S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 19, №2, стр. 809–814 (2022) УДК 512.54
DOI 10.33048/semi.2022.19.068 MSC 20E07

MUTUAL EMBEDDINGS OF RIGHT-ANGLED ARTIN GROUPS AND GENERALIZED BAUMSLAG-SOLITAR GROUPS

F.A. DUDKIN

Abstract. A finitely generated group G acting on a tree so that all vertex and edge stabilizers are infinite cyclic groups is called a *generalized Baumslag–Solitar group* (GBS group). In this paper, we study when a given GBS group can be embedded in a *right-angled Artin group* ($RAAG$) and vice versa. An exhaustive description has been obtained in both cases. If an embedding exists, then we discuss its construction.

Keywords: right-angled Artin group, generalized Baumslag–Solitar group, embedding problem.

1. Introduction

In combinatorial group theory, one can distinguish groups whose presentations are somehow connected with some graphs built on generating elements as on vertices. One of the most well-known and studied classes of such groups is the class of Artin groups, which includes braid groups, *right-angled Artin group*, Coxeter groups, and some other classes of groups. Right-angled Artin groups are also called *free partially commutative* or *graph groups*. We will briefly call groups from this class $RAAG$. This is an interesting and actively studied class of groups.

In what follows, we will use letters Γ, Λ to denote undirected graphs without loops and multiple edges. Let $V(\Gamma)$ and $E(\Gamma)$ denote the sets of vertices and edges of the graph Γ. Any such graph defines $RAAG$ $A(\Gamma)$ as follows:

$$A(\Gamma) \cong \langle\ V(\Gamma) \mid [a,b]=1,\ \text{for all } \{a,b\} \in E(\Gamma)\ \rangle\,.$$

Dudkin, F.A., Mutual embeddings of right-angled Artin groups and generalized Baumslag-Solitar groups.

The work was supported by the Russian Science Foundation, project 19-11-00039.

Received August, 26, 2022, published November, 11, 2022.

The class $RAAG$ includes different groups from free groups to free Abelian groups. They have applications not only in various areas of mathematics, but also in computer science and robotics. These groups were studied by A. Baudisch, C. Droms, and many other authors (see, for example, the survey [1]).

Another important and interesting class of groups defined by graphs is the class of generalized Baumslag–Solitar groups (GBS groups). By the Bass–Serre theorem, G is presentable as $\pi_1(\mathbb{A})$, the fundamental group of a graph of groups $\mathbb{A}$ (see [2]) whose vertex and edge groups are infinite cyclic groups.

To each GBS-group G we can associate a labeled graph $\mathbb{A}$, a particular kind of a graph of groups. This labeled graph corresponds to an action of G on a tree and defines a presentation of G (see [3] for more details about labeled graphs and their properties).

As D. Robinson [4] noted, GBS groups occupy an important position in combinatorial group theory due to the following properties: non-cyclic GBS groups are exactly those finitely generated groups of cohomological dimension 2 that have a commensurate infinite cyclic subgroup; GBS groups are coherent (every finitely generated subgroup admits a finite presentation).

Earlier, embeddings of GBS groups [5, 6] and embeddings of $RAAGs$ [7] were actively studied. In this paper, we will study when a GBS group can be embedded in a $RAAG$ and vice versa. Our approaches to the study of embedding provide not only a criterion for the existence of an embedding, but also allow us to construct such an embedding if it exists.

Theorem 1. *If a GBS group G is embedded in a RAAG H, then $G \cong \mathbb{F}_n \times \mathbb{Z}, n \geqslant 0$. The embedding of $G \cong \mathbb{F}_n \times \mathbb{Z}$ in a RAAG H can be constructed algorithmically.*

Theorem 2. *If a finitely generated RAAG H embeds in a GBS group G, then H is either free or $H \cong \mathbb{F}_n \times \mathbb{Z}, n \geqslant 0$. The embedding of a RAAG H in G can be constructed algorithmically.*

2. Preliminaries

To describe GBS groups, it is convenient to use the class of oriented pseudographs (loops and multiple edges allowed). A *pseudograph* A is a set of vertices $V(A)$, set of edges $E(A)$, maps $\partial_0, \partial_1 \colon E(A) \to V(A)$ — endpoints of the edge and inversion $^{-}\colon E(A) \to E(A)$ such that $\partial_0(\bar{e}) = \partial_1(e), \partial_1(\bar{e}) = \partial_0(e), \bar{\bar{e}} = e, \bar{e} \neq e$.

Given a GBS-group G, we can represent the corresponding graph of groups by a *labeled graph* $\mathbb{A} = (A, \lambda)$, where A is a finite connected pseudograph and $\lambda \colon E(A) \to \mathbb{Z} \setminus \{0\}$ labels the edges of A. The label $\lambda(e)$ of an edge e with the origin $v = \partial_0(e)$ defines the embedding $\alpha_e \colon e \to v^{\lambda(e)}$ of the cyclic edge group $\langle e\rangle$ into the cyclic vertex group $\langle v\rangle$.

The fundamental group $\pi_1(\mathbb{A})$ *of a labeled graph* $\mathbb{A} = (A, \lambda)$ is described by generators and defining relations. Denote by $\overline{A}$ the undirected pseudograph obtained from A by identifying e and $\bar{e}$. Denote by $\beta(A)$ the number of edges outside the maximal subtree in the pseudograph $\overline{A}$. Each maximal subtree T in $\overline{A}$ defines the following presentation:

$$(1) \qquad \left\langle \begin{array}{l} g_v, v \in V(\overline{A}), \\ t_e, e \in E(\overline{A}) \setminus E(T) \end{array} \;\middle|\; \begin{array}{l} g_{\partial_0(e)}^{\lambda(e)} = g_{\partial_1(e)}^{\lambda(\bar{e})}, e \in E(T), \\ t_e^{-1} g_{\partial_0(e)}^{\lambda(e)} t_e = g_{\partial_1(e)}^{\lambda(\bar{e})}, e \in E(\overline{A}) \setminus E(T) \end{array} \right\rangle.$$

For different maximal subtrees, corresponding presentations define isomorphic groups [2].

A generalized Baumslag–Solitar group is called *elementary* if it is isomorphic to $\mathbb{Z}, \mathbb{Z}^2$ or the Klein bottle group $\langle\ a, t \mid t^{-1}at = a^{-1}\ \rangle$.

If two labeled graphs $\mathbb{A}$ and $\mathbb{B}$ define isomorphic GBS groups $\pi_1(\mathbb{A}) \cong \pi_1(\mathbb{B})$, and group $\pi_1(\mathbb{A})$ is not elementary, there is a finite sequence of *expansions* and *collapses* (see Fig. 1) connecting $\mathbb{A}$ and $\mathbb{B}$ [8]. A labeled graph is called *reduced* if it does not allow collapses (the label ± 1 can appear only on a loop).

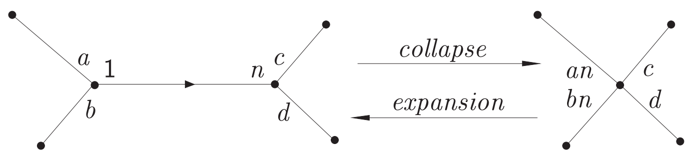

Fig. 1. Collapse and expansion

We need some definitions and statements from [7]. For a given graph Γ, we construct its *extension* — the graph Γ^e [7]. Vertices of the graph Γ^e are words in the group $A(\Gamma)$ conjugate to the vertices Γ. Two vertices in $A(\Gamma^e)$ commute if and only if they commute as words in $A(\Gamma)$.

Lemma 1. [7, Corollary 8] *Let Λ and Γ be two finite graphs and Λ be a forest. Then $A(\Lambda)$ embeds in $A(\Gamma)$ if and only if Λ embeds in Γ^e.*

3. Auxiliary statements

Lemma 2. *A free group $\mathbb{F}_2$ of rank 2 does not embed into a GBS group G if and only if the group G is a solvable Baumslag–Solitar group*

$$BS(1, n) \cong \langle\ a, t \mid t^{-1}at = a^n\ \rangle, n \in \mathbb{Z}.$$

Proof. Note that $BS(1,0) \cong \mathbb{Z}, BS(1,1) \cong \mathbb{Z}^2$, and $BS(1,-1)$ is isomorphic to the Klein bottle group. Hence, all elementary groups are solvable Baumslag–Solitar groups. Moreover, the group $BS(1,n)$ is metabelian for $n \neq 0$. Therefore the group $BS(1,n)$ cannot contain a free group of rank 2 as a subgroup.

Assume that the GBS group G is given by a reduced labeled graph $\mathbb{A}$ and does not contain subgroups isomorphic to $\mathbb{F}_2$. If $\beta(A) \geqslant 2$, then G contains a free subgroup of rank 2 generated by generators of the second type $t_e, e \in E(\overline{A}) \setminus E(T)$, see (1). So $\beta(A) \leqslant 1$.

If $\beta(A) = 0$ and the group G is not elementary, then the center of G is isomorphic to $\mathbb{Z}$. In this case G contains $\mathbb{F}_2 \times \mathbb{Z}, n \geqslant 2$ [9]. You can understand how embedding works from [2, 9].

It remains to analyze the case $\beta(A) = 1$. If, moreover, G is not a solvable Baumslag–Solitar group $BS(1,n)$, then either $G \cong BS(m,n)$ and $m, n \notin \{-1, 0, 1\}$; or the labeled graph contains at least two vertices.

In the first case, the group G contains a subgroup

$$\langle\ a, b, c \ \mid\ a^m = b^n, b^m = c^n\ \rangle$$

and, arguing as if $\beta(A) = 0$, we get an embedding.

In the second case, the group G contains the subgroup

$$\langle\ a, b, c \ \mid\ a^m = b^n, b^k = c^l\ \rangle$$

and again a free group of rank 2 embeds in G. $\square$

Lemma 3. *Let $n \geqslant 2$ be an integer and RAAG H be given by a finite graph Γ. A RAAG $G = A(\Gamma_n) \cong \mathbb{F}_n \times \mathbb{Z}$ embeds in a RAAG H if and only if the graph Γ has a subgraph Γ_2 (see Fig. 2).*

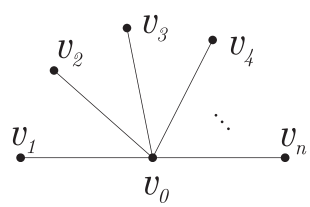

FIG. 2. Graph Γ_n

Proof. Indeed, if Γ_2 is a subgraph of the graph Γ, then $\mathbb{F}_2 \times \mathbb{Z}$ is embedded in H. Group $\mathbb{F}_n \times \mathbb{Z}, n \geqslant 2$ is embedded in the $\mathbb{F}_2 \times \mathbb{Z}$ group.

Now suppose that the group $G \cong \mathbb{F}_n \times \mathbb{Z}$ is embedded in a *RAAG* H. If the graph Γ does not have a subgraph Γ_2, then any three points generate one of the graphs shown in the Fig. 3.

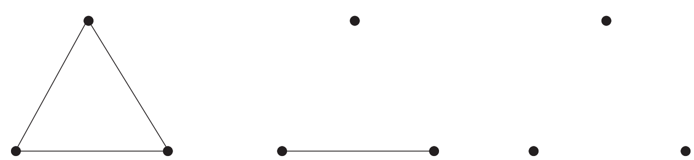

FIG. 3. Subgraphs on three points

In this case, the graph Γ decomposes into a union of cliques. Therefore the group H is a free product of free abelian groups.

Note that the graph Γ^e (not necessarily finite) is also a union of cliques. Therefore, the graph Γ_2 does not embed into Γ^e. Now it follows from Lemma 1 that $A(\Gamma_2) \cong \mathbb{F}_2 \times \mathbb{Z}$ is not embedded in H. Therefore, G does not embed into H either. This contradiction completes the proof of Lemma. $\square$

4. Proofs of embedding theorems

Proof of Theorem 1. Let *RAAG* H be given by the graph Γ. It is known that any *RAAG* is residually torsion-free nilpotent group (see [10, 11]). Since every torsion-free nilpotent group is a residually $\mathcal{F}_p$-group for every prime p, every *RAAG* is a residually $\mathcal{F}_p$-group for every prime p. Therefore the group G is a residually $\mathcal{F}_p$-group for every prime p.

In the paper [12] a criterion for the $\mathcal{F}_\pi$-residuality of *GBS* groups is given for an arbitrary set of primes π. Using this criterion we can see that the group G is represented by a reduced labeled graph $\mathbb{A}$ such that all labels on its edges are equal to 1. Such a labeled graph can only be a bouquet of n circles with labels 1 (see Fig. 4). So $G \cong \mathbb{F}_n \times \mathbb{Z}$.

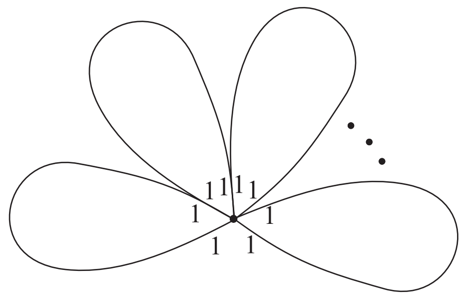

FIG. 4. Bouquet of n circles with labels 1

If $n = 0$, then $G \cong \mathbb{Z}$. The embedding of G in H is obvious.

If $n = 1$, then $G \cong \mathbb{Z} \times \mathbb{Z}$. Such a group is embedded in a *RAAG* H if and only if the corresponding graph has at least one edge. The embedding in this case is obvious.

If $n \geqslant 2$, then Lemma 3 gives an embedding criterion. □

Proof of Theorem 2. If a *RAAG* H is embedded in a *GBS* group G, then the image of the embedding is a finitely generated subgroup of G. It acts on the same tree as G. Moreover, the vertex stabilizers of this action are subgroups of the vertex stabilizers of the action G. Hence the stabilizers are either trivial or cyclic.

If vertex stabilizers are trivial, then H acts freely on the tree, so it is itself free. If $H \cong \mathbb{Z}$ then H is always embedded in G. If $H \cong \mathbb{F}_n, n \geqslant 2$, then H embeds in G if and only if the group G is not solvable Baumslag-Solitar group by Lemma 2. Embeddings are discussed in the proof of Lemma 2.

If vertex stabilizers are cyclic, then H is a *GBS* group. Arguing as in Theorem 1, we get that $H \cong \mathbb{F}_n \times \mathbb{Z}, n \geqslant 0$. Hence H is represented by the only reduced labeled graph. In this case, the problem of embedding H in G is algorithmically decidable [6]. □

REFERENCES

[1] A.J. Duncan, V.N. Remeslennikov, A.V. Treier, *A survey of free partially commutative groups*, J. Phys.: Conf. Ser., **1441** (2020), 012136.

[2] J.P. Serre, *Trees*, Springer, Berlin etc., 1980. Zbl 0548.20018

[3] M. Clay, M. Forester, *On the isomorphism problem for generalized Baumslag–Solitar groups*, Algebr. Geom. Topol., **8**:4 (2008), 2289–2322. Zbl 1191.20021

[4] D.J.S. Robinson, *Generalized Baumslag-Solitar groups: a survey of recent progress*, in Campbell, C.M. (ed.) et al., *Groups St Andrews 2013. Selected papers of the conference, St. Andrews, UK, August 3–11, 2013*, LMS, Lecture Note Series, **422**, Cambridge University Press, Cambridge, 2016, 457–469. Zbl 1346.20040

[5] G. Levitt, *Quotients and subgroups of Baumslag–Solitar groups*, J. Group Theory, **18**:1 (2015), 1–43. Zbl 1317.20030

[6] F.A. Dudkin, *On the embedding problem for generalized Baumslag–Solitar groups*, J. Group Theory, **18**:4 (2015), 655–684. Zbl 1402.20043

[7] S.-H Kim, T. Koberda, *Embedability between right-angled Artin groups*, Geom. Topol., **17**:1 (2013), 493–530. Zbl 1278.20049

[8] M. Forester, *Deformation and rigidity of simplicial group actions on trees*, Geom. Topol., **6** (2002), 219–267. Zbl 1118.20028

[9] G. Levitt, *Generalized Baumslag-Solitar groups: rank and finite index subgroups*, Ann. Inst. Fourier, **65**:2 (2015), 725–762. Zbl 1375.20029

[10] G. Duchamp, D. Krob, *The lower central series of the free partially commutative group*, Semigroup Forum, **45**:3 (1992), 385–394. Zbl 0814.20025

[11] E.I. Timoshenko, *Mal'tsev bases for partially commutative nilpotent group*, Int. J. Algebra Comput., **32**:1 (2022), 1–9. Zbl 7492573

[12] F.A. Dudkin, $\mathcal{F}_\pi$*-residuality of generalized Baumslag-Solitar groups*, Arch. Math., **114**:2 (2020), 129–134. Zbl 1480.20071

Fedor Anatol'evich Dudkin
Sobolev Institute of Mathematics, 4, pr. Koptyuga, Novosibirsk, 630090, Russia
Novosibirsk State University, 1, Pirogova str., Novosibirsk, 630090, Russia
Email address: dudkinf@math.nsc.ru