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ON FUNCTION SPACES. II

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ABSTRACT. For certain properties $\mathfrak{P}$ of topological T_0-spaces, we prove that a T_0-space $\mathbb{Y}$ has property $\mathfrak{P}$ if and only if the function space $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ endowed with a particular topology $\mathcal{T}$ possesses $\mathfrak{P}$ for some T_0-space $\mathbb{X}$.

Keywords: A-space, core-compact space, d-space, essentially complete space, function space, injective space, sober space, T_0-space.

1. Introduction

This paper continues [10]. In [10], the interplay of different topological properties for a T_0-space $\mathbb{Y}$ and its function space $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ (endowed with the pointwise convergence topology) was investigated. Here, we extend some results from [10] as well as from [9] considering different topologies on the set $C(\mathbb{X}, \mathbb{Y})$ of continuous functions from a T_0-space $\mathbb{X}$ to a T_0-space $\mathbb{Y}$.

Our main results are Theorems 9, 12, 18, 20, 22, 30, 31, Propositions 27 and 32, and Corollaries 17, 21, and 23.

For all the notions and notation which are not defined here, we refer to the monographs [9, 12, 13] as well as to [10].

2. Topologies on $C(\mathbb{X}, \mathbb{Y})$

Given topological T_0-spaces $\mathbb{X}$ and $\mathbb{Y}$, let $C(\mathbb{X}, \mathbb{Y})$ denote the set of all continuous functions from $\mathbb{X}$ to $\mathbb{Y}$. An arbitrary function $f \in C(\mathbb{X}, \mathbb{Y})$ can be considered as an element of the Cartesian power Y^X. *The pointwise convergence topology* on $C(\mathbb{X}, \mathbb{Y})$

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is induced by the Tychonoff topology on Y^X. Therefore, the collection of all sets of the form

$$V_{x,U} = \{f \in C(\mathbb{X}, \mathbb{Y}) \mid f(x) \in U\}, \text{ where } x \in X \text{ and } \varnothing \neq U \in \mathcal{T}(\mathbb{Y}),$$

forms a subbasis of the pointwise convergence topology. In what follows, we denote this topology by $\mathcal{P}$.

If $\mathcal{T}$ is a topology on $C(\mathbb{X}, \mathbb{Y})$ then we denote the space $\langle C(\mathbb{X}, \mathbb{Y}), \mathcal{T}\rangle$ by $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ and by $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ when $\mathcal{T} = \mathcal{P}$. The specialization order on $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is denoted by $\leq_{\mathcal{T}}$ or just by $\leq$ in case when $\mathcal{T} = \mathcal{P}$. For f, $g \in C(\mathbb{X}, \mathbb{Y})$, the relation $f \leq_{\mathcal{P}} g$ holds if and only if $f(x) \leq_{\mathbb{Y}} g(x)$ for all $x \in X$. We write $f \leq g$ instead of $f \leq_{\mathcal{P}} g$. It is clear that $f \leq_{\mathcal{T}} g$ implies that $f \leq g$ whenever $\mathcal{P} \subseteq \mathcal{T}$.

We also consider the mapping

$$\xi\colon \mathbb{Y} \to C(\mathbb{X}, \mathbb{Y}); \quad \xi\colon y \mapsto \xi_y, \quad \text{where } \xi_y(x) = y \text{ for all } x \in X.$$

Lemma 1. *Let $\mathbb{X}, \mathbb{Y}$ be T_0-spaces. If $\mathcal{T}$ is a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\leq_{\mathcal{P}})$ then $\leq_{\mathcal{T}}$ coincides with $\leq_{\mathcal{P}}$. In this case, the following holds.*

1. *$\mathbb{Y}$ contains a least element with respect to the specialization order $\leq_{\mathbb{Y}}$ if and only if $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ contains a least element with respect to $\leq_{\mathcal{T}}$ for some (equivalently, for each) T_0-space $\mathbb{X}$.*
2. *$\mathbb{Y}$ contains a greatest element with respect to $\leq_{\mathbb{Y}}$ if and only if $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ contains a greatest element with respect to $\leq_{\mathcal{T}}$ for some (equivalently, for each) T_0-space $\mathbb{X}$.*

Proof. The pointwise convergence topology $\mathcal{P}$ is finer than the Alexandroff topology $\mathcal{T}_A(\leq_{\mathcal{P}})$, see for example Lemma 1.2.8 in [9]. As the specialization order of $\mathcal{T}_A(\leq_{\mathcal{P}})$ coincides with $\leq_{\mathcal{P}}$, we obtain that $\leq_{\mathcal{T}}$ coincides with $\leq_{\mathcal{P}}$; we denote this order simply by $\leq$.

(1) Let $\langle Y; \leq_{\mathbb{Y}}\rangle$ have a least element $\perp$. It is clear that $\xi_\perp \in C(\mathbb{X}, \mathbb{Y})$ is a least element in $\langle C(\mathbb{X}, \mathbb{Y}); \leq\rangle$. Conversely, suppose that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\leq)$ and $o \in C(\mathbb{X}, \mathbb{Y})$ is a least element in $\langle C(\mathbb{X}, \mathbb{Y}); \leq\rangle$. Choose arbitrary elements $x, x' \in X$ and suppose that $o(x) \in U \in \mathcal{T}(\mathbb{Y})$. Then $o \in V_{x,U} \in \mathcal{P} \subseteq \mathcal{T}$. If $y \in Y$ then $o \leq \xi_y$, whence $\xi_y \in V_{x,U}$ and $y \in U$. As we can choose y arbitrary, we conclude that $U = Y$ and $o(x') \in U$. Thus, we proved that $o(x) \leq_{\mathbb{Y}} o(x')$. Symmetrically, $o(x') \leq_{\mathbb{Y}} o(x)$, whence $o = \xi_\perp$ for some $\perp \in Y$. Since $\xi_\perp = o \leq \xi_y$, we conclude that $\perp \leq y$ for all $y \in Y$. Therefore, $\langle Y; \leq_{\mathbb{Y}}\rangle$ has a least element.

The proof of (2) is similar but simpler. □

Given topological spaces $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, let $M(\mathbb{X}, \mathbb{Y})$ denote the set of all functions from X to Y. We put

$$\lambda\colon M(\mathbb{Z} \times \mathbb{X}, \mathbb{Y}) \to M(\mathbb{Z}, M(\mathbb{X}, \mathbb{Y})), \quad \lambda(f)\colon z \mapsto f(z, x);$$
$$\lambda^*\colon M(\mathbb{Z}, M(\mathbb{X}, \mathbb{Y})) \to M(\mathbb{Z} \times \mathbb{X}, \mathbb{Y}), \quad \lambda^*(g)\colon (z, x) \mapsto [g(z)](x).$$

It is not hard to verify that λ and λ^* are mutually inverse mappings. Hence they establish a one-to-one correspondence between sets $M(\mathbb{Z}\times\mathbb{X}, \mathbb{Y})$ and $M(\mathbb{Z}, M(\mathbb{X}, \mathbb{Y}))$.

For the following definition, we refer to [1]. A topology $\mathcal{T}$ on $C(\mathbb{X}, \mathbb{Y})$ is *proper*, if $\lambda(C(\mathbb{Z} \times \mathbb{X}, \mathbb{Y})) \subseteq C(\mathbb{Z}, \mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y}))$ for an arbitrary space $\mathbb{Z}$. A topology $\mathcal{T}$ on $C(\mathbb{X}, \mathbb{Y})$ is *admissible*, if $\lambda^*(C(\mathbb{Z}, \mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y}))) \subseteq C(\mathbb{Z} \times \mathbb{X}, \mathbb{Y})$ for an arbitrary space $\mathbb{Z}$. A topology $\mathcal{T}$ on $C(\mathbb{X}, \mathbb{Y})$ is *exponential*, if $\mathcal{T}$ is both proper and admissible.

The *compact-open topology* $\mathcal{K}$ on $C(\mathbb{X},\mathbb{Y})$ is defined by the subbasis of open sets $V_{K,U} = \{f \in C(\mathbb{X},\mathbb{Y}) \mid f(K) \subseteq U\}$, where $K \subseteq X$ is compact and $\varnothing \neq U \in \mathcal{T}(\mathbb{Y})$, The *Isbell topology* $\mathcal{I}$ on $C(\mathbb{X},\mathbb{Y})$ is defined by the subbasis of open sets

$$V_{\mathcal{H},U} = \{f \in C(\mathbb{X},\mathbb{Y}) \mid f^{-1}(U) \in \mathcal{H}\},$$

where $\mathcal{H}$ is a Scott-open set in $\langle \mathcal{T}(\mathbb{X}); \subseteq\rangle$ and $\varnothing \neq U \in \mathcal{T}(\mathbb{Y})$. The *core-open topology* $\mathcal{C}$ on $C(\mathbb{X},\mathbb{Y})$ is defined by the subbasis of open sets $V_{U,W} = \{f \in C(\mathbb{X},\mathbb{Y}) \mid U \ll f^{-1}(W)\}$, where $\varnothing \neq U \in \mathcal{T}(\mathbb{X})$ and $\varnothing \neq W \in \mathcal{T}(\mathbb{Y})$. Let $\mathcal{T}$ be a T_0-topology on $\mathcal{T}(\mathbb{X})$ such that the specialization order $\leq_{\mathcal{T}}$ coincides with $\subseteq$. Consider the topology on $C(\mathbb{X},\mathbb{Y})$ defined by the subbasis of open sets

$$V_{\mathcal{O},U} = \{f \in C(\mathbb{X},\mathbb{Y}) \mid f^{-1}(U) \in \mathcal{O}\},$$

where $\mathcal{O} \in \mathcal{T}$ and $U \in \mathcal{T}(\mathbb{Y})$. We say that this topology is *induced by* $\mathcal{T}$ and denote it by $\mathcal{T}^\sharp$. It is clear that $\mathcal{T}_0 \subseteq \mathcal{T}_1$ implies that $\mathcal{T}_0^\sharp \subseteq \mathcal{T}_1^\sharp$. Moreover, it follows from the definition above that $\mathcal{I}$ is induced by the Scott topology on $\langle \mathcal{T}(\mathbb{X}); \subseteq\rangle$.

Lemma 2. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces. Then $\mathcal{T}_A(\subseteq)^\sharp \subseteq \mathcal{T}_A(\leq)$.*

Proof. It suffices to show that $V_{\mathcal{O},W} \in \mathcal{T}_A(\leq)$ for all upper cones $\mathcal{O}$ in $\langle \mathcal{T}(\mathbb{X}); \subseteq\rangle$ and all open sets $W \in \mathcal{T}(\mathbb{Y})$. Indeed, let $f \in V_{\mathcal{O},W}$ and let $f \leq g$ in $C(\mathbb{X},\mathbb{Y})$. We prove first that $f^{-1}(W) \subseteq g^{-1}(W)$. Indeed, let $x \in f^{-1}(W)$; then $f(x) \in W$. Since W is an upper cone with respect to $\leq_{\mathbb{Y}}$, we obtain that $g(x) \in W$ and thus, $x \in g^{-1}(W)$ which is our desired conclusion.

Since $f^{-1}(W) \in \mathcal{O}$ and $f^{-1}(W) \subseteq g^{-1}(W)$, we conclude that $g^{-1}(W) \in \mathcal{O}$ as $\mathcal{O}$ is an upper cone with respect to $\subseteq$. This yields that $g \in V_{\mathcal{O},W}$ and $V_{\mathcal{O},W} \in \mathcal{T}_A(\leq)$. □

In what follows, we consider also the *Scott topology* $\mathcal{S} = \mathcal{T}_S(\leq_{\mathcal{P}})$ on $C(\mathbb{X},\mathbb{Y})$.

Lemma 3. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces. If $\mathcal{T}$ is a topology on $C(\mathbb{X},\mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$ then ξ is a homeomorphic embedding of $\mathbb{Y}$ into $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$.*

Proof. It is straightforward to verify that $\xi(Y) \cap V_{x,U} = \xi(U)$ for all $x \in X$ and $U \in \mathcal{T}(\mathbb{Y})$. Therefore, ξ is open. Moreover, for all $W \in \mathcal{T}(\mathbb{Y})$ and all $y \in Y$, we have

$$\xi_y^{-1}(W) = \begin{cases} X, & \text{if } y \in W; \\ \varnothing, & \text{if } y \notin W. \end{cases}$$

Furthermore, let $\mathcal{O} \in \mathcal{T}_A(\subseteq)$ and let $W \in \mathcal{T}(\mathbb{Y})$. Then we have

$$\xi^{-1}(V_{\mathcal{O},W}) = \{y \in Y \mid \xi_y \in V_{\mathcal{O},W}\} = \{y \in Y \mid \xi_y^{-1}(W) \in \mathcal{O}\} =$$
$$= \begin{cases} Y, & \text{if } \mathcal{O} = \mathcal{T}(\mathbb{X}); \\ W, & \text{if } \mathcal{O} \notin \{\mathcal{T}(\mathbb{X}), \varnothing\}; \\ \varnothing, & \text{if } \mathcal{O} = \varnothing. \end{cases}$$

In any case, $\xi^{-1}(V_{\mathcal{O},W}) \in \mathcal{T}(\mathbb{Y})$. Therefore, $\xi^{-1}(V) \in \mathcal{T}(\mathbb{Y})$ for all $V \in \mathcal{T}_A(\subseteq)^\sharp$, whence the desired statement follows. □

Let $\mathbb{X}$ be a T_0-space. If $\langle I; \leqslant\rangle$ is an up-directed poset and $\mathfrak{x} = \{x_i \mid i \in I\} \subseteq X$ then we say that $\mathfrak{x}$ is a *net* in $\mathbb{X}$. If $A \subseteq X$, then $\mathfrak{x}$ is *eventually in* A if there is $i \in I$ such that $x_j \in A$ for all $j \geqslant i$. Then $\mathfrak{x}$ converges to some $x \in X$ if $x \in U \in \mathcal{T}(\mathbb{X})$ implies that $\mathfrak{x}$ is eventually in U for all $U \in \mathcal{T}(\mathbb{X})$. Further, if $\mathbb{Y}$ is also a T_0-space and $\mathfrak{f} = \{f_j \mid j \in J\}$ is a net in $C(\mathbb{X},\mathbb{Y})$ for some up-directed poset

$\langle J; \sqsubseteq \rangle$, then $\mathfrak{f}$ *converges continuously to* $f \in C(\mathbb{X}, \mathbb{Y})$ if for arbitrary $x \in X$, the net $\{f_j(x_i) \mid (i,j) \in I \times J\}$ converges to $f(x)$ whenever $\mathfrak{x} = \{x_i \mid i \in I\}$ converges to x.

The *natural topology* $\mathcal{T}_*$ on $C(\mathbb{X}, \mathbb{Y})$ consists of those sets $U \subseteq C(\mathbb{X}, \mathbb{Y})$ which satisfy the following condition for all nets $\mathfrak{f}$ in $C(\mathbb{X}, \mathbb{Y})$ and all $f \in C(\mathbb{X}, \mathbb{Y})$:

if $\mathfrak{f}$ converges continuously to $f \in U$ then $\mathfrak{f}$ is eventually in U.

For the following facts, we refer to Chapters 6 and 15 in [9] as well as to Sections 5.3–5.4 in [13] and to Section II-4 in [12].

Proposition 4. *Let* $\mathbb{X}$, $\mathbb{Y}$ *be* T_0*-spaces.*

(1) *If* $\mathcal{T}_0$ *is a proper and* $\mathcal{T}_1$ *is an admissible topology on* $C(\mathbb{X}, \mathbb{Y})$ *then* $\mathcal{T}_0 \subseteq \mathcal{T}_1$.
(2) *If* $\mathcal{T}_0 \subseteq \mathcal{T}_1$ *and* $\mathcal{T}_1$ *is a proper topology on* $C(\mathbb{X}, \mathbb{Y})$ *then* $\mathcal{T}_0$ *is also a proper topology on* $C(\mathbb{X}, \mathbb{Y})$.
(3) *If* $\mathcal{T}_0 \subseteq \mathcal{T}_1$ *and* $\mathcal{T}_0$ *is an admissible topology on* $C(\mathbb{X}, \mathbb{Y})$ *then* $\mathcal{T}_1$ *is also an admissible topology on* $C(\mathbb{X}, \mathbb{Y})$.
(4) $\mathcal{P} \subseteq \mathcal{K} \subseteq \mathcal{I} \subseteq \mathcal{T}_*$ *and* $\mathcal{K} \subseteq \mathcal{C}$.
(5) *The natural topology* $\mathcal{T}_*$ *is the finest proper topology on* $C(\mathbb{X}, \mathbb{Y})$.

For the next statement, we refer to [4] and to [9, Theorem 6.2.1], see also [13, Theorem 5.4.4].

Theorem 5. *The following conditions are equivalent for a* T_0*-space* $\mathbb{X}$.

(1) $\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$ *is a continuous poset.*
(2) $\langle \mathcal{T}(\mathbb{X}), \mathcal{T}_S(\subseteq) \rangle$ *is an* α*-space.*
(3) *For an arbitrary* T_0*-space* $\mathbb{Y}$, *there is an exponential topology on* $C(\mathbb{X}, \mathbb{Y})$.
(4) *There is an exponential topology on* $C(\mathbb{X}, \mathbb{S})$.

A topological space $\mathbb{X}$ which satisfies the equivalent conditions of Theorem 5 is called a *core-compact space*. A topological space $\mathbb{X}$ is *locally compact*, if for each $x \in X$ and for each $U \in \mathcal{T}(\mathbb{X})$ with $x \in U$, there is a compact set $K \subseteq X$ and an open set $V \in \mathcal{T}(\mathbb{X})$ such that $x \in V \subseteq K \subseteq U$.

For the next statement, we refer to [2], [12, Lemma II-4.2], to [9, Theorem 6.3.3], and to [13, Section 5.4].

Proposition 6. *Let* $\mathbb{X}$, $\mathbb{Y}$ *be* T_0*-spaces.*

(1) *If* $\mathbb{X}$ *is core-compact then* $\mathcal{C} = \mathcal{I} = \mathcal{T}_*$ *and this topology is exponential on* $C(\mathbb{X}, \mathbb{Y})$.
(2) *If* $\mathbb{X}$ *is locally compact then* $\mathcal{K} = \mathcal{C} = \mathcal{I} = \mathcal{T}_*$ *and this topology is exponential on* $C(\mathbb{X}, \mathbb{Y})$.
(3) *If* $\mathbb{X}$ *is an* α^**-space then* $\mathcal{P} = \mathcal{K} = \mathcal{C} = \mathcal{I} = \mathcal{T}_*$ *and this topology is exponential on* $C(\mathbb{X}, \mathbb{Y})$.

Corollary 7. *For a core-compact space* $\mathbb{X}$ *and a* T_0*-space* $\mathbb{Y}$ *the exponential topology on* $C(\mathbb{X}, \mathbb{Y})$ *is defined by the subbasis of open sets*

$$V_{U,W} = \{f \in C(\mathbb{X}, \mathbb{Y}) \mid U \prec f^{-1}(W)\},$$

where $\varnothing \neq U \in \mathcal{T}(\mathbb{X})$ *and* $\varnothing \neq W \in \mathcal{T}(\mathbb{Y})$.

3. d-SPACES $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$

The proof of the following lemma repeats the proof of [10, Lemma 2].

Lemma 8. *Let $\mathbb{X}$ and $\mathbb{Y}$ be T_0-spaces, let $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\leq_{\mathcal{P}})$, and let*

$$F = \{f_i \in C(\mathbb{X}, \mathbb{Y}) \mid i \in I\}$$

be an up-directed set with respect to $\leq_{\mathcal{P}}$. If a function $f\colon X \to Y$ is such that $f(x) = \sup_{\mathbb{Y}}\{f_i(x) \mid i \in I\}$ and $f(x) \in \mathrm{cl}_{\mathbb{Y}}\{f_i(x) \mid i \in I\}$ for all $x \in X$, then $f = \sup F$ in $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$.

Theorem 9. *For T_0-spaces $\mathbb{X}$ and $\mathbb{Y}$, let $\mathcal{T}$ satisfy one of the following conditions:*

(a) *$\mathcal{T}$ is a proper topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T}$;*
(b) *$\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{S}$.*

Then $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a d-space if and only if $\mathbb{Y}$ is a d-space.

Proof. According to [9, Lemma 6.1.9], $\mathcal{T}_A(\leq_{\mathcal{P}})$ is an admissible topology. By Proposition 4(1), we obtain that $\mathcal{T} \subseteq \mathcal{T}_A(\leq_{\mathcal{P}})$. Therefore, $\leq_{\mathcal{T}}$ and $\leq$ agree by Lemma 1.

Suppose first that $\mathbb{Y}$ is a d-space and consider the case when $\mathcal{T}$ is a proper topology finer than $\mathcal{P}$ (condition (a)). According to Lemma 8, each nonempty set $F = \{f_i \in C(\mathbb{X}, \mathbb{Y}) \mid i \in I\}$ which is up-directed with respect to $\leq$ has a least upper bound f. It remains to prove that $f \in \mathrm{cl}_{\mathcal{T}} F$.

We establish that F converges continuously to f. Let a net $\mathfrak{x} = \{x_j \mid j \in J\} \subseteq X$ with J being up-directed converge to some $x \in X$ and let $f(x) \in U \in \mathcal{T}(\mathbb{Y})$. Then $x \in f^{-1}(U) \in \mathcal{T}(\mathbb{X})$ and thus $\mathfrak{x}$ is eventually in $f^{-1}(U)$. This means that $\{f(x_j) \mid j \in J\}$ is eventually in U. Since $f(x_j) \in \mathrm{cl}_{\mathbb{Y}}\{f_i(x_j) \mid i \in I\}$ for all $j \in J$ by Lemma 8 and F is up-directed with respect to $\leq$, we conclude that $\{f_i(x_j) \mid (i, j) \in I \times J\}$ is eventually in U which proves our desired statement.

Let $f \in W \in \mathcal{T}$. By Proposition 4(5), we conclude that $W \in \mathcal{T}_*$. Since F converges continuously to f, we obtain that F is eventually in W. This yields that f is a limit point for F and $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a d-space.

Suppose next that $\mathbb{Y}$ is a d-space and consider the case when $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{S}$ (condition (b)). In the notation above, we have to prove again that $f \in \mathrm{cl}_{\mathcal{T}} F$. Indeed, if $f \in W \in \mathcal{T} \subseteq \mathcal{S}$ then by the definition of the Scott topology, we obtain that $f_i \in W$ for some $i \in I$ which is our desired conclusion.

Suppose now that $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is a d-space. In what follows, we use only the assumption that $\mathcal{P} \subseteq \mathcal{T}$. Let a set $D \subseteq Y$ be up-directed with respect to the specialization order. Then the set $\xi(D) = \{\xi_d \mid d \in D\}$ is also up-directed with respect to $\leq$. According to our assumption, there exists $f = \sup \xi(D) \in C(\mathbb{X}, \mathbb{Y})$ and $f \in \mathrm{cl}_{\mathcal{T}} \xi(D)$. We show that $f \in \xi(Y)$. Indeed, let $f(x_0) \in U \in \mathcal{T}(\mathbb{Y})$. We also choose an arbitrary element $x_1 \in X$. Then $f \in V_{x_0,U} \in \mathcal{P} \subseteq \mathcal{T}$. Thus, there exists $d \in D$ such that $\xi_d \in V_{x_0,U}$, whence $f(x_1) \geq \xi_d(x_1) = d = \xi_d(x_0) \in U$. This yields that $f(x_1) \in U$ whence $f(x_0) \leq f(x_1)$. A similar argument shows that $f(x_1) \leq f(x_0)$. Therefore $f = \xi_y$ for some $y \in Y$. It is clear that $y = \sup_{\mathbb{Y}} D$. If $y \in U \in \mathcal{T}(\mathbb{Y})$, then $\xi_y \in V_{x,U} \in \mathcal{P} \subseteq \mathcal{T}$, where $x \in X$. As $\xi_y \in \mathrm{cl}_{\mathcal{T}} \xi(D)$, we conclude that $\xi_d \in V_{x,U}$ for some $d \in D$. Thus, $d = \xi_d(x) \in U$ and $y \in \mathrm{cl}_{\mathbb{Y}} D$. □

The fact that $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is a d-space whenever $\mathbb{Y}$ is a d-space was established in [5]. The fact that $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ is a d-space whenever $\mathbb{Y}$ is a d-space is established in [12, Lemma II-4-3(i)]. Theorem 3.3 in [14] shows that $\mathbb{Y}$ is a d-space provided that $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ is a d-space for some T_0-space $\mathbb{X}$. For the case $\mathcal{T} = \mathcal{P}$, Theorem 9 was proved in [10], see also [9, Corollary 8.4.2].

4. Sobriety in function spaces

The proof of the following statement almost repeats the proof of [10, Theorem 6]. Nonetheless, we give it here for the sake of completeness.

Proposition 10. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces and let $\mathcal{T}$ be a topology on $C(\mathbb{X},\mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ is a sober space then $\mathbb{Y}$ is also sober.*

Proof. Let $\mathbb{Z}$ be a T_0-space and let a subspace $\mathbb{Z}_0 \leq \mathbb{Z}$ be such that $Z = \mathrm{sob}_\mathbb{Z} Z_0$. Consider an arbitrary continuous function $f_0\colon \mathbb{Z}_0 \to \mathbb{Y}$. By Lemma 3, $\xi\colon \mathbb{Y} \to \mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ is continuous. Thus, the function $\xi f_0\colon \mathbb{Z}_0 \to \mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ is continuous. As $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ is a sober space, there is a continuous function $f\colon \mathbb{Z} \to \mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ such that $f|_{\mathbb{Z}_0} = \xi f_0$. We prove that $f(Z) \subseteq \xi(Y)$. Indeed, let $z \in Z$; then $z = \sup_\mathbb{Z}(\downarrow z \cap Z_0)$ and $z \in \mathrm{cl}_\mathbb{Z}(\downarrow z \cap Z_0)$. It is straightforward to verify that $f(z) \in \mathrm{cl}_\mathbb{Y} f(\downarrow z \cap Z_0)$ and $f(z) = \sup f(\downarrow z \cap Z_0)$, see Lemma 1.5.3 in [9]. Let x_0, $x_1 \in X$ and let $[f(z)](x_0) \in U \in \mathcal{T}(\mathbb{Y})$. Then $f(z) \in V_{x_0,U} \in \mathcal{P} \subseteq \mathcal{T}$, whence $z \in f^{-1}(V_{x_0,U}) \in \mathcal{T}(\mathbb{Z})$. Therefore there is $z_0 \in \downarrow z \cap Z_0$ such that $f(z_0) \in V_{x_0,U}$. Since the function $f(z_0)$ is constant, $f(z_0) \in V_{x_1,U} \in \mathcal{P} \subseteq \mathcal{T}$. But then $f(z) \in V_{x_1,U}$, as $f(z_0) \leq f(z)$. Therefore $[f(z)](x_1) \in U$. Similarly, $[f(z)](x_1) \in U$ implies that $[f(z)](x_0) \in U$; that is, $[f(z)](x_0) = [f(z)](x_1)$, which is our desired conclusion. Inclusion $f(Z) \subseteq \xi(Y)$ implies that $(\xi^{-1} f)|_{\mathbb{Z}_0} = f_0$. In view of [9, Theorem 5.3.2], $\mathbb{Y}$ is a sober space. □

For the following statement, we refer to [10, Proposition 16] or to [9, Proposition 6.1.15].

Proposition 11. *Let $\mathcal{T}$ be an exponential topology on $C(\mathbb{X},\mathbb{Y})$.*

(1) *If $\mathbb{Y}$ is [densely] injective, then $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ is [densely] injective.*
(2) *If $\mathbb{Y}$ is sober, then $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ is sober.*

The following statement generalizes Theorem 6 in [10].

Theorem 12. *For a T_0-space $\mathbb{Y}$, the following conditions are equivalent.*

(1) *$\mathbb{Y}$ is sober.*
(2) *$\mathbb{C}_\mathcal{J}(\mathbb{X},\mathbb{Y})$ is sober for each core-compact space $\mathbb{X}$.*
(3) *$\mathbb{C}_\mathcal{J}(\mathbb{X},\mathbb{Y})$ is sober for some core-compact space $\mathbb{X}$.*
(4) *$\mathbb{C}_\mathcal{J}(\mathbb{X},\mathbb{Y})$ is sober for some T_0-space $\mathbb{X}$.*
(5) *$\mathbb{C}(\mathbb{X},\mathbb{Y})$ is sober for each T_0-space $\mathbb{X}$.*
(6) *$\mathbb{C}(\mathbb{X},\mathbb{Y})$ is sober for some T_0-space $\mathbb{X}$.*
(7) *$\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ is sober for some [core-compact] T_0-space $\mathbb{X}$ and some topology $\mathcal{T}$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$.*

Proof. (1) implies (2) by Proposition 11(2). (2) implies (3) and (3) implies (4) in a trivial way. (4) implies (1) by Proposition 10. Statements (1), (5), and (6) are equivalent by [10, Theorem 6]. Furthermore, (3) obviously implies (7) and (7) implies (1) by Proposition 10. □

5. Essential completeness in function spaces

As in [6], we consider the following properties of a T_0-space $\mathbb{X}$.

(H_0) $\mathbb{X}$ has a least element 0 with respect to the specialization order $\leq$.
(H_1) $\mathbb{X}$ is a join semilattice with respect to $\leq$; $\vee$ denotes the join operation in X.

(H$_2$) The join operation $\vee\colon \mathbb{X}^2 \to \mathbb{X}$ is continuous; that is, $\langle X, \vee, \mathcal{T}(\mathbb{X})\rangle$ is a topological join-semilattice.

In [6], the following statement was proved, see also [9, Corollary 10.4.2].

Theorem 13. [9, Corollary 10.4.2] *A T_0-space $\mathbb{X}$ is essentially complete if and only if $\mathbb{X}$ is a d-space with properties* (H$_0$)–(H$_2$).

Proposition 14. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces and let $\mathcal{T}$ be a topology on $C(\mathbb{X},\mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ has the properties* (H$_1$)–(H$_2$) *then $\mathbb{Y}$ also possesses the properties* (H$_1$)–(H$_2$).

Proof. According to our assumption, for arbitrary $y_0, y_1 \in Y$, there is $f = \xi_{y_0} \vee \xi_{y_1} \in C(\mathbb{X},\mathbb{Y})$. We prove that f is a constant function. Indeed, let $f(x_0) \in U \in \mathcal{T}(\mathbb{Y})$; then $f \in V_{x_0,U} \in \mathcal{P} \subseteq \mathcal{T}$. We may assume without loss of generality that $U \neq Y$. We choose an arbitrary element $x_1 \in X$. Since $\vee$ is a continuous function, there are $V_0, V_1 \in \mathcal{T}$ such that $\xi_{y_0} \in V_0$, $\xi_{y_1} \in V_1$, and $V_0 \cap V_1 \subseteq V_{x_0,U}$. As $U \neq Y$, we conclude that $V_{x_0,U} \neq C(\mathbb{X},\mathbb{Y})$. There are three possible cases.

Case 1: $V_{1-i} = C(\mathbb{X},\mathbb{Y})$ for some $i < 2$. In this case, $\xi_{y_i} \in V_i = V_0 \cap V_1 \subseteq V_{x_0,U}$ and thus $y_i \in U$. Therefore, $\xi_{y_i} \in V_{x_1,U}$. As $\xi_{y_i} \leq f$, we obtain that $f \in V_{x_1,U}$ and $f(x_1) \in U$.

Case 2: $V_0, V_1 \neq C(\mathbb{X},\mathbb{Y})$. Since $\mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$ and $V_0, V_1 \in \mathcal{T}$, we conclude that there are $k_0 > 0$ and $k_1 > 0$, there are upper cones $\mathcal{H}_{i0}, \ldots, \mathcal{H}_{ik_i}$ in $\langle \mathcal{T}(\mathbb{X}); \subseteq\rangle$, and there are open sets $W_{i0}, \ldots, W_{ik_i} \in \mathcal{T}(\mathbb{Y})$, $i < 2$, such that

$$\xi_{y_i} \in V_{\mathcal{H}_{i0},W_{i0}} \cap \ldots \cap V_{\mathcal{H}_{ik_i},W_{ik_i}} \subseteq V_i \quad \text{for all } i < 2.$$

Therefore, $\xi_{y_i}^{-1}(W_{ij}) \in \mathcal{H}_{ij}$ for all $i < 2$ and all $j \leqslant k_i$. We put $U_0 = W_{00} \cap \ldots \cap W_{0k_0}$ and $U_1 = W_{10} \cap \ldots \cap W_{1k_1}$; then $U_0, U_1 \in \mathcal{T}(\mathbb{Y})$. As $\varnothing \notin \{V_0, V_1\}$, we conclude that $\mathcal{H}_{ij} \neq \varnothing$ for all $i < 2$ and all $j \leqslant k_i$. As $V_0, V_1 \neq C(\mathbb{X},\mathbb{Y})$, we may assume that $\mathcal{H}_{ij} \neq \mathcal{T}(\mathbb{X})$ for all $i < 2$ and all $j \leqslant k_i$. This yields that $\xi_{y_i}^{-1}(W_{ij}) \neq \varnothing$ whence $\xi_{y_i}^{-1}(W_{ij}) = X$ for all $i < 2$ and all $j \leqslant k_i$. Hence, $y_i \in W_{ij}$ for all $i < 2$ and all $j \leqslant k_i$ and therefore, $y_0 \in U_0$ and $y_1 \in U_1$.

Since ξ_{y_0}, ξ_{y_1} are constant functions, we have $\xi_{y_0} \in V_{x_1,U_0}$ and $\xi_{y_1} \in V_{x_1,U_1}$. Therefore $f \in V_{x_1,U_0} \cap V_{x_1,U_1} = V_{x_1,U_0\cap U_1}$, whence $f(x_1) \in U_0 \cap U_1 \in \mathcal{T}(\mathbb{Y})$. Moreover, if $y' \in U_0 \cap U_1$ then

$$\xi_{y'} \in V_{\mathcal{H}_{00},W_{00}} \cap \ldots \cap V_{\mathcal{H}_{0k_0},W_{0k_0}} \cap V_{\mathcal{H}_{10},W_{10}} \cap \ldots \cap V_{\mathcal{H}_{1k_1},W_{1k_1}} \subseteq V_0 \cap V_1 \subseteq V_{x_0,U},$$

whence $y' \in U$. We have therefore proved that $f(x_1) \in U_0 \cap U_1 \subseteq U$. This means that $f(x_0) \leq f(x_1)$. A symmetric argument shows that $f(x_1) \leq f(x_0)$. We obtain that $f = \xi_y$ for some $y \in Y$, whence $y = y_0 \vee y_1$ in $\mathbb{Y}$. Thus, $\mathbb{Y}$ has the property (H$_1$).

Finally, we prove that $\mathbb{Y}$ has the property (H$_2$). Suppose that $y = y_0 \vee y_1 \in U \in \mathcal{T}(\mathbb{Y})$. Then $\xi_{y_0} \vee \xi_{y_1} = \xi_y \in V_{x,U}$ for each $x \in X$. Since $\vee$ is a continuous function on $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$, there are open sets $V_0, V_1 \in \mathcal{T}$ such that $\xi_{y_0} \in V_0$, $\xi_{y_1} \in V_1$, and $V_0 \cap V_1 \subseteq V_{x_0,U}$. As $\mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$, we conclude that there are $k_0 > 0$ and $k_1 > 0$, there are upper cones $\mathcal{H}_{i0}, \ldots, \mathcal{H}_{ik_i}$ in $\langle \mathcal{T}(\mathbb{X}); \subseteq\rangle$, and there are open sets $W_{i0}, \ldots, W_{ik_i} \in \mathcal{T}(\mathbb{Y})$, $i < 2$, such that

$$\xi_{y_i} \in V_{\mathcal{H}_{i0},W_{i0}} \cap \ldots \cap V_{\mathcal{H}_{ik_i},W_{ik_i}} \subseteq V_i \quad \text{for all } i < 2.$$

Therefore, $\xi_{y_i}^{-1}(W_{ij}) \in \mathcal{H}_{ij}$ for all $i < 2$ and all $j \leqslant k_i$. In particular, $\mathcal{H}_{ij} \neq \varnothing$ for all $i < 2$ and all $j \leqslant k_i$. We put

$$U_0 = \bigcap \bigl\{W_{0j} \mid j \leqslant k_0,\ \mathcal{H}_{0j} \neq \mathcal{T}(\mathbb{X})\bigr\} \text{ and } U_1 = \bigcap \bigl\{W_{1j} \mid j \leqslant k_1,\ \mathcal{H}_{1j} \neq \mathcal{T}(\mathbb{X})\bigr\}.$$

Then we have that $U_0, U_1 \in \mathcal{T}(\mathbb{Y})$. Moreover, $\xi_{y_i}^{-1}(W_{ij}) \neq \varnothing$ whence $\xi_{y_i}^{-1}(W_{ij}) = X$ for all $i < 2$ and all $j \leqslant k_i$ such that $\mathcal{H}_{ij} \neq \mathcal{T}(\mathbb{X})$. As above, we obtain that $y_0 \in U_0$ and $y_1 \in U_1$ and $U_0 \cap U_1 \subseteq U$. Therefore, $\mathbb{Y}$ has the property (H_2). □

The following statement has quite a straightforward proof and is to find as Proposition 3.2 in [3].

Proposition 15. *Let $\mathbb{X}$ be a core-compact space and let $\mathcal{B} \subseteq \mathcal{T}(\mathbb{X})$ be an additive basis of $\mathcal{T}(\mathbb{X})$. If open sets $U, U', V_0, \ldots, V_n \in \mathcal{B}$, $n < \omega$, are such that $U \prec U' \prec V_0 \cup \ldots \cup V_n$ then there are $W_0, \ldots, W_n \in \mathcal{B}$ such that*

$$W_i \prec V_i \text{ for all } i \leqslant n \text{ and } U \subseteq W_0 \cup \ldots \cup W_n \subseteq U'.$$

Proof. For each $x \in U'$, there is $i \leqslant n$ such that $x \in U' \cap V_i \in \mathcal{T}(\mathbb{X})$. As $\mathbb{X}$ is core-compact and as $\mathcal{B}$ is a basis, there is $W_x \in \mathcal{B}$ such that $x \in W_x \prec U' \cap V_i$. Hence, $U \prec U' \subseteq \bigcup_{x \in U'} W_x$. This yields that $U \subseteq \bigcup_{j \leqslant k} W_{x_j} \subseteq U'$ for some $x_0, \ldots, x_k \in U'$. For each $i \leqslant n$, we put $W_i = \bigcup\{W_{x_j} \mid j \leqslant k,\ W_{x_j} \prec U' \cap V_i\}$. Then it is clear that $W_i \prec V_i$ for all $i \leqslant n$ and that $U \subseteq W_0 \cup \ldots \cup W_n \subseteq U'$. □

Proposition 16. *Let $\mathbb{X}$ be a core-compact space and let $\mathbb{Y}$ possess the properties* (H_1)–(H_2). *Then $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ also possesses the properties* (H_1)–(H_2).

Proof. Let $f_0, f_1 \in C(\mathbb{X}, \mathbb{Y})$. For each $x \in X$, we put

$$g(x) = f_0(x) \vee f_1(x).$$

We prove that $g \in C(\mathbb{X}, \mathbb{Y})$. Indeed, let $V \in \mathcal{T}(\mathbb{Y})$ and let $x \in g^{-1}(V)$. As $f_0(x) \vee f_1(x) \in V$ and $\vee$ is continuous in $\mathbb{Y}$, we conclude that there are open sets $V_0, V_1 \in \mathcal{T}(\mathbb{Y})$ such that $f_0(x) \in V_0$, $f_1(x) \in V_1$, and $V_0 \cap V_1 \subseteq V$. But then

$$x \in U = f_0^{-1}(V_0) \cap f_1^{-1}(V_1) \in \mathcal{T}(\mathbb{X}).$$

If $z \in U$ then $g(z) = f_0(z) \vee f_1(z) \geq f_0(z) \in V_0$ and $g(z) \geq f_1(z) \in V_1$, whence $g(z) \in V_0 \cap V_1 \subseteq V$ which implies that $x \in U \subseteq g^{-1}(V)$. The fact that $U \in \mathcal{T}(\mathbb{X})$ proves that $g^{-1}(V) \in \mathcal{T}(\mathbb{Y})$. Therefore, g is continuous and $g = f_0 \vee f_1$ in $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$. Hence, $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ has the property (H_1).

To prove that $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ has the property (H_2), we consider two functions $f_0, f_1 \in C(\mathbb{X}, \mathbb{Y})$ and open sets $U \in \mathcal{T}(\mathbb{X})$, $W \in \mathcal{T}(\mathbb{Y})$ such that

$$g = f_0 \vee f_1 \in V_{U,W}.$$

This means that $U \prec g^{-1}(W)$. Since $\mathbb{Y}$ possesses (H_2), for each element $x \in g^{-1}(W)$, there are open sets $V_{0x}, V_{1x} \in \mathcal{T}(\mathbb{Y})$ such that $f_0(x) \in V_{0x}$, $f_1(x) \in V_{1x}$, and $V_{0x} \cap V_{1x} \subseteq W$. We claim that

$$g^{-1}(W) = \bigcup \bigl\{f_0^{-1}(V_{0x}) \cap f_1^{-1}(V_{1x}) \mid x \in g^{-1}(W)\bigr\}.$$

Indeed it is clear that $g^{-1}(W) \subseteq \bigcup\bigl\{f_0^{-1}(V_{0x}) \cap f_1^{-1}(V_{1x}) \mid x \in g^{-1}(W)\bigr\}$. To prove the reverse inclusion, we consider an arbitrary element $x \in g^{-1}(W)$ and establish that $U(x) = f_0^{-1}(V_{0x}) \cap f_1^{-1}(V_{1x}) \subseteq g^{-1}(W)$. Indeed, let $z \in U(x)$. Then $g(z) \geq f_0(z) \in V_{0x}$ and $g(z) \geq f_1(z) \in V_{1x}$, whence $g(z) \in V_0 \cap V_1 \subseteq W$ which implies that $z \in g^{-1}(W)$.

Furthermore, since $U \prec g^{-1}(W) = \bigcup\{U(x) \mid x \in g^{-1}(W)\}$, we conclude that there are $x_0, \ldots, x_k \in g^{-1}(W)$ such that $U \prec U(x_0) \cup \ldots \cup U(x_k)$. As $\mathbb{X}$ is core-compact, there is $V \in \mathcal{T}(\mathbb{X})$ such that

$$U \prec V \prec U(x_0) \cup \ldots \cup U(x_k).$$

Taking $\mathcal{B} = \mathcal{T}(\mathbb{X})$ and applying Proposition 15, we obtain that there are open sets $S_0, \ldots, S_k$, $T_0, \ldots, T_k \in \mathcal{T}(\mathbb{X})$ such that

$$S_i \prec T_i \prec U(x_i) \text{ for all } i \leqslant k \text{ and } U \subseteq S_0 \cup \ldots \cup S_n \subseteq V.$$

For each $i \leqslant k$, we have $T_i \prec U(x_i) = f_0^{-1}(V_{0x_i}) \cap f_1^{-1}(V_{1x_i})$ whence

$$f_0 \in W_0 = V_{T_0, V_{0x_0}} \cap \ldots \cap V_{T_k, V_{0x_k}} \in \mathcal{I};$$
$$f_1 \in W_1 = V_{T_0, V_{1x_0}} \cap \ldots \cap V_{T_k, V_{1x_k}} \in \mathcal{I}.$$

To establish (H_2), it suffices to prove that $W_0 \cap W_1 \subseteq V_{U,W}$. Indeed, if $h \in W_0 \cap W_1$ then $h \in V_{T_i, V_{0x_i}} \cap V_{T_i, V_{1x_i}}$ for each $i \leqslant k$. This implies that

$$S_i \prec T_i \subseteq h^{-1}(V_{0x_i}) \cap h^{-1}(V_{1x_i}) = h^{-1}(V_{0x_i} \cap V_{1x_i}) \subseteq h^{-1}(W)$$

for all $i \leqslant k$. Hence, $U \subseteq S_0 \cup \ldots \cup S_k \prec h^{-1}(W)$ yielding $h \in V_{U,W}$, which is our desired conclusion. □

Corollary 17. *For a T_0-space $\mathbb{Y}$, the following statements are equivalent.*

(1) $\mathbb{Y}$ *is a topological semilattice with respect to* $\leq_{\mathbb{Y}}$.
(2) $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ *is a topological semilattice with respect to* $\leq$ *for each core-compact space* $\mathbb{X}$.
(3) $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ *is a topological semilattice with respect to* $\leq$ *for some core-compact space* $\mathbb{X}$.
(4) $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ *is a topological semilattice with respect to* $\leq$ *for some T_0-space* $\mathbb{X}$.
(5) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ *is a topological semilattice with respect to* $\leq$ *for each T_0-space* $\mathbb{X}$.
(6) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ *is a topological semilattice with respect to* $\leq$ *for some T_0-space* $\mathbb{X}$.
(7) $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ *is a topological semilattice with respect to* $\leq$ *for some [core-compact] T_0-space $\mathbb{X}$ and some topology $\mathcal{T}$ such that* $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$.

Proof. (1) implies (2) by Proposition 16. (2) implies (3) and (3) implies (4) in a trivial way. (4) implies (1) by Proposition 14. Statements (1), (5), and (6) are equivalent by [10, Proposition 9]. Furthermore, (3) obviously implies (7) and (7) implies (1) by Proposition 14. □

The next theorem is a generalization of Theorem 10 from [10].

Theorem 18. *For a T_0-space $\mathbb{Y}$, the following statements are equivalent.*

(1) $\mathbb{Y}$ *is essentially complete.*
(2) $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ *is essentially complete for each core-compact space* $\mathbb{X}$.
(3) $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ *is essentially complete for some core-compact space* $\mathbb{X}$.
(4) $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ *is essentially complete for some T_0-space* $\mathbb{X}$.
(5) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ *is essentially complete for each T_0-space* $\mathbb{X}$.
(6) $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ *is essentially complete for some T_0-space* $\mathbb{X}$.
(7) $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ *is essentially complete for some [core-compact] T_0-space $\mathbb{X}$ and some topology $\mathcal{T}$ such that* $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$.

Proof. If $\mathbb{X}$ is a core-compact space and $\mathbb{Y}$ is an essentially complete space, then $\mathbb{Y}$ is a d-space and possesses the properties (H_0)–(H_2) by Theorem 13. By Theorem 9, $\mathbb{C}_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$ is a d-space. By Proposition 16, $\mathbb{C}_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$ has the properties (H_1)–(H_2). By Lemma 1 and Lemma 2, $\mathbb{C}_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$ has the property (H_0). Applying Theorem 13 again, we obtain that $\mathbb{C}_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$ is essentially complete. Therefore, (1) implies (2).

It is straightforward that (2) implies (3), (3) implies (4), and (3) implies (7).

Suppose now that $\mathbb{C}_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$ is an essentially complete space for some T_0-spaces $\mathbb{X}$ and $\mathbb{Y}$. According to Theorem 13, $\mathbb{C}_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$ is a d-space and has the properties (H_0)–(H_2). By the proof of Theorem 9, $\mathbb{Y}$ is also a d-space. By Proposition 14, $\mathbb{Y}$ has the properties (H_1)–(H_2). By Lemma 1 and Lemma 2, $\mathbb{Y}$ has the property (H_0). By Theorem 13, $\mathbb{Y}$ is essentially complete and (4) implies (1). Then (7) implies (1) in a similar way.

Statements (1) and (5)–(6) are equivalent by [10, Theorem 10]. □

6. A-spaces

Definition 1. A T_0-space $\mathbb{X}$ is an *A-space* [an *f-space*] if it has a basic subspace $\mathbb{X}_0$ [consisting of compact elements, respectively] which is a *parus* with respect to the specialization order $\leq_{\mathbb{X}}$; that is, if $x, y \in X_0$ and $x, y \leq a$ for some $a \in X$, then there exists $x \vee y \in X_0$. In this case, $\mathbb{X}_0$ is called an *A-basic subspace of* $\mathbb{X}$ [an *f-basic subspace of* $\mathbb{X}$, respectively].

The following theorem was established in [9].

Theorem 19. [9, Theorem 7.3.4] *Let $\mathbb{X}$ be an α^*-space and let $\mathbb{Y}$ be an A-space with a least element. Then $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is also an A-space with a least element.*

The next statement shows that the requirement in Theorem 19 on $\mathbb{Y}$ to have a least element can be weakened under certain circumstances. The proof of Theorem 20 follows the same lines as the one of Theorem 7.3.4 in [9]. In the proof, we also use some ideas from [15].

Theorem 20. *Let $\mathbb{X}$ be a core-compact space and let $\mathbb{Y}$ be a T_0-space such that $\downarrow y$ has a least element for all $y \in Y$. Then $\downarrow f$ has a least element with respect to $\leq_{\mathcal{P}}$ for each $f \in C(\mathbb{X},\mathbb{Y})$. Moreover, if $\mathbb{Y}$ is an A_d-space then $\mathbb{C}_{\mathcal{S}}(\mathbb{X},\mathbb{Y})$ is an A_d-space.*

Proof. As usual, we write $\leq$ instead of $\leq_{\mathcal{P}}$ for the sake of simplicity. For each $y \in Y$, let $\perp_y$ denote the least element of the set $\downarrow y$ in $\mathbb{Y}$ and let $\mathbb{Y}_0$ be an A-basic subspace in $\mathbb{Y}$.

According to Theorem 9, $\mathbb{C}_{\mathcal{S}}(\mathbb{X},\mathbb{Y})$ is a d-space.

Claim 1. Let $\mathbb{Z}$ be an arbitrary α-space and let $z \in Z$ be such that $\downarrow_{\mathbb{Z}} z$ has a least element $\perp_z$. Then $\perp_z \prec_{\mathbb{Z}} \perp_z$.

Proof of Claim. Indeed, let $a \in \uparrow\perp_z$. Since $a \in Z$ and $\mathbb{Z}$ is a basic subspace in $\mathbb{Z}$, we conclude that there is $b \in Z$ such that $b \prec a$. But then $b \in \downarrow a$ and thus $\perp_z \leq_{\mathbb{Z}} b \prec a$. This implies that $a \in \operatorname{int}\uparrow b \subseteq \uparrow\perp_z$ which yields that $\uparrow\perp_z \in \mathcal{T}(\mathbb{Z})$. □

It follows from Claim 1 that $\perp_y \in Y_0$ for all $y \in Y$.

Claim 2. $\downarrow f$ has a least element with respect to $\leq$ for each $f \in C(\mathbb{X},\mathbb{Y})$.

Proof of Claim. Consider the mapping

$$\perp_f \colon X \to Y, \quad \perp_f \colon x \mapsto \perp_{f(x)}.$$

It is straightforward that $\bot_f \leq g$ for each $g \leq f$. So, it remains to prove that $\bot_f$ is continuous. Indeed, let $V \in \mathcal{T}(\mathbb{Y})$; then applying Claim 1 and using the continuity of f, we have

$$\bot_f^{-1}(V) = \{x \in X \mid \bot_{f(x)} \in V\} = \bigcup\big\{f^{-1}(\uparrow\bot_y) \mid y \in Y,\ \bot_y \in V\big\} \in \mathcal{T}(\mathbb{X}),$$

which completes the proof. □

Similarly to the proof of [9, Theorem 7.3.4], we consider the set

$$\begin{aligned}\mathcal{W} =& \Big\{\langle f, \varnothing, \varnothing\rangle \mid f \in C(\mathbb{X}, \mathbb{Y})\Big\} \cup \\ & \cup \Big\{\big\langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\}\big\rangle \mid \\ & \quad f \in C(\mathbb{X}, \mathbb{Y}),\ n < \omega,\ U_0, \dots, U_n \in \mathcal{T}(\mathbb{X})\backslash\{\varnothing\},\ y_0, \dots, y_n \in Y_0\Big\},\end{aligned}$$

where each $\big\langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\}\big\rangle \in \mathcal{W}$ possesses the following properties:

($*$) the set $\{U_0, \dots, U_n\}$ is closed under nonempty intersections;

($**$) if $U_i \subseteq U_j$ for some $i, j \leqslant n$ then $y_j \leq_{\mathbb{Y}} y_i$.

($***$) $\bot_{f(x)} \leq y_i$ for all $x \in \bigcup_{j\leqslant n} U_j$, where $i \leqslant n$ is such that $U_i = \bigcap\{U_j \mid j \leqslant n,\ x \in U_j\}$.

For each $W = \big\langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\}\big\rangle \in \mathcal{W}$, we define a mapping $f_W : X \to Y$ by the following rule:

$$f_W(x) = \begin{cases} y_i, & \text{if } U_i = \bigcap\{U_j \mid j \leqslant n,\ x \in U_j\}; \\ \bot_{f(x)}, & \text{if } x \notin \bigcup_{j\leqslant n} U_j. \end{cases}$$

It follows from the definition that $f_W(X) \subseteq Y_0$.

Claim 3. $f_W \in C(\mathbb{X}, \mathbb{Y})$ for each $W \in \mathcal{W}$.

Proof of Claim. Let $W = \big\langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\}\big\rangle$. Since $\mathbb{Y}_0$ is a basic subspace of $\mathbb{Y}$, it suffices to show that $f_W^{-1}(\operatorname{int}\uparrow a) \in \mathcal{T}(\mathbb{X})$ for all $a \in Y_0$. Indeed, for a fixed element $a \in Y_0$, the following three cases are possible.

Case 1: $\bot_{f(x)} \in \operatorname{int}\uparrow a$ for some $x \in X$. We have in this case that $a \prec \bot_{f(x)}$ whence $a = \bot_{f(x)}$ as $\bot_{f(x)}$ is a least element in $\downarrow f(x)$. We claim that $f_W^{-1}(\operatorname{int}\uparrow a) = f^{-1}(\operatorname{int}\uparrow a)$.

Indeed, let $x' \in f_W^{-1}(\uparrow\bot_{f(x)}) = f_W^{-1}(\operatorname{int}\uparrow a)$. This means that $\bot_{f(x)} \leq_{\mathbb{Y}} f_W(x')$. There are two subcases.

Case 1.1: $x' \in \bigcup_{i\leqslant n} U_i$. Let $j \leqslant n$ be such that $U_j = \bigcap\{U_i \mid i \leqslant n,\ x' \in U_i\}$. Then $\bot_{f(x)} \leq f_W(x') = y_j$. As $\bot_{f(x')} \leq y_j$ by ($***$), we conclude that $a = \bot_{f(x)} = \bot_{f(x')} \leq f(x')$. This implies that $x' \in f^{-1}(\operatorname{int}\uparrow a)$.

Case 1.2: $x' \notin \bigcup_{i\leqslant n} U_i$. In this case, $\bot_{f(x)} \leq f_W(x') = \bot_{f(x')}$ whence $a = \bot_{f(x)} \prec \bot_{f(x)} = \bot_{f(x')} \leq f(x')$. Therefore, $x' \in f^{-1}(\operatorname{int}\uparrow a)$ again.

Thus, $f_W^{-1}(\operatorname{int}\uparrow a) \subseteq f^{-1}(\operatorname{int}\uparrow a)$. To prove the reverse inclusion, suppose that $x' \in f^{-1}(\operatorname{int}\uparrow a)$. This yields that $a = \bot_{f(x)} \leq f(x')$ whence $\bot_{f(x)} = \bot_{f(x')}$. If $x' \in \bigcup_{i\leqslant n} U_i$ then there is $j \leqslant n$ such that $U_j = \bigcap\{U_i \mid i \leqslant n,\ x' \in U_i\}$. Thus, $a = \bot_{f(x)} = \bot_{f(x')} \leq y_j = f_W(x')$ by ($***$). If $x \notin \bigcup_{i\leqslant n} U_i$ then $a = \bot_{f(x)} = \bot_{f(x')} = f_W(x')$ and $x' \in f_W^{-1}(\operatorname{int}\uparrow a)$ in any case.

Therefore, $f^{-1}(\operatorname{int}\uparrow a) \subseteq f_W^{-1}(\operatorname{int}\uparrow a)$ and $f_W^{-1}(\operatorname{int}\uparrow a) = f^{-1}(\operatorname{int}\uparrow a) \in \mathcal{T}(\mathbb{X})$ in Case 1.

Case 2: $\{x \in X \mid a \prec \perp_{f(x)}\} = \varnothing$ and $I = \{i \leqslant n \mid a \prec y_i\} \neq \varnothing$. We put $U = \bigcup_{i \in I} U_i$; then $U \in \mathcal{T}(\mathbb{X})$. We show that $U = f_W^{-1}(\operatorname{int}\uparrow a)$. Indeed, if $x \in U$ then $x \in U_j$ for some $j \in I$. But then $f_W(x) = y_i$ for some $i \leqslant n$ such that $x \in U_i \subseteq U_j$. Applying $(**)$, we obtain that $a \prec y_j \leq y_i = f_W(x)$. This means that $x \in f_W^{-1}(\operatorname{int}\uparrow a)$ and $U \subseteq f_W^{-1}(\operatorname{int}\uparrow a)$. Conversely, let $x \in f_W^{-1}(\operatorname{int}\uparrow a)$. As $\{x \in X \mid a \prec \perp_{f(x)}\} = \varnothing$, we conclude that $y_i = f_W(x) \in \operatorname{int}\uparrow a$ whence $i \in I$. It follows that $x \in U_i \subseteq U$. Therefore, $f_W^{-1}(\operatorname{int}\uparrow a) \subseteq U$ which is our desired conclusion.

Case 3: $\{x \in X \mid a \prec \perp_{f(x)}\} = \{i \leqslant n \mid a \prec y_i\} = \varnothing$. In this case, $f_W^{-1}(\operatorname{int}\uparrow a) = \varnothing \in \mathcal{T}(\mathbb{X})$. □

The following statement is a corollary of the definition of f_W.

Claim 4. For $f \in C(\mathbb{X}, \mathbb{Y})$ and $W = \langle f, \{U_0, \dots, U_n\}, \{y_0, \dots, y_n\} \rangle \in \mathcal{W}$, we have $f_W \leq f$ if and only if $y_i \leq f(x)$ for all $x \in \bigcup_{j \leqslant n} U_j$, where $i \leqslant n$ is such that $U_i = \bigcap \{U_j \mid j \leqslant n,\ x \in U_j\}$.

Claim 5. Let $f \in C(\mathbb{X}, \mathbb{Y})$ and let $W = \langle f, \{U\}, \{y\} \rangle$ be such that $y \in Y_0$, $U \in \mathcal{T}(\mathbb{X})$, and $U \prec f^{-1}(\operatorname{int}\uparrow y)$. Then $f_W \ll f$.

Proof of Claim. First of all, we notice that $W \in \mathcal{W}$ provided that $y \prec f(x)$ for all $x \in U$. Indeed, for each $x \in U$, we have $\perp_{f(x)} = \perp_y \leq y \prec f(x)$ which implies that W possesses the property $(***)$. The conditions $(*)$–$(**)$ are satisfied in a trivial way. Moreover, it is straightforward to see that $f_W \leq f$.

Suppose that $f \leq g = \bigvee_{i \in I} g_i$ for some family $\{g_i \in C(\mathbb{X}, \mathbb{Y}) \mid i \in I\}$ which is up-directed with respect to $\leq_{\mathcal{P}}$. According to Theorem 9, $h(x) = \bigvee_{i \in I} g_i(x)$ exists for all $x \in X$. It is clear that $h(x) \leq g(x)$ for all $x \in X$. Conversely, if $g(x) \in U \in \mathcal{T}(\mathbb{Y})$ then $g \in V_{x,U} \in \mathcal{P} \subseteq \mathcal{S}$. Since $\mathbb{C}_{\mathcal{S}}(\mathbb{X}, \mathbb{Y})$ is a d-space, we conclude that $g_i \in V_{x,U}$ for some $i \in I$, whence $h(g) \geq g_i(x) \in U$. Therefore, $h(x) \in U$ and $g(x) \leq h(x)$; that is, $g(x) = \bigvee_{i \in I} g_i(x)$ for all $x \in X$. Hence, we have by our assumption that

$$U \prec f^{-1}(\operatorname{int}\uparrow y) \subseteq g^{-1}(\operatorname{int}\uparrow y) = \bigcup_{i \in I} g_i^{-1}(\operatorname{int}\uparrow y).$$

This yields that $U \subseteq g_i^{-1}(\operatorname{int}\uparrow y)$ and $g_i(U) \subseteq \uparrow y$ for some $i \in I$. We claim that $f_W \leq g_i$. Indeed, if $x \in U$ then $f_W(x) = y \leq g_i(x)$. Suppose now that $x \notin U$. Since $f(x) \leq g(x)$ and $g_i(x) \leq g(x)$, we conclude that $\perp_{f(x)} = \perp_{g(x)} = \perp_{g_i(x)}$. Therefore, $f_W(x) = \perp_{f(x)} = \perp_{g_i(x)} \leq g_i(x)$ and $f_W \leq g_i$ which proves that $f_W \ll f$. □

Claim 6. We have

$$f = \bigvee \Big\{ f_{\langle f, \{U\}, \{y\} \rangle} \mid y \prec f(x) \text{ for some } x \in X,\ U \prec f^{-1}(\operatorname{int}\uparrow y) \Big\}$$

for each function $f \in C(\mathbb{X}, \mathbb{Y})$. In particular, $\langle C(\mathbb{X}, \mathbb{Y}); \leq_{\mathcal{P}} \rangle$ is a continuous poset.

Proof of Claim. According to Claim 5, f is an upper bound for the family

$$\mathcal{F}_f = \Big\{ f_{\langle f, \{U\}, \{y\} \rangle} \mid y \prec f(x) \text{ for some } x \in X,\ U \prec f^{-1}(\operatorname{int}\uparrow y) \Big\}.$$

Let g be another upper bound for $\mathcal{F}_f$. It suffices to establish that $f \leq g$. Indeed, let $x \in X$. As $\mathbb{Y}_0$ is a basic subspace of $\mathbb{Y}$, we conclude that $f(x) = \bigvee \{y \in Y_0 \mid y \prec f(x)\}$. Fix an element $y \in Y_0$ such that $y \prec f(x)$. Then we have $x \in f^{-1}(\operatorname{int}\uparrow y) \in \mathcal{T}(\mathbb{X})$. As $\mathbb{X}$ is core-compact, the poset $\langle \mathcal{T}(\mathbb{X}); \subseteq \rangle$ is continuous. Therefore, there is a nonempty set $U_y \in \mathcal{T}(\mathbb{X})$ such that $x \in U_y \prec f^{-1}(\operatorname{int}\uparrow y)$. We put $W(y) =$

$\langle f, \{U_y\}, \{y\}\rangle$. Then we have $f_{W(y)} \in \mathcal{F}_f$ and thus, $y = f_{W(y)}(x) \leq g(x)$. This implies that $f(x) = \bigvee\{y \in Y_0 \mid y \prec f(x)\} \leq g(x)$. Hence, $f \leq g$ which is our desired conclusion.

The last statement follows from Claim 5. □

From Claim 6 and Lemma 1.10.5 in [9], we obtain the following statement.

Claim 7. Let $f \in C(\mathbb{X}, \mathbb{Y})$ and let $W = \langle f, \{U\}, \{y\}\rangle$ be such that $y \in Y_0$, $U \in \mathcal{T}(\mathbb{X})$, and $U \prec f^{-1}(\operatorname{int}\uparrow y)$. Then $f_W \prec_{\mathcal{S}} f$.

We put $C_0 = \{f_W \mid W \in \mathcal{W}\}$.

Claim 8. If Y_0 is a parus in $\langle Y; \leq_{\mathbb{Y}}\rangle$ then $\mathbb{C}_0$ is a parus in $\langle C(\mathbb{X}, \mathbb{Y}); \leq_{\mathcal{P}}\rangle$.

Proof of Claim. We consider arbitrary sequences

$$W_0 = \langle g_0, \{U_0, \ldots, U_m\}, \{a_0, \ldots, a_m\}\rangle \in \mathcal{W},$$
$$W_1 = \langle g_1, \{V_0, \ldots, V_n\}, \{b_0, \ldots, b_n\}\rangle \in \mathcal{W}$$

and assume that $f_0 = f_{W_0}$, $f_1 = f_{W_1} \leq h$ for some $h \in C(\mathbb{X}, \mathbb{Y})$. As $\mathbb{Y}_0$ is a parus in $\mathbb{Y}$, for all $x \in X$, there is $f_0(x) \vee f_1(x) \in Y_0$. We put

$$\begin{aligned} U =& \{U_0, \ldots, U_m\} \cup \{V_0, \ldots, V_n\} \cup \{U_i \cap V_j \mid i \leqslant m,\ j \leqslant n,\ U_i \cap V_j \neq \varnothing\} = \\ =& \{Z_0, \ldots, Z_k\}; \\ F =& \{y_0, \ldots, y_k\}; \\ W =& \langle h, U, F\rangle, \end{aligned}$$

where y_i, $i \leqslant k$, is defined in the following way. For each $i \leqslant k$, we have $\bigcap\{U_j \mid j \leqslant m,\ Z_i \subseteq U_j\} \neq \varnothing$. Hence, $\bigcap\{U_j \mid j \leqslant m,\ Z_i \subseteq U_j\} = U_{m_i}$ for some $m_i \leqslant m$. Symmetrically, $\bigcap\{V_j \mid j \leqslant n,\ Z_i \subseteq V_j\} \neq \varnothing$, whence $\bigcap\{V_j \mid j \leqslant n,\ Z_i \subseteq V_j\} = V_{n_i}$ for some $n_i \leqslant n$. Thus for each element $x \in Z_i$, we have $a_{m_i} \leq f_0(x) \leq h(x)$ and $b_{n_i} \leq f_1(x) \leq h(x)$. As $\mathbb{Y}_0$ is a parus in $\mathbb{Y}$, there is an element $y_i = a_{m_i} \vee b_{n_i} \in Y_0$.

By our definition, W possesses the property $(*)$. Suppose that $Z_{i_0} \subseteq Z_{i_1}$ for some $i_0, i_1 \leqslant k$. For each $j < 2$, let $m_j \leqslant m$ and $n_j \leqslant n$ be such that

$$U_{m_j} = \bigcap\{U_t \mid t \leqslant m,\ Z_{i_j} \subseteq U_t\} \quad \text{and} \quad V_{n_j} = \bigcap\{V_t \mid t \leqslant n,\ Z_{i_j} \subseteq V_t\}.$$

Since $Z_{i_0} \subseteq Z_{i_1}$, we obtain that $U_{m_0} \subseteq U_{m_1}$ and $V_{m_0} \subseteq V_{m_1}$. We have therefore by our definition that $y_{i_0} = a_{m_0} \vee b_{n_0} \geq a_{m_1} \vee b_{n_1} = y_{i_1}$ as W_0 and W_1 have the property $(**)$. This yields that W possesses the property $(**)$. To establish that W also possesses the property $(***)$, we consider an arbitrary element $x \in \bigcup_{i \leqslant k} Z_i$. Let $j \leqslant k$ be such that $Z_j = \bigcap\{Z_t \mid t \leqslant k,\ x \in Z_t\}$ and let

$$U_{m_j} = \bigcap\{U_t \mid t \leqslant m,\ Z_j \subseteq U_t\}; \quad V_{n_j} = \bigcap\{V_t \mid t \leqslant n,\ Z_j \subseteq V_t\}.$$

If $x \in U_s$ for some $s \leqslant m$, then $Z_j \subseteq U_s$ as $U_s \in U$. This implies that $x \in Z_j \subseteq U_{m_j} \subseteq U_s$. Similarly, $x \in V_s$ for some $s \leqslant n$ implies that $x \in Z_j \subseteq V_{n_j} \subseteq V_s$. Therefore, we obtain that

$$U_{m_j} = \bigcap\{U_t \mid t \leqslant m,\ x \in U_t\}; \quad V_{n_j} = \bigcap\{V_t \mid t \leqslant n,\ x \in V_t\}.$$

As W_0 and W_1 possess the property $(***)$, we obtain

$$y_j = a_{m_j} \vee b_{n_j} \geq a_{m_j} = f_0(x) \geq \bot_{g_0(x)};$$
$$y_j = a_{m_j} \vee b_{n_j} \geq b_{n_j} = f_1(x) \geq \bot_{g_1(x)};$$
$$h(x) \geq f_0(x) = a_{m_j};$$
$$h(x) \geq f_1(x) = b_{n_j}.$$

Hence $\bot_{g_0(x)} = \bot_{h(x)} = \bot_{g_1(x)}$ and

$$h(x) \geq y_j = f_W(x) \geq \bot_{h(x)} = \bot_{g_0(x)} = \bot_{g_1(x)}.$$

This proves that W also has the property $(***)$.

Finally, we prove that $f_W = f_0 \vee f_1$ in $\mathbb{C}(\mathbb{X}, \mathbb{Y})$. We prove first that $f_0, f_1 \leq f_W$. Indeed, let $x \in X$. Two cases are possible. In what follows, we use the notation introduced above.

Case 1: $x \in \bigcup_{i \leqslant k} Z_k$. In this case, $f_0(x) = a_{m_j}$, $f_1(x) = b_{n_j}$, $f_W(x) = y_j = a_{m_j} \vee b_{n_j} \geq a_{m_j} = f_0(x)$. Similarly, $f_W(x) \geq b_{n_j} = f_1(x)$.

Case 2: $x \notin \bigcup_{i \leqslant k} Z_i$. This implies that $x \notin \bigcup_{i \leqslant m} U_i \cup \bigcup_{i \leqslant n} V_i$. Therefore,

$$\bot_{g_0(x)} = f_0(x) \leq h(x); \quad \bot_{g_1(x)} = f_1(x) \leq h(x),$$

whence $f_0(x) = f_1(x) = \bot_{g_0(x)} = \bot_{g_1(x)} = \bot_{h(x)} = f_W(x)$.

Therefore, f_W is indeed an upper bound of f_0 and f_1. Suppose that $f_0, f_1 \leq f$ for some $f \in C(\mathbb{X}, \mathbb{Y})$; we have to establish that $f_W \leq f$. Indeed, for $x \in X$, two cases are possible.

Case 1: $x \in \bigcup_{i \leqslant k} Z_k$. According to the considerations in *Case 1* above, we have $f_W(x) = y_j = a_{m_j} \vee b_{n_j} = f_0(x) \vee f_1(x) \leq f(x)$ in this case.

Case 2: $x \notin \bigcup_{i \leqslant k} Z_i$. This implies as in *Case 2* above that $f_W(x) = \bot_{g_0(x)} = \bot_{g_1(x)} = f_0(x) = f_1(x) \leq f(x)$.

Therefore, $f_W = f_0 \vee f_1$ and the proof of Claim is complete. □

Claim 9. $\mathbb{C}_0$ is a basic subspace in $\mathbb{C}_{\mathcal{S}}(\mathbb{X}, \mathbb{Y})$.

Proof of Claim. Let $f \in V \in \mathcal{S}$. According to Claim 6, $f = \bigvee \mathcal{F}_f$, where

$$\mathcal{F}_f = \Big\{ f_{\langle f, \{U\}, \{y\} \rangle} \mid y \prec f(x) \text{ for some } x \in X,\ U \prec f^{-1}(\operatorname{int}{\uparrow} y) \Big\}.$$

According to Claim 8, for each finite nonempty subfamily $\mathcal{G} \subseteq \mathcal{F}_f$, there is $W(\mathcal{G}) \in \mathcal{W}$ such that $f_{\mathcal{G}} = f_{W(\mathcal{G})} = \bigvee \mathcal{G} \leq f$. Thus, the family

$$\mathcal{D} = \{ f_{\mathcal{G}} \mid \mathcal{G} \subseteq \mathcal{F}_f,\ 0 < |\mathcal{G}| < \omega \}$$

is up-directed with respect to $\leq_{\mathcal{P}}$ and $f = \bigvee \mathcal{D}$. This implies that $f_{\mathcal{G}} \in V$ for some finite nonempty family $\mathcal{G} \subseteq \mathcal{F}_f$. Moreover, $f \in \operatorname{int}_{\mathcal{S}} {\uparrow} g$ for all $g \in \mathcal{G}$ according to Claim 6. Therefore,

$$f \in \bigcap \{ \operatorname{int}_{\mathcal{S}} {\uparrow} g \mid g \in \mathcal{G} \} = \operatorname{int}_{\mathcal{S}} {\uparrow} f_{\mathcal{G}},$$

which implies that $f_{\mathcal{G}} \prec_{\mathcal{S}} f$. □

The desired statements follow from Claims 2, 8, and 9. □

Corollary 21. *The following statements are equivalent for a T_0-space $\mathbb{Y}$.*

(1) $\mathbb{Y}$ *is an A_d-space such that ${\downarrow} y$ has a least element for each $y \in Y$.*

(2) *For each core-compact space* $\mathbb{X}$*,* $\mathbb{C}_{\mathcal{S}}(\mathbb{X},\mathbb{Y})$ *is an* A_d*-space such that* $\downarrow f$ *has a least element for each* $f \in C(\mathbb{X},\mathbb{Y})$*.*

Proof. (1) implies (2) by Theorem 20. Let $\mathbb{T}$ be a trivial (one-element) topological space; $\mathbb{T}$ is obviously core-compact. If (2) holds, then $\mathbb{Y} \cong \mathbb{C}_{\mathcal{S}}(\mathbb{T},\mathbb{Y})$ is an A_d-space such that $\downarrow f$ has a least element for each $f \in C(\mathbb{X},\mathbb{Y})$. Hence, (1) also holds. □

Theorem 22. *If* $\mathbb{X}$ *is a core-compact space and* $\mathbb{Y}$ *is an* A*-space with a least element then* $\mathbb{C}_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$ *is an* A*-space with a least element.*

Proof. The proof is very close to the one of Theorem 7.3.4 in [9]; we use certain parts of it here.

Let $\bot$ denote the least element in $\mathbb{Y}$ and let $\mathbb{Y}_0$ be an A-basic subspace in $\mathbb{Y}$; in particular, $\bot \in Y_0$. It follows from Lemma 1 that $\mathbb{C}_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$ has a least element.

Similarly to the proof of [9, Theorem 7.3.4], we consider the set

$$\mathcal{W} = \big\{\langle\varnothing,\varnothing\rangle\big\} \cup \big\{\big\langle\{U_0,\dots,U_n\},\{y_0,\dots,y_n\}\big\rangle \mid n<\omega,\ U_0,\dots,U_n \in \mathcal{T}(\mathbb{X})\backslash\{\varnothing\},\ y_0,\dots,y_n \in Y_0\big\},$$

where each $\big\langle\{U_0,\dots,U_n\},\{y_0,\dots,y_n\}\big\rangle \in \mathcal{W}$ possesses the following properties:

($*$) the set $\{U_0,\dots,U_n\}$ is closed under nonempty intersections;

($**$) if $U_i \subseteq U_j$ for some $i,j \leqslant n$ then $y_j \leq_{\mathbb{Y}} y_i$.

For each $W = \big\langle\{U_0,\dots,U_n\},\{y_0,\dots,y_n\}\big\rangle \in \mathcal{W}$, we define a mapping $f_W\colon X \to Y$ by the following rule:

$$f_W(x) = \begin{cases} y_i, & \text{if } U_i = \bigcap\{U_j \mid j \leqslant n,\ x \in U_j\};\\ \bot, & \text{if } x \notin \bigcup_{j\leqslant n} U_j. \end{cases}$$

We put $C_0 = \{f_W \mid W \in \mathcal{W}\}$.

Claims 1–3 were established in the proof of [9, Theorem 7.3.4]

Claim 1. $f_W \in C(\mathbb{X},\mathbb{Y})$ for each $W \in \mathcal{W}$.

Claim 2. For $f \in C(\mathbb{X},\mathbb{Y})$ and $W = \big\langle\{U_0,\dots,U_n\},\{y_0,\dots,y_n\}\big\rangle \in \mathcal{W}$, we have $f_W \leq f$ if and only if $y_i \leq f(x)$ for all $x \in \bigcup_{j\leqslant n} U_j$, where $i \leqslant n$ is such that $U_i = \bigcap\{U_j \mid j \leqslant n,\ x \in U_j\}$.

Claim 3. If Y_0 is a parus in $\langle Y;\leq_{\mathbb{Y}}\rangle$ then $\mathbb{C}_0$ is a parus in $\big\langle C(\mathbb{X},\mathbb{Y});\leq_{\mathcal{P}}\big\rangle$.

Claim 4. $\mathbb{C}_0$ is a basic subspace in $\mathbb{C}_{\mathcal{I}}(\mathbb{X},\mathbb{Y})$.

Proof of Claim. Suppose first that $U \in \mathcal{T}(\mathbb{X})$, $W \in \mathcal{T}(\mathbb{Y})$, and $f \in C(\mathbb{X},\mathbb{Y})$ are such that $f \in V_{U,W} \in \mathcal{I}$; then $U \prec f^{-1}(W)$. Since $\mathbb{Y}_0$ is a basic subspace in $\mathbb{Y}$, $f^{-1}(W) = \bigcup_{y\in Y_0\cap W} f^{-1}(\operatorname{int}\uparrow y)$. Therefore, $U \prec f^{-1}(\operatorname{int}\uparrow y_0) \cup \ldots \cup f^{-1}(\operatorname{int}\uparrow y_n)$ for some $y_0,\dots,y_n \in Y_0 \cap W$. Since $\mathbb{X}$ is core-compact, there is $U' \in \mathcal{T}(\mathbb{X})$ with $U \prec U' \prec f^{-1}(\operatorname{int}\uparrow y_0)\cup\ldots\cup f^{-1}(\operatorname{int}\uparrow y_n)$. We put $\mathcal{B} = \mathcal{T}(\mathbb{X})$ and apply Proposition 15. According to it and the fact that $\mathbb{X}$ is core-compact, there are $W_0,\dots,W_n$, $S_0,\dots,S_n \in \mathcal{T}(\mathbb{X})$ such that

$$W_i \prec S_i \prec f^{-1}(\operatorname{int}\uparrow y_i) \text{ for all } i \leqslant n \text{ and } U \subseteq W_0 \cup \ldots \cup W_n \subseteq U'.$$

We claim that $\bigcap_{i\leqslant n} V_{W_i,W} \subseteq V_{U,W}$. Indeed, if $g \in \bigcap_{i\leqslant n} V_{W_i,W}$ then $W_i \prec g^{-1}(W)$. Hence, $U \subseteq W_0 \cup \ldots \cup W_n \prec g^{-1}(W)$ which implies that $g \in V_{U,W}$.

For each $i \leqslant n$, we put $f_i = f_{\langle\{S_i\},\{y_i\}\rangle}$; then $f_i \in C_0$. Moreover, if $x \in S_i$ then $f_i(x) = y_i \prec f(x)$ as $S_i \prec f^{-1}(\operatorname{int}\uparrow y_i)$. Furthermore, $y_i \in W$ whence $W_i \prec S_i \subseteq$

$f_i^{-1}(W)$ and $f_i \in V_{W_i,W}$. Therefore, $f_i \leq f$ for all $i \leqslant n$ by Claim 2. According to Claim 3, there is $h = f_0 \vee \ldots \vee f_n \leq f$. As $f_i \in V_{W_i,W}$ for all $i \leqslant n$, we conclude that $h \in \bigcap_{i \leqslant n} V_{W_i,W} \subseteq V_{U,W}$.

Finally, $f \in \bigcap_{i \leqslant n} V_{S_i,\mathrm{int}\uparrow y_i}$ as $S_i \prec f^{-1}(\mathrm{int}\uparrow y_i)$ for each $i \leqslant n$. Suppose that $g \in \bigcap_{i \leqslant n} V_{S_i,\mathrm{int}\uparrow y_i}$; then for all $i \leqslant n$ and all $x \in S_i$, one has $f_i(x) = y_i \prec g(x)$, whence $f_i \leq g$ by Claim 2. This implies that $h \leq g$. Summarizing, we obtain that $f \in \bigcap_{i \leqslant n} V_{S_i,\mathrm{int}\uparrow y_i} \subseteq \uparrow h$; that is, $h \prec_{\mathcal{I}} f$.

To complete the proof, we assume that $f \in V \in \mathcal{I}$. Then there is $k < \omega$ and there are open sets $U_0, \ldots, U_k \in \mathcal{T}(\mathbb{X})$ and $W_0, \ldots, W_k \in \mathcal{T}(\mathbb{Y})$ such that

$$f \in V_{U_0,W_0} \cap \ldots \cap V_{U_k,W_k} \subseteq V.$$

According to what we have just proved, there are $h_0, \ldots, h_k \in C(\mathbb{X}, \mathbb{Y})$ such that $h_i \prec_{\mathcal{I}} f$ and $h_i \in V_{U_i,W_i}$ for all $i \leqslant k$. By Claim 3, there is $h = h_0 \vee \ldots \vee h_n \leq f$. In particular, $h \in V_{U_0,W_0} \cap \ldots \cap V_{U_k,W_k} \subseteq V$ and $h \prec_{\mathcal{I}} f$. □

The desired statement follows from Claims 3–4. □

We notice that Theorem 22 generalizes Theorem 7.3.4 in [9] (see Theorem 19) as $\mathcal{I} = \mathcal{P}$ on $C(\mathbb{X}, \mathbb{Y})$ by Proposition 6(3) whenever $\mathbb{X}$ is an α^*-space.

Corollary 23. *The following statements are equivalent for a T_0-space $\mathbb{Y}$.*

(1) *$\mathbb{Y}$ is an A-space with a least element.*
(2) *For each core-compact space $\mathbb{X}$, $\mathbb{C}_{\mathcal{I}}(\mathbb{X}, \mathbb{Y})$ is an A-space with a least element.*
(3) *For each α^*-space $\mathbb{X}$, $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is an A-space with a least element.*

Proof. (1) implies (2) by Theorem 22. (1) implies (3) as $\mathcal{I} = \mathcal{P}$ for an α^*-space $\mathbb{X}$. A trivial (one-element) topological space $\mathbb{T}$ is obviously an α^*-space and thus core-compact. If (2) holds, then $\mathbb{Y} \cong \mathbb{C}_{\mathcal{I}}(\mathbb{T}, \mathbb{Y})$ is an A-space with a least element. Thus, (1) also holds and each of (2) and (3) implies (1). □

7. Injectivity in function spaces

We will make use of the following results.

Theorem 24. [9, Theorem 4.2.3] *A topological T_0-space $\mathbb{X}$ is injective if and only if the following conditions hold:*

(1) *$\mathbb{X}$ is a d-space;*
(2) *$\mathbb{X}$ is an α-space;*
(3) *$\langle X; \leq_{\mathbb{X}} \rangle$ is a complete lattice.*

Theorem 25. [9, Theorem 4.2.4] *A topological T_0-space $\mathbb{X}$ is densely injective if and only if the following conditions hold:*

(1) *$\mathbb{X}$ is a d-space;*
(2) *$\mathbb{X}$ is an α-space;*
(3) *$\mathbb{X}$ is a* bc*-domain.*

Corollary 26. *A topological T_0-space $\mathbb{X}$ is [densely] injective if and only if $\mathbb{X}$ is a A_d-space with a least and a greatest element [with a least element].*

Proposition 27. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces and let $\mathcal{T}$ be a topology on $C(\mathbb{X}, \mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})$ is an α^*-space [an α-space] for some space $\mathbb{X}$, then $\mathbb{Y}$ is also an α^*-space [an α-space].*

Proof. Let $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ be an α^*-space. Suppose that $y \in U \in \mathcal{T}(\mathbb{Y})$ and fix an element $x \in X$; then $\xi_y \in V_{x,U} \in \mathcal{P} \subseteq \mathcal{T}$. According to our assumption, there are continuous functions $f_0, \ldots, f_m \in V_{x,U}$ and a set $V \in \mathcal{T}$ such that $\xi_y \in V \subseteq \uparrow f_0 \cup \ldots \cup \uparrow f_m$. Since $V \in \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$, there are upper cones $\mathcal{H}_0, \ldots, \mathcal{H}_n$ in $\langle \mathcal{T}(\mathbb{X}); \subseteq\rangle$ and open sets $W_0, \ldots, W_n \in \mathcal{T}(\mathbb{Y})$ such that

$$\xi_y \in V_{\mathcal{H}_0,W_0} \cap \ldots \cap V_{\mathcal{H}_n,W_n} \subseteq V \subseteq \uparrow f_0 \cup \ldots \cup \uparrow f_m \subseteq V_{x,U}.$$

Therefore, $\xi_i^{-1}(W_j) \in \mathcal{H}_j$ for all $j \leqslant n$. In particular, $\mathcal{H}_j \neq \varnothing$ for all $j \leqslant n$. We put

$$W = \bigcap \{W_j \mid j \leqslant n,\ \mathcal{H}_j \neq \mathcal{T}(\mathbb{X})\}.$$

Then we have that $W \in \mathcal{T}(\mathbb{Y})$. Moreover, $\xi_y^{-1}(W_j) \neq \varnothing$ whence $\xi_y^{-1}(W_j) = X$ for all $j \leqslant n$ such that $\mathcal{H}_j \neq \mathcal{T}(\mathbb{X})$. This implies that $y \in W$. If $y' \in W$, then $\xi_{y'}^{-1}(W_j) = X$ for all $j \leqslant n$ such that $\mathcal{H}_j \neq \mathcal{T}(\mathbb{X})$. Therefore,

$$\xi_{y'} \in V_{\mathcal{H}_0,W_0} \cap \ldots \cap V_{\mathcal{H}_n,W_n} \subseteq V \subseteq \uparrow f_0 \cup \ldots \cup \uparrow f_m \subseteq V_{x,U}$$

and thus $y' \in U$. We established therefore that $y \in W \subseteq \uparrow f_0(x) \cup \ldots \cup \uparrow f_m(x)$ and $f_0(x), \ldots, f_m(x) \in U$. This implies that $\mathbb{Y}$ is an α^*-space.

If $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is an α-space, we use in the argument above with $m = 0$. □

Proposition 28. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces and let $\mathcal{T}$ be a topology on $C(\mathbb{X},\mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is a densely injective space then $\mathbb{Y}$ is also densely injective.*

Proof. According to Theorem 25, $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is an α-space, a d-space, and a bc-domain. According to Proposition 27, $\mathbb{Y}$ is an α-space. According to the proof of Theorem 9, $\mathbb{Y}$ is a d-space. In view of Theorem 25, in order to prove the dense injectivity of $\mathbb{Y}$, it suffices to show that $\mathbb{Y}$ is a partial join-semilattice with respect to the specialization order $\leq_{\mathbb{Y}}$.

Let y_0, $y_1 \leq y$ in $\mathbb{Y}$. This means that ξ_{y_0}, $\xi_{y_1} \leq \xi_y$ in $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$. As $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is a bc-domain, there is a continuous function $f = \xi_{y_0} \vee \xi_{y_1}$. We prove that f is constant. Indeed, let $f(x_0) \in U \in \mathcal{T}(\mathbb{Y})$ and let $x_1 \in X$ be an arbitrary element. Then $f \in V_{x_0,U} \in \mathcal{P} \subseteq \mathcal{T}$. By Corollary 26, $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is an A_d-space. By [10, Lemma 7], there are open sets $V_0, V_1 \in \mathcal{T}$ such that $\xi_{y_0} \in V_0$, $\xi_{y_1} \in V_1$, and $V_0 \cap V_1 \subseteq V_{x_0,U}$. Using the assumption that $\mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$ and applying the same argument as in the proof of Proposition 14, we obtain that $f = \xi_y$ for some $y \in Y$, whence $y = y_0 \vee y_1$ in $\mathbb{Y}$.

Applying Theorem 25 again, we conclude that $\mathbb{Y}$ is densely injective. □

Proposition 29. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces and let $\mathcal{T}$ be a topology on $C(\mathbb{X},\mathbb{Y})$ such that $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is an injective space then $\mathbb{Y}$ is also injective.*

Proof. According to Theorem 24 $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is an α-space, a d-space, and a complete lattice with respect to the specialization order $\leq$. By Proposition 27, $\mathbb{Y}$ is an α-space. By the proof of Theorem 9, $\mathbb{Y}$ is a d-space. Since $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is an A_d-space having a least and a greatest element by Corollary 26, it possesses the properties (H_0)–(H_2) by [10, Lemma 7]. By Lemma 1 and Proposition 14, the space $\mathbb{Y}$ also possesses the properties (H_0)–(H_2). This means that $\mathbb{Y}$ is a complete lattice with respect to the specialization order $\leq_{\mathbb{Y}}$. Applying Theorem 24 again, we conclude that $\mathbb{Y}$ is an injective space. □

The following statement generalizes Theorem 19 in [10].

Theorem 30. *For a T_0-space $\mathbb{Y}$, the following conditions are equivalent.*

(1) $\mathbb{Y}$ *is densely injective.*
(2) $\mathbb{C}_{\mathcal{J}}(\mathbb{X},\mathbb{Y})$ *is densely injective for each core-compact space* $\mathbb{X}$.
(3) $\mathbb{C}_{\mathcal{J}}(\mathbb{X},\mathbb{Y})$ *is densely injective for some core-compact space* $\mathbb{X}$.
(4) $\mathbb{C}_{\mathcal{J}}(\mathbb{X},\mathbb{Y})$ *is densely injective for some T_0-space* $\mathbb{X}$.
(5) $\mathbb{C}_{\mathcal{S}}(\mathbb{X},\mathbb{Y})$ *is densely injective for each core-compact space* $\mathbb{X}$.
(6) $\mathbb{C}(\mathbb{X},\mathbb{Y})$ *is densely injective for each α^*-space* $\mathbb{X}$.
(7) $\mathbb{C}(\mathbb{X},\mathbb{Y})$ *is densely injective for some α^*-space* $\mathbb{X}$.
(8) $\mathbb{C}(\mathbb{X},\mathbb{Y})$ *is densely injective for some T_0-space* $\mathbb{X}$.
(9) $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ *is densely injective for some [core-compact] T_0-space $\mathbb{X}$ and some topology $\mathcal{T}$ such that $\mathcal{P}\subseteq\mathcal{T}\subseteq\mathcal{T}_A(\subseteq)^\sharp$.*

Proof. (1) implies (2) by Proposition 11(1). (2) implies (3) implies (4), (6) implies (7) implies (8), and (3) implies (9) in a trivial way. (4) implies (1) and each of (8) and (9) implies (1) by Proposition 28. Furthermore, (2) implies (5) as $\mathcal{J}=\mathcal{S}$ whenever $\mathbb{C}_{\mathcal{J}}(\mathbb{X},\mathbb{Y})$ is densely injective, see [9, Lemma 1.8.6].

Let $\mathbb{T}$ be a trivial (one-element) topological space; $\mathbb{T}$ is obviously core-compact. If (5) holds, then $\mathbb{Y}\cong\mathbb{C}_{\mathcal{S}}(\mathbb{T},\mathbb{Y})$ is a densely injective space. Hence, (1) also holds, and (5) implies (1).

Statements (1), (6), and (7) are equivalent by [10, Theorem 19]. □

The following statement generalizes Theorem 20 in [10].

Theorem 31. *For a T_0-space $\mathbb{Y}$, the following conditions are equivalent.*

(1) $\mathbb{Y}$ *is injective.*
(2) $\mathbb{C}_{\mathcal{J}}(\mathbb{X},\mathbb{Y})$ *is injective for each core-compact space* $\mathbb{X}$.
(3) $\mathbb{C}_{\mathcal{J}}(\mathbb{X},\mathbb{Y})$ *is injective for some core-compact space* $\mathbb{X}$.
(4) $\mathbb{C}_{\mathcal{J}}(\mathbb{X},\mathbb{Y})$ *is injective for some T_0-space* $\mathbb{X}$.
(5) $\mathbb{C}_{\mathcal{S}}(\mathbb{X},\mathbb{Y})$ *is injective for each core-compact space* $\mathbb{X}$.
(6) $\mathbb{C}(\mathbb{X},\mathbb{Y})$ *is injective for each α^*-space* $\mathbb{X}$.
(7) $\mathbb{C}(\mathbb{X},\mathbb{Y})$ *is injective for some α^*-space* $\mathbb{X}$.
(8) $\mathbb{C}(\mathbb{X},\mathbb{Y})$ *is injective for some T_0-space* $\mathbb{X}$.
(9) $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ *is injective for some [core-compact] T_0-space $\mathbb{X}$ and some topology $\mathcal{T}$ such that $\mathcal{P}\subseteq\mathcal{T}\subseteq\mathcal{T}_A(\subseteq)^\sharp$.*

Proof. (1) implies (2) by Proposition 11(1). (2) implies (3) implies (4), (6) implies (7) implies (8), and (3) implies (9) in a trivial way. (4) implies (1) and each of (8)and (9) implies (1) by Proposition 29. Furthermore, (2) implies (5) as $\mathcal{J}=\mathcal{S}$ whenever $\mathbb{C}_{\mathcal{J}}(\mathbb{X},\mathbb{Y})$ is injective.

Let $\mathbb{T}$ be a trivial (one-element) topological space; $\mathbb{T}$ is obviously core-compact. If (5) holds, then $\mathbb{Y}\cong\mathbb{C}_{\mathcal{S}}(\mathbb{T},\mathbb{Y})$ is a injective space, which yields (1).

Statements (1), (6), and (7) are equivalent by [10, Theorem 20]. □

The fact that (1) implies (2) in Theorems 30 and 31 was established in [12, Proposition II-4.6]. The fact that (3) implies (1) in Theorems 30 and 31 was established in [14, Theorems 4.6, 4.7].

8. Δ-SPACES $\mathbb{C}(\mathbb{X},\mathbb{Y})$

Definition 2. [7] A continuous function $\delta\colon\mathbb{X}\to\mathbb{X}$ from a topological space $\mathbb{X}$ into itself is a *deflation*, if the set $\delta(X)$ is finite and $\delta(x)\leq_{\mathbb{X}}x$ for all $x\in X$.

A T_0-space $\mathbb{X}$ is a Δ-*space*, if there is an up-directed family $\{\delta_i\colon \mathbb{X}\to\mathbb{X}\mid i\in I\}$ of deflations of $\mathbb{X}$ with the property that for every $U\in\mathcal{T}(\mathbb{X})$ and every $x\in U$, there is $i\in I$ such that $\delta_i(x)\in U$.

A space $\mathbb{X}$ is a Δ_d-*space*, if $\mathbb{X}$ is a Δ-space and a d-space simultaneously.

The following partial generalization of Theorem 23 from [10] holds.

Proposition 32. *Let $\mathbb{X}$, $\mathbb{Y}$ be T_0-spaces and let $\mathcal{T}$ be a topology on $C(\mathbb{X},\mathbb{Y})$ such that $\mathcal{P}\subseteq\mathcal{T}\subseteq\mathcal{T}_A(\subseteq)^\sharp$. If $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ is a Δ-space [Δ_d-space] then $\mathbb{Y}$ is also a Δ-space [Δ_d-space].*

Proof. Let D be an up-directed family of deflations of the space $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$ which satisfies all the requirements of Definition 2. We fix an element $a\in X$. For each $\delta\in D$, consider the mapping

$$\kappa_\delta\colon \mathbb{Y}\to\mathbb{Y},\quad \kappa_\delta\colon y\mapsto\delta(\xi_y)(a).$$

The set $\kappa_\delta(Y)$ is finite as the set $\{\delta(\xi_y)\mid y\in Y\}$ is finite. Moreover, for every $y\in Y$, we have $\kappa_\delta(y)=\delta(\xi_y)(a)\le\xi_y(a)=y$, since δ is a deflation of $\mathbb{C}_\mathcal{T}(\mathbb{X},\mathbb{Y})$. Let $U\in\mathcal{T}(\mathbb{X})$; then

$$\begin{aligned}\kappa_\delta^{-1}(U)&=\{y\in Y\mid \delta(\xi_y)(a)\in U\}=\{y\in Y\mid \delta(\xi_y)\in V_{a,U}\in\mathcal{T}\}=\\&=\{y\in Y\mid \xi_y\in\delta^{-1}(V_{a,U})\in\mathcal{T}\}=\xi^{-1}\delta^{-1}(V_{a,U})\in\mathcal{T}(\mathbb{Y})\end{aligned}$$

as ξ is continuous by Lemma 3 and δ is continuous by Definition 2. Thus, κ_δ is continuous, whence it is a deflation of $\mathbb{Y}$. Moreover, if $\delta,\delta'\in D$ are such that $\delta\le\delta'$, then $\kappa_\delta(y)=\delta(\xi_y)(a)\le\delta'(\xi_y)(a)=\kappa_{\delta'}(y)$ for all $y\in Y$. Therefore, $\{\kappa_\delta\mid\delta\in D\}$ is an up-directed family of deflations of $\mathbb{C}_\mathcal{T}(\mathbb{Y},\mathbb{Y})$ by Lemma 1. Finally, if $y\in U\in\mathcal{T}(\mathbb{Y})$ then $\xi_y\in V_{a,U}\in\mathcal{P}\subseteq\mathcal{T}$. In view of the choice of D, there is a deflation $\delta\in D$ such that $\delta(\xi_y)\in V_{a,U}$. Then $\kappa_\delta(y)=\delta(\xi_y)(a)\in U$, and the proof is complete.

The statement about Δ_d-spaces follows with the use of Theorem 9. □

Proposition 32 was established for the case $\mathcal{T}=\mathcal{I}$ in [14, Theorem 5.5].

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