N. Zhelyabin, *Mikheev's construction for Mal'tsev coalgebras*, Algebra Logic **51**:5 (2012), 445–447. Zbl 1283.17024

[13] M.E. Goncharov, V.N. Zhelyabin, *Embedding Mal'tsev coalgebras into Lie coalgebras with triality*, Algebra Logic, **52**:1 (2013), 24–40. Zbl 1323.17022

[14] D.Kh. Kozybaev, *Right-alternative and right-symmetric coalgebras*, South-Kazakhstan Science and Education. **19**:12 (2000), 155–163.

[15] D. Kozybaev, U. Umirbaev, V. Zhelyabin, *Some examples of nonassociative coalgebras and supercoalgebras*, Linear Algebra Appl., **643** (2022), 235–257. Zbl 7501865

[16] G. Santos Filho, L. Murakami, I. Shestakov, *Locally finite coalgebras and the locally nilpotent radical II*, Commun. Algebra, **49**:12 (2021), 5472–5482. Zbl 7431311

[17] W. Michaelis, *An example of a non-zero Lie coalgebra M for which Loc(M) = 0*, J. Pure Appl. Algebra., **68**:3 (1990), 341–348. Zbl 0721.17022

[18] W.D. Nichols, *The structure of the dual Lie coalgebra of the Witt algebra*, J. Pure Appl. Algebra., **68**:3 (1990), 359–364. Zbl 0721.17023

[19] W.D. Nichols, *On Lie and associative duals*, J. Pure Appl. Algebra, **87**:3 (1993), 313–320. Zbl 0790.17010

[20] G. Griffing, *The dual coalgebra of certain infinite-dimensional Lie algebras*, Commun. Algebra, **30**:12 (2002), 5715–5724. Zbl 1043.17013

[21] V.N. Zhelyabin, *Dual coalgebras of Jordan bialgebras and superalgebras*, Sib. Math. J., **46**:6 (2005), 1050–1061. Zbl 1118.17009

[22] I.M. Gel'fand, I.Ya. Dorfman, *Hamiltonian operators and algebraic structures related to them*, Funkts. Anal. Prilozh., **13**:4 (1979), 13–30. Zbl 0428.58009

[23] A.A. Balinskii, S.P. Novikov, *Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras*, Sov. Math., Dokl., **32** (1985), 228–231. Zbl 0606.58018

[24] D. Kozybaev, U. Umirbaev, *Identities of the left-simmtric Witt algebras*, Int. J. Algebra Comput., **26**:2 (2016), 435–450. Zbl 1337.16018

[25] I.L. Kantor, *Jordan and Lie superalgebras determined by a Poisson algebra*, in Aleksandrov, I. A. (ed.) et al., Proceedings of the second Siberian school, Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., **151** (1992), 55–80. Zbl 0803.17012

[26] D. King, K. McCrimmon, *The Kantor doubling process revisited*, Comm. Algebra, **23**:1 (1995), 357–372. Zbl 0852.17034

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