S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 19, №2, стр. 835–851 (2022) УДК 519.86
DOI 10.33048/semi.2022.19.070 MSC 46N10

NONLINEAR INPUT-OUTPUT BALANCE AND YOUNG DUALITY: ANALYSIS OF COVID-19 MACROECONOMIC IMPACT ON KAZAKHSTAN

A. BORANBAYEV N. OBROSOVA A. SHANANIN

Abstract. We discuss the possibilities of the new approach to the inter-industry linkages modeling for the analysis of regional macroeconomic effects of Covid-19. Our approach is based on the mathematical framework of nonlinear input-output balance that allows to find the equilibrium point in the set of industry inputs and prices by solving the primal nonlinear resource allocation problem and the Young dual problem of prices formation. We identify and calibrate the model on the base of aggregated official input-output statistics of Kazakhstan. Given the scenario conditions for primal factors prices and final consumption in the economy the model allows to evaluate the new competitive equilibrium in the production network. The advantage of the model is non-linearity of balances and technologies that allows substitution of industry inputs. In the case of technologies with constant elasticity of substitution (CES) we apply the model to analysis of macroeconomic responses of the Kazakhstan economy to the Covid-19 pandemic.

Keywords: resource allocation problem, Young duality, Covid-19 macroeconomic shocks, input-output table, supply network.

1. Introduction

The Covid-19 pandemic has been a shock to the global economy. The macroeconomic consequences of the pandemic for each country are determined by both internal and external factors. Internal factors are associated with changes in labor supply

---

Boranbayev, A. Obrosova, N., Shananin, A., Nonlinear input output balance and Young duality.

This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09261118).

Received October, 17, 2022, published November, 11, 2022.

and final demand of products of sectors of the economy as a result of morbidity and internal lockdowns. External factors are related to the drop of export-import flows as a result of external lockdowns and violations of supply chains, as well as changes in export goods income as a result of changes in their prices. Various mathematical models are used to analyze the impact of the pandemic on macroeconomic indicators. On the one hand, the well known SIR models can be used to analyze the impact of the epidemic and policies on labor supply and labor efficiency in the economy. The large amount of the literature discusses economic and social effects of the Covid-19 crisis in terms of the classical SIR models and modifications (for ex., see [1], [2],[3]). On the other hand, models should take into account the features of external and internal links in the production network of the region to analyze the spread of external and internal shocks in the supply networks. The topical question is whether external or internal factors are the drivers of the macroeconomic indicators dynamics for specific country during the Covid-19 pandemic. The appropriate way to answer to this question is the Input-Output (IO) analysis. Input-Output (IO) models allow one to identify drivers of the economy, to analyze the interindustry connections and to evaluate the impact of shocks in supply networks on macroeconomic indicators. The base linear Input-Output Leontief model [4],[5] has been successfully used since the 40s of the last century to analyze the inter-industry relations, calculate economic multipliers, identify industries that are growth drivers in the economy [6], [7]. W.Leontief was awarded the Nobel Prize in Economics (1973) for the development of this model. Number of analysis based on the linear Leontief input-output model and its modifications for the number of regional economics and global supply networks was presented in the last two years (for ex., see [8]-[11]). Some of them use the results of SIR model for the scenario parameters calculations [1], [3]. The strong assumption of the linear Leontief model is the constancy of direct input coefficients in a supply network. Since the 1980s difficulties with Leontief method occured because of the increasing substitution of inputs of sectors for developed economies. So in modern conditions the main hypothesis of the Leontief model has violated. The shock of Covid-19 pandemic and previous economic crises of the last quarter of century has focused the world attention on the nonlinearities in economic reactions. The actual problem is the analysis of the nonlinear response of production networks and of the aggregate output to the supply and demand shocks that came with the pandemic. In this paper we suggest an IO analysis approach considering nonlinear links in complex supply networks, which helps to give answer on above question: what consequences of the pandemic (internal or external) were key for the considered region. We apply our approach to analyze the dominant macroeconomic consequences of Covid-19 pandemic for the economy of Kazakhstan on the base of the official statistics of the region. It seems that analysis of pandemic responses is particularly important for so-called catching-up economies, because of need for stable growth for innovative development of the economy.

Methods that are presented in this paper are related to ideas from [12] and the previous non-covid papers from the whole cycle of publications about production networks which are summarized in [13]. That works discuss the spread of economic shocks in production networks taking into account the nonlinearities in intersectoral connections in a competitive equilibrium framework with nonlinear production technologies.

Our approach differs from previous studies since we develop a framework that describes analytically the formation of input-output flows and input prices dynamics in the production network when the final consumption and external characteristics of the economy are shocked. Note that in contrast to [12] we consider an open economy and take into account the export-import flows of the region. Besides that we don't need to operate in terms of approximations of output of the first or the second order near the equilibrium to evaluate the macroeconomic indicies with the model as it is considered in [12]. Our approach allows to calculate from the obtained analytical expressions the changed equilibrium (intersectoral flows, prices, outputs of sectors) in various scenario conditions of external and internal shocks with high accuracy.

The mathematical foundation of our approach is presented in papers [14], [15]. The inter-industry connection analysis is based on the solution of the optimal resource allocation problem and the Young dual problem for input prices in the supply network. Once identified and verified on the base of the official IO statistics our model allows to provide the scenario calculations to evaluate the response of the regional economy to external or internal shocks. In the case of Constant Elasticity of Substitution (CES) production function (or Cobb-Douglas as a special case) one of the benefits of this method is rather simple algorithm for calculation of the competitive equilibrium in the production network for a specific scenario. Given the shocked final consumption of industries and the primary factors prices we solve the primal and the dual problem for the new initial data set and evaluate the changed competitive equilibrium in the production network taking into account the substitution of inputs. Some application of our methodology for the analysis of aggregated inter-industry relations for different regions are given in the papers [16]-[19].

The paper is organized as follows: the next section summarizes the framework description and the main results about the identification of the model in the case of CES technologies and utility. In the section 3 we compare the inter-industry connections in the Kazakhstan economy for the pandemic year 2020 and for the previous years and identify aggregate industry complexes that differs by their responses to Covid-19 pandemic. We calibrate the model on the base of aggregated IO statistics 2013-2019. Then we present scenario evaluations with our model and discuss the dominant consequences of the Covid-19 pandemic for the Kazakhstan economy.

2. Non-Linear Inter-Industry Balance

In this section we give briefly the results that are proved in the papers [14],[15],[18]

2.1. **The mathematical framework.** Consider a production network with the following structure:

- m pure industries,
- n primary production factors, which are not produced by industries of the economy,
- one aggregate final consumer, which includes the all types of final consumption (households, government, export, etc.)

The output of any industry j is produced using intermediate outputs of industries $i = 1, .., m$. The j-th output is sold as an intermediate input to industries and as a final good to the final consumer. Let fix the base year t. Further we use the following notation and basic assumptions:

- $X^j = \left(X_1^j, \dots, X_m^j\right)$ - the vector of intermediate inputs of industry j in current prices for the base year t, where X_i^j is the amount of good i used for the production of good j;
- $l^j = \left(l_1^j, \dots, l_n^j\right)$ - primary production factors in prices for the year t, that are consumed by the industry j;
- each factor i is totally limited by the value $l_i \geq 0$, therefore
$$\sum_{j=1}^{m} l_i^j \leq l_i,\ i = 1, .., n, \quad l = (l_1, \dots, l_n) \geq 0$$
- $X^0 = \left(X_1^0, \dots, X_m^0\right)$ - vector of final consumption of products in prices for the year t;
- Φ_m is the class of concave, monotonically nondecreasing, continuous and positively homogeneous of degree one functions on $R_{\geq 0}^{m+n}$, that vanish at the origin;
- $F_j\left(X^j, l^j\right) \in \Phi_{m+n}$ - production functions of the industry j, $F_j(0,0) = 0$;
- $F_0\left(X^0\right) \in \Phi_m$ - utility function of the final consumer, $F_0(0) = 0$.

We assume that the elements of vectors X^j, X^0, l^j are valued in current basic prices for the base year t. Set a problem of optimal resources allocation: given the vector $l = (l_1, \dots, l_n) \geq 0$ of factors limit determine the values X^j, X^0, l^j as a solution of the following problem

$$F_0\left(X^0\right) \to \max \tag{1}$$

$$F_j\left(X^j, l^j\right) \geq \sum_{i=0}^{m} X_j^i, j = 1, \dots, m \tag{2}$$

$$\sum_{j=1}^{m} l^j \leq l \tag{3}$$

$$X^0 \geq 0, \quad X^1 \geq 0, \dots, X^m \geq 0, \quad l^1 \geq 0, \dots, l^m \geq 0. \tag{4}$$

Assumption 1. Productivity condition. The supply chain is productive, i.e., there exists $\hat{X}^1 \geq 0, \dots, \hat{X}^m \geq 0,\ \hat{l}^1 \geq 0, \dots, \hat{l}^m \geq 0$ such that

$$F_j\left(\hat{X}^j, \hat{l}^j\right) > \sum_{i=1}^{m} \hat{X}_j^i,\ j = 1, ..., m.$$

Corollary 1. If the set of sectors is productive and $l = (l_1, \dots, l_n) > 0$, then the optimization problem (1)-(4) satisfies to the Slater condition.
Assumption 2. Denote

$$\mathrm{A}(l) = \Big\{X^0 = \left(X_1^0, \dots, X_m^0\right) \geq 0 \,\Big|\, X_j^0 \leq F_j\left(X^j, l^j\right) - \textstyle\sum_{i=1}^{m} X_j^i,\ j = 1...m; \\ \textstyle\sum_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{l^j} \leq l,\ X^1 \geq 0, \dots, X^m \geq 0,\ l^1 \geq 0, \dots, l^m \geq 0\Big\}.$$

There exists $\hat{l} \in \mathrm{R}^n_{>0}$ such that the set $\mathrm{A}\left(\hat{l}\right)$ is bounded.
Corollary 2. The set $\mathrm{A}(l)$ is bounded, convex and closed for any $l \in \mathrm{R}^n_{\geq 0}$.

Theorem 1. *([15]). The set of vectors* $\left\{\hat{X}^0,\, \hat{X}^1, ..., \hat{X}^m, \hat{l}^1, ..., \hat{l}^m\right\}$, *which satisfy to the constraints (2)–(4) is the solution of the optimization problem (1)–(4) if and only if there exist Lagrange multipliers* $p_0 > 0$, $p = (p_1, ..., p_m) \geq 0$ *and* $s = (s_1, ..., s_n) \geq 0$ *such that*

(5) $\left(\hat{X}^j, \hat{l}^j\right) \in Arg\max\{p_j F_j\left(X^j, l^j\right) - pX^j - sl^j \mid X^j \geq 0, l^j \geq 0\},\ j = 1, \ldots, m$

$$p_j\left(F_j\left(\hat{X}^j, \hat{l}^j\right) - \hat{X}^0_j - \sum_{i=1}^m \hat{X}^i_j\right) = 0,\ j = 1, \ldots, m \tag{6}$$

$$s_k\left(l_k - \sum_{j=1}^m \hat{l}^j_k\right) = 0,\ k = 1, ..., n \tag{7}$$

$$\hat{X}^0 \in Arg\max\{p_0 F_0(X^0) - pX^0 \mid X^0 \geq 0\}. \tag{8}$$

The Theorem 1 provides that the optimal resources allocation corresponds to a market equilibrium mechanisms (supply and demand are equal) and the Lagrange multipliers to constraints (2) and (3) can be interpreted as follows

- $p = (p_1, \ldots, p_m)$ - prices of goods;
- $s = (s_1, \ldots, s_n)$ - prices of factors.

To construct the dual problem we introduce the cost function $q_j(p, s)$ of the industry j and the consumer price index $q_0(q)$ that can be find as the Young transform of the corresponding production and utility functions (for details see [20])

$$q_j(p, s) = \inf\left\{\frac{pX^j + sl^j}{F_j(X^j, l^j)}\left|X^j \geq 0, l^j \geq 0, F_j\left(X^j, l^j\right) > 0\right.\right\},$$

$$q_0(q) = \inf\left\{\frac{qX^0}{F_0(X^0)}\left|X^0 \geq 0, F_0\left(X^0\right) > 0\right.\right\}$$

Note that $q_j(p, s) \in \Phi_{m+n}$, $q_0(q) \in \Phi_m$ and the Young transform is an involution, i.e.

$$F_0\left(X^0\right) = \inf\left\{\frac{qX^0}{q_0(q)} \mid q \geq 0, q_0(q) > 0\right\}$$

,

$$F_j\left(X^j, l^j\right) = \inf\left\{\frac{pX^j + sl^j}{q_j(p, s)} \mid p \geq 0, s \geq 0, q_j(p, s) > 0.\right\}.$$

The optimal value of (1) depending on $l = (l_1, .., l_n)$ (right part of (3)) in the problem (1)-(4) is called the aggregate production function $F^A(l) \in \Phi_n$. The Young transform of $F^A(l)$ is the aggregate cost function and

$$q_A(s) = \inf\left\{\frac{sl}{F^A(l)}\,\middle|\, l \geq 0,\, F^A(l) > 0.\right\} \in \Phi_n,$$

$$F^A(l) = \inf\left\{\frac{sl}{q_A(s)}\middle| s \geq 0,\, q_A(s) > 0\right\}.$$

Theorem 2. *([14],[15]) If Lagrange multipliers $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_m) \geq 0$, $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_n) \geq 0$ to the problem (1)-(4) satisfy to (5)-(8) then $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_m) \geq 0$ is the solution of the following problem*

$$q_0(p) \to \max_p \tag{9}$$

$$q_j(\hat{s}, p) \geq p_j,\ j = 1, \ldots, m \tag{10}$$

$$p = (p_1, \ldots, p_m) \geq 0. \tag{11}$$

Moreover, the aggregate cost function $q_A(\hat{s}) = q_0(\hat{p}(\hat{s}))$.

The convex programming problem (9)-(11) is called the Young dual problem to the problem (1)-(4). The solution of the primal problem of resource allocation (1)-(4) and the corresponding solution of the dual problem of price formation (9)-(11) give us the equilibrium point $\left\{\hat{X}^0, \hat{X}^1, ..., \hat{X}^m, \hat{l}^1, ..., \hat{l}^m, \hat{p}_1, \ldots, \hat{p}_m, \hat{s}_1, \ldots, \hat{s}_n\right\}$ in the production network. The shock of final consumption $\hat{X}_0$ or of prices of resources $\hat{s}$ results in shifting the equilibrium point. Thus, if the model is identified and calibrated on the base of the official input-output statistics we can evaluate the response of the production network to the various shocks.

2.2. **Identification of the model. The case of CES technologies and utility.** . The initial data for identification and calibration of the model is the set of symmetric Input-Output (IO) tables $Z(t)$ of the state over the period of years. Note that the symmetric IO table is yearly published as a part of national accounts system of Kazakhstan.

A symmetric IO table $Z(t)$ - the data on financial flows in terms of pure industries (products) that reflects the generation of products and their allocation among the various components of intermediate and final demand for the year t. The table $Z(t)$ consists of three quadrants (I,II,III):

$$\begin{array}{cc} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} \mathrm{I} & \mathrm{II} \\ \mathrm{III} & \end{pmatrix} \end{array}$$

Quadrant I: the element $Z_i^j,\ i,j = 1..m$ is the amount of money that industry i received from industry j for the intermediate inputs produced by i and consumed by j.

Quadrant II includes the column vectors of final consumption (households, government,export etc.), i.e., Z_i^j is the payment of final consumer $(j - m)$ for a good i, $i = 1..m$, $j = m + 1..m + k$

Quadrant III: the element Z_i^j $i = m + 1..m + n$, $j = 1..m$ denotes the payment of industry j for the intermediate consumption of primary factor $(i - m)$ that are not produced by industries. In this paper we consider two types of factors: labor (roughly we associate it with the Gross Value Added of industries) and Import $(n = 2)$.

Denote

$$Z^0 = \left(Z_1^0, \ldots, Z_m^0\right)^T,\ Z_i^0 = \sum_{j=m+1}^{m+k} Z_i^j,\ i = 1...m,$$

$$A_j = \sum_{i=1}^{m+n} Z_i^j,\ \ j = 1..m.$$

Values A_j correspond to the total inputs (intermediate and primary), consumed by the pure industry $j = 1, \ldots, m$. Due to the symmetry of the IO table the value A_j equals to the total consumption of product $j = 1, \ldots, m$, i.e. we have the following balance qualities

$$A_j = \sum_{i=1}^{m+k} Z_j^i, \; j = 1, \ldots, m,$$

$$\sum_{j=1}^{m} \sum_{i=m+1}^{m+n} Z_i^j = \sum_{j=1}^{m} \sum_{i=m+1}^{m+k} Z_j^i.$$

The inverse problem of identification of the model for a base year t is as follows: to construct the problem of allocation of resources (1)-(4), which solution reproduces the inter-industry financial flows in the economy for the base year. We solve the problem in the class of Constant Elasticity of Substitution (CES) production and utility functions

$$F_j\left(X^j, l^j\right) = \left(\sum_{i=1}^{m}\left(\frac{X_i^j}{w_i^j}\right)^{-\rho_j} + \sum_{k=1}^{n}\left(\frac{l_k^j}{w_{m+k}^j}\right)^{-\rho_j}\right)^{-\frac{1}{\rho_j}} j = 1, \ldots, n \tag{12}$$

$$F_0\left(X^0\right) = \left(\sum_{i=1}^{m}\left(\frac{X_i^0}{w_i^0}\right)^{-\rho_0}\right)^{-\frac{1}{\rho_0}} \tag{13}$$

where $\rho_j, \rho_0 \in (-1, 0) \cup (0, +\infty)$, $w_1^j > 0, \ldots, w_{m+n}^j > 0$, $j = 0, \ldots, m$. Constant elasticity of substitution of the industry j equals to $\sigma_j = \frac{1}{1+\rho_j}$, $j = 0, 1, \ldots, m$. The Young transform of the CES production function is the CES cost function, i.e.

$$q_j(p, s) = \left(\sum_{i=1}^{m}\left(w_i^j p_i\right)^{\frac{\rho_j}{1+\rho_j}} + \sum_{k=1}^{n}\left(w_{m+k}^j s_k\right)^{\frac{\rho_j}{1+\rho_j}}\right)^{\frac{1+\rho_j}{\rho_j}}, \; j = 1, \ldots, m. \tag{14}$$

The Young transform of the CES utility function is CES consumer price index

$$q_0(p) = \left(\sum_{i=1}^{m}\left(w_i^0 p_i\right)^{\frac{\rho_0}{1+\rho_0}}\right)^{\frac{1+\rho_0}{\rho_0}}. \tag{15}$$

Theorem 3. *Given the IO table $Z(t)$ for the base year t we define the parameters of CES functions $F_j\left(X^j, l^j\right)$ (12) and $F_0\left(X^0\right)$ (13) as follows*

$$w_i^j = \left(Z_i^j\right)^{\frac{1+\rho_j}{\rho_j}} \left(\sum_{k=1}^{m+n} Z_k^j\right)^{-\frac{1+\rho_j}{\rho_j}}, i = 1, \ldots, m+n; j = 1, \ldots, m,$$

$$w_i^0 = \left(Z_i^0\right)^{\frac{1+\rho_0}{\rho_0}} \left(\sum_{k=1}^{m} Z_k^0\right)^{-\frac{1+\rho_0}{\rho_0}}, i = 1, \ldots, m$$

and the vector of factors constraint

$$l = (l_1, \ldots, l_n), l_i = \sum_{j=1}^{m} Z_{m+i}^j, i = 1, \ldots, n.$$

Then the set of values

$$\left\{\hat{X}_i^0 = Z_i^0, \hat{X}_i^j = Z_i^j, \hat{l}_t^j = Z_{m+t}^j, i = 1, \ldots, m; j = 1, \ldots, m, t = 1, \ldots, n\right\},$$

is the solution of the convex programming problem (1)-(4).

The Theorem 3 obviously follows from the Theorem 1 with $p = (1, \ldots, 1)$, $s = (1, \ldots, 1)$.

Thus, the problem (1) - (4) explains the first (I) quadrant and the third quadrant (III) of the symmetric IO table $Z(t)$ in the base year t for any values ρ_j, $j = 1, .., m$.

Denote

$$a_{\text{ij}} = \left(w_i^j\right)^{\frac{\rho_j}{1+\rho_j}} = \frac{Z_i^j}{\sum_{k=1}^{m+n} Z_k^j}, i = 1, \ldots, m, j = 1, \ldots, m,$$

$$b_{\text{kj}} = \left(w_{m+k}^j\right)^{\frac{\rho_j}{1+\rho_j}} = \frac{Z_{m+k}^j}{\sum_{k=1}^{m+n} Z_k^j},\ i = 1, \ldots, m, j = 1, \ldots, m, k = 1, \ldots, n,$$

Note that the $(m \times m)$-matrix $A = \|a_{\text{ij}}\|$ is a Leontief matrix of technology coefficients. Denote $B = \|b_{\text{kj}}\|$ -$(n \times m)$- matrix. Note that the coefficients of matrices A, B don't depend on ρ_j, $j = 1, .., m$. Solving the Young dual problem (9)-(11) in the case of CES cost functions (14), (15) we obtain the following theorem.

Theorem 4. *In the case of CES cost functions (14) and consumer price index (15) the vector of price indexes $p = (p_1, \ldots, p_m) \geq 0$ is the solution of the Young dual problem (9)-(11) for the given vector of price indexes of resources $s = (s_1, \ldots, s_n)$ if and only if $p = (p_1, \ldots, p_m)$ is the solution of the following nonlinear system of equations*

$$\left(\sum_{i=1}^{m} a_{ij} \left(p_i\right)^{\frac{\rho_j}{1+\rho_j}} + \sum_{k=1}^{n} b_{kj} \left(s_k\right)^{\frac{\rho_j}{1+\rho_j}}\right)^{\frac{1+\rho_j}{\rho_j}} = p_j,\ j = 1, \ldots, m. \tag{16}$$

Thus, $p_j(s)$, $j = 1, \ldots, m$ we can find for any given values of $s = (s_1, \ldots, s_n)$ and for any fixed value of ρ_j, $j = 1, \ldots, m$.

2.3. **Comparative statics analysis.** Now we can provide comparative statics analysis with the help of the developed framework. We assume that the CES technologies are stable over several years with the fixed elasticities of substitution of inputs ρ_j. The method of verification of ρ_j depends on the completeness of the statistics of IO data for the region and can be varied due to the purposes of the comparative statics analysis. Recall that the purpose of this paper is to analyze and range the macroeconomic responses of the Kazakhstan economy to the Covid-19 pandemic. In the next section we discuss the method of the calibration of the model with the initial data set of IO tables of Kazakhstan, that allows to evaluate ρ_j for the aggregate industrial complexes of the region. In this section we assume that the values of ρ_j are fixed.

Let fix the base year t with the IO statistics $Z(t)$ and identify the coefficients of CES functions (12), (13), (14), (15) according to the Theorem 3. The initial data (statistics or scenario) for comparative statics analysis with the model is

- price indexes on factors (in relation to the base year t) $s = (s_1, \ldots, s_n)$
- aggregate spending of final consumers (in current prices of the considered year) $\hat{Z}^0 = \left(\hat{Z}_1^0, \ldots, \hat{Z}_m^0\right)$

Given the initial data set that differs from the data of the base year t we can resolve the resource allocation problem (1)-(4) and the Young dual problem (9)-(11). As a result we obtain the new values of interindustry flows and price indices of intermediate inputs that correspond to the shifting equilibrium state in the model.

Note that in terms of the model the output of the j-th industry has the form

$$\hat{Y}_j = p_j F_j \left(X_1^j, \ldots, X_m^j, l_1^j, \ldots, l_n^j \right), \ j = 1, \ldots, m$$

and the new values of intermediate inputs are equal to

$$\hat{Z}_i^j = p_j X_i^j$$

. Denote $\Lambda = \|\lambda_{ij}\|$ is $(m \times m)$-matrix, where

$$\lambda_{\mathrm{ij}} = a_{\mathrm{ij}} \left(\frac{p_i(s)}{p_j(s)} \right)^{\frac{\rho_j}{1+\rho_j}}, i = 1, \ldots, m, j = 1, \ldots, m.$$

Assume that the matrix Λ is productive. Then $(E - \Lambda)^{-1} \geq 0$ exists, where E is the identity $(m \times m)$-matrix.

Then the evaluated vector of total output $\hat{Y} = \left(\hat{Y}_1, .., \hat{Y}_m \right)$ equals to

$$\hat{Y} = (E - \Lambda)^{-1} \hat{Z}^0$$

and the evaluated new coefficients of the I-st and the III-d IO table quadrant equal to

$$\hat{Z}_i^j = \hat{Y}_j \left(w_i^j \frac{p_i(s)}{p_j(s)} \right)^{\frac{\rho_j}{1+\rho_j}} = a_{\mathrm{ij}} \hat{Y}_j \left(\frac{p_i(s)}{p_j(s)} \right)^{\frac{\rho_j}{1+\rho_j}}, i = 1, \ldots, m, j = 1, \ldots, m.$$

$$\hat{Z}_{m+k}^j = \hat{Y}_j \left(w_{m+k}^j \frac{s_k}{p_j(s)} \right)^{\frac{\rho_j}{1+\rho_j}} = b_{k,j} \hat{Y}_j \left(\frac{s_k}{p_j(s)} \right)^{\frac{\rho_j}{1+\rho_j}} \ k = 1, \ldots, n, j = 1, \ldots, m.$$

Here p_i is the solution of the dual problem (9)-(11) for the new value of s. As a result we evaluate the I-st $\left\| \hat{Z}_i^j \right\|, i = 1, \ldots, m, j = 1, \ldots, m$ and the III-d $\left\| \hat{Z}_i^j \right\|, i = m + 1, \ldots, m + n, j = 1, \ldots, m$ quadrant of the IO table for the new scenario conditions s and $\hat{Z}^0$.

3. Comparative statics analysis of the macroeconomic consequences of Covid-19 pandemic for the Kazakhstan economy.

For the purpose of the macroeconomic effects analysis of Covid-19 pandemic for Kazakhstan with the help of developed model we should reveal the sectors that had a different values of responses to the pandemic. We discuss the aggregation principles for the initial 68 branches of the official IO statistics of Kazakhstan that we use for model evaluations. This aggregation allows us to estimate with the model the importance of internal and external pandemic shocks for the main macroeconomic characteristics of the Kazakhstan economy.

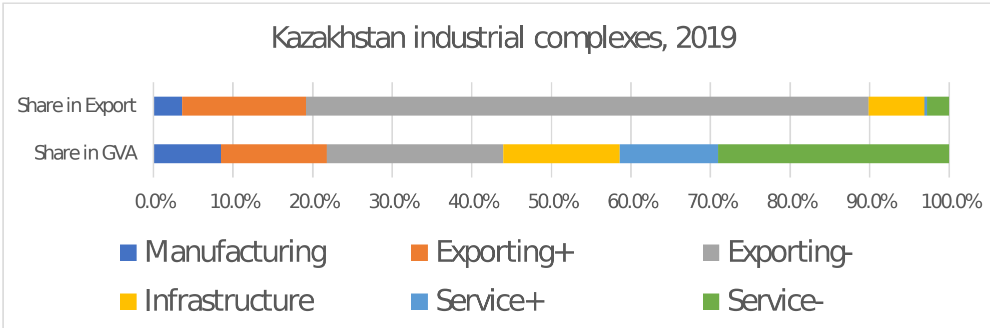

FIGURE 1. Export and GVA share of Kazakhstan aggregated industrial complexes, 2019

3.1. **The main features of the Kazakhstan industrial complexes. Aggregation of the initial IO tables.** The data base for the analysis includes IO tables for 68 pure industries of the economy of Kazakhstan 2013-2020 published as the part of National Accounts by the Agency for Strategic planning and reforms (https://stat.gov.kz). The main feature is that the Kazakhstan economy depends sufficiently on raw material exports. The export of raw material generates economic rent that is the main source of the government expenses. The export volume is about 36% of GDP according to 2019 but the share of processed product is 29% of the total exports. According to that features we suggest the following aggregation principles for 68 branches of the Kazakhstan economy:

(1) Exporting complex collects the most export oriented branches in exporting complex;
(2) Manufacturing complex collects the inertial manufacturing branches oriented on domestic market with limited capacity;
(3) Service complex aggregates the group of services that can growth or fall dramatically depending on consumption;
(4) Infrastructure complex aggregates the group of natural monopolies and infrastructure branches that produce intermediate goods and services with low share of final consumption in the total output.

Additionally, in purpose of pandemic impact studying we divide Exporting (1) and Service (3) sectors in two parts with symbol - and + each while these sectors contain branches that suffered in pandemic (for ex., gas and oil production) and thriving ones (for ex., medicine and education services). We implement the aggregation based on the statistics of non-pandemic 2019 IO table and pandemic 2020 IO table. So for the evaluations with the model we consider the following six aggregate complexes of the Kazakhstan economy: Exporting+, Exporting-, Manufacturing, Service+, Service-, Infrastructure. The adequacy of the suggested aggregation principles is supported by the structure of the Kazakhstan economy that is showed in Fig.3.1.

3.2. **Model calibration. Evaluation of the elasticities of inputs.** We calibrate the model on the base of IO statistics 2013-2019, having 2013 as the base year. Elasticities of substitution of inputs for each of six industrial complexes are the calibration parameters. We find the parameters of elasticities $\rho = (\rho_1, .., \rho_6)$

minimizing the residual sum of squares of Total output, Gross Value Added (GVA) and Intermediate Import consumption (Import used) values of the whole Kazakhstan economy, i.e.,

(17)
$$\sum_{t=2013}^{2019}\left[\left(\hat{Y}(t)(\rho)-Y(t)\right)^2+\left(\hat{V}(t)(\rho)-V(t)\right)^2+\left(\hat{I}(t)(\rho)-I(t)\right)^2\right]\to\min_{\rho},$$

where $Y(t), I(t), V(t)$ are the statistics of Total output, GVA and Import used values, and $\hat{Y}(t)(\rho), \hat{V}(t)(\rho), \hat{I}(t)(\rho)$ are the corresponding values which we evaluate with the model.

We solve the 6-dimensional optimizing problem (17) by the alternating-variable descent method implementation. The solution $\rho = (\rho_1, .., \rho_6)$ of the optimizing problem and corresponding elasticities of substitution of inputs are presented in the Fig. 2

	Manufacturing	Exporting+	Exporting-	Infrastructure	Service+	Service-
Optimal ρ_j	-0.4	0.2	0.3	0.85	0.9	-0.35
Elasticity	1.7	0.8	0.8	0.5	0.5	1.5

FIGURE 2. Evaluated elasticities

Evaluation that corresponds to the optimum value of ρ for 2013 - 2019 years we present in the Fig. 3

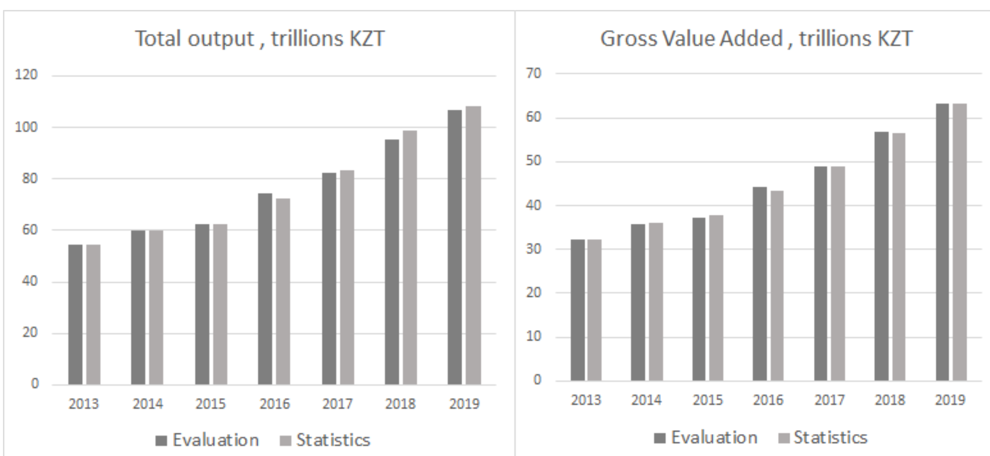

FIGURE 3. Statistic and evaluation parameters in 2013-2019 years

3.3. **Comparative static analysis. Macroeconomic responses of Kazakhstan to Covid-19 pandemic.** We consider the two sides of Covid-19 pandemic shock: external shock that is connected to the violation of export supplies and the fall in world energy prices and internal shock caused by the shift of domestic final consumption .

We take 2019 as a non-pandemic year for shock evaluation of export and final consumption values 2020. We consider two scenarios for 2020 year, so called "External shock" and "Internal shock". Both shock impact values we take from 2020 pandemic year statistics as the vector of price indexes of factors $s = (s_1, s_2) = (1.07, 1.08)$ (s_1 - consumer price index, s_2 - currency exchange rate index). The external shock corresponds to export changes due to reduction of the world business activity and changing in export structure. Note that export of oil and gas decreased in contrast to medical and education services export that increased. Export in 2020 (in prices of 2019) compare to 2019 has changed significantly different in various sectors. The values of discrepancy we show in the Table 1. Thus for the external shock sce-

Table 1. Export change 2020vs2019

Sector	Export change 2020 vs 2019	Coefficient of shock compensation
Manufacturing	+11%	0.89
Exporting+	+6%	0.94
Exporting-	-30%	1.30
Infrastructure	-18%	1.18
Service+	+27%	0.73
Service-	+7%	0.93

nario we multiply the export part of final demand 2020 for every sector on the corresponding coefficient to compensate the external shock (the last column of the Tab.1). Note that we remain the part of final domestic consumption at the level of 2020 in this scenario.

The internal shock reflects the domestic final consumption changing in the pandemic 2020 year. Households and government consumption of drugs, education services increased and demand for travelling, food services and some other services and goods fell. Domestic final consumption in 2020 (in prices of 2019) compare to 2019 changed on the following values that are presented in the Table 2. Thus

Table 2. Domestic consumption change

Sector	Consumption change 2020 vs 2019	Coefficient of shock compensation
Manufacturing	+11%	0.89
Exporting+	+13%	0.87
Exporting-	-1%	1.01
Infrastructure	-26%	1.26
Service+	+21%	0.79
Service-	-9%	1.09

for the internal shock scenario we multiply the internal part of final demand 2020 (government and household consumption) in every sector on the corresponding coefficient to compensate the external shock (the last column of the Tab. 2). Note that we remain the export part of the final consumption at the level of 2020 in this scenario.

In Fig.4 we present the results of Export shock scenario model evaluation and the actual 2020 data for the aggregated characteristics of Kazakhstan. Note that "Statistics, 2020" column in Fig.4 expose to the both types of shocks while the

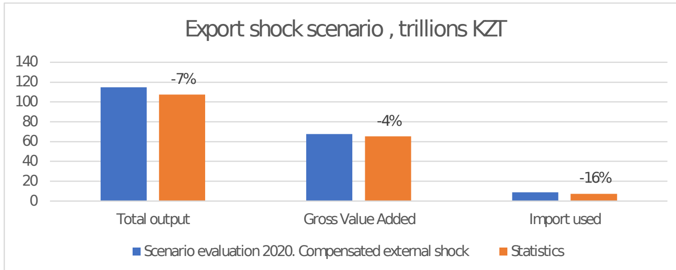

FIGURE 4. Export shock impact on Kazakhstan economy

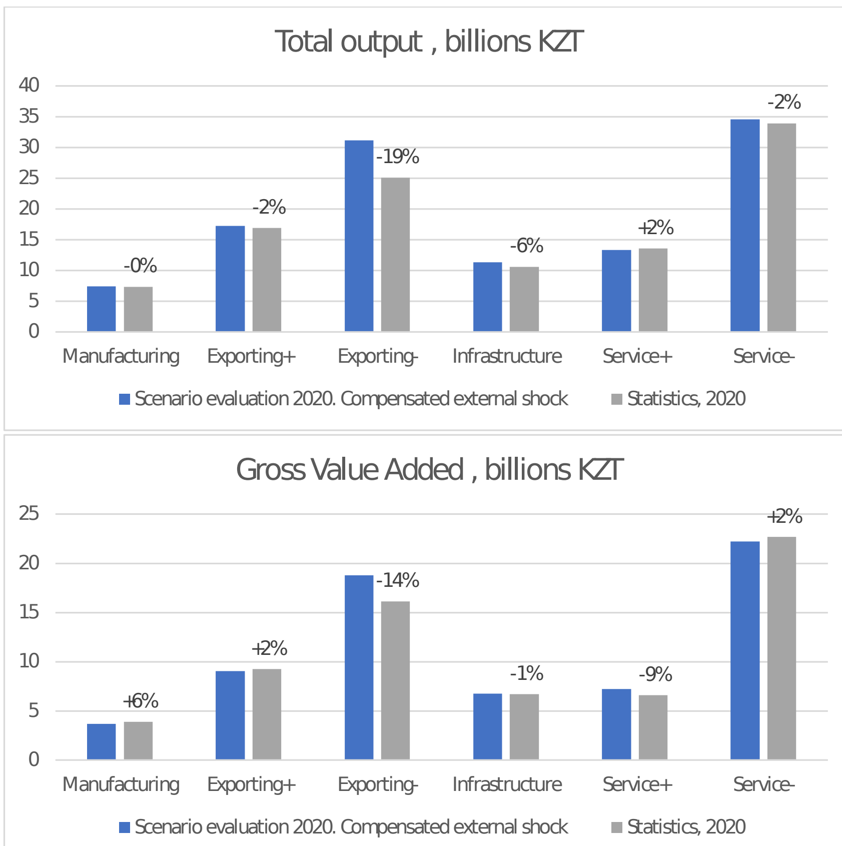

FIGURE 5. Export shock impact on sectors

scenario evaluation has the pandemic shock without external part of it. Correspondingly the percentage on the top of columns “Statistics, 2020” in the Fig.5 shows the value of external shock impact on the macroeconomic characteristics of each industrial complex.

As we can see, the pandemic export shock leads to the recession of the Kazakhstan economy. The significant fall in Total outputs of Exporting- and Infrastructure complexes implies decline in the Total Output of the whole Kazakhstan economy. Note that the Total Outputs in Manufacturing, Service+, Service-, Exporting+ demonstrate almost no changes. This result shows that the Kazakhstan economy has a significant dependency of export, and has sectors that don't depend on export but their influence to the macroeconomic parameters of the region is not determinant. Recall that we divide the economy of Kazakhstan on industrial complexes depending on their foreign trade intensity and pandemic impact. Export shock scenario calculation shows that the pandemic export shock results in disproportional decrease in complexes depending on their linkages with Exporting- complex.

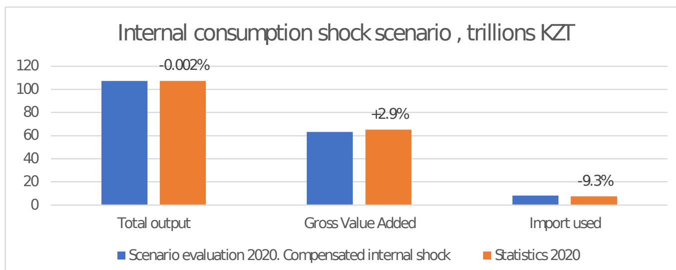

FIGURE 6. Consumption shock impact on Kazakhstan economy

Internal shock scenario. In Fig.6 we present the results of Internal shock scenario model evaluation and the actual 2020 data for the aggregated characteristics of Kazakhstan. Note that "Statistics, 2020" column in Fig.6 expose to the both types of shocks while the scenario evaluation has the pandemic shock without internal part of it. Correspondingly the percentage on the top of columns "Statistics, 2020" in the Fig.7 shows the value of internal shock impact on the macroeconomic characteristics of each industrial complex.

The internal consumption pandemic shock doesn't affect Total output of the whole Kazakhstan economy, but leads to multidirectional changes in Total output in industrial complexes: significant fall in Total outputs of Export- and Infrastructure complexes in contrast to increase in Manufacturing sector and the spike in Service+ complex that includes medical and education services which was worthy financed by the Kazakhstan government. This result shows that the Kazakhstan economy policy almost compensated pandemic impact on internal consumption in terms of total economy output.

4. CONCLUSION

On the base of scenario evaluations with the model we can conclude that the external shock of Covid-19 pandemic was more significant for the economy of Kazakhstan than the internal shock. On the one hand, this indicates the competent economic and social policy of the state during the pandemic. On the other hand the recession of the economy as a result of the export shock indicates economic risks and the need to reduce the dependence of the economy on exports of raw materials. Another result of the analysis is that the effects of the Covid-19 pandemic for the

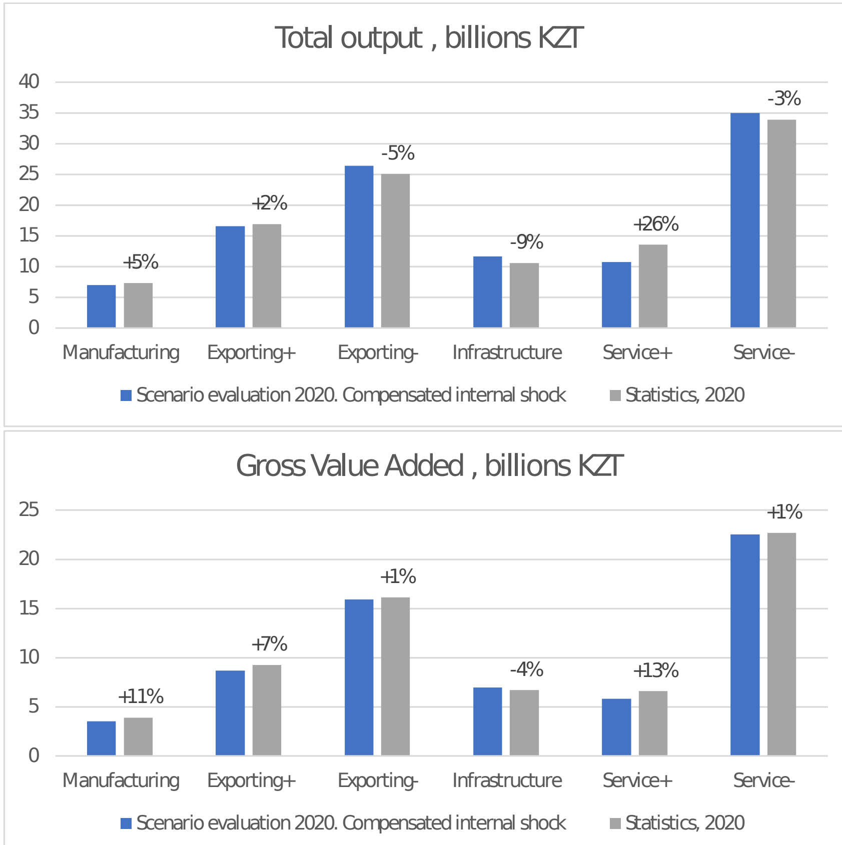

FIGURE 7. Consumption shock impact on sectors

Kazakhstan economy should be analyzed not only using models of the spread of epidemics (SIR model and its modifications) but also using macroeconomic models such as Computable General Equilibrium (CGE) models [21] that allow analyzing indirect responses of the economy to external and internal shocks. As a result of the paper investigation we can say that the developed model of nonlinear inter-industry linkages is a useful tool for the comparative statics analysis of internal and external shocks impact on the economy taking into account the substitution of inputs in the production network.

REFERENCES

[1] M.S. Eichenbaum, S. Rebelo, M. Trabandt, *The macroeconomics of epidemics*, Rev. Finance Studies, **34**:11 (2021), 5149–5187.

[2] D. Acemoglu, V. Chernozhukov, I. Werning, M.D. Whinston, *A multi-risk SIR model with optimally targeted lockdown*, NBER Working Paper No. 27102, 2020.

[3] F.E. Alvarez, D. Argente, F. Lippi, *A simple planning problem for Covid-19 lockdown*, NBER Working Paper No. 26981, 2020.

[4] W. Leontief, *Quantitative input-output relations in the economic system of the United States*, Review of Economics and Statistics, **18**:3 (1936), 105–125.

[5] W.W. Leontief, *The structure of American economy, 1919-1939: An empirical application of equilibrium analysis*, Oxford University Press, New York, 1951
[6] R.E. Miller, P.D. Blair, *Input-output analysis. Foundations and extensions*, Cambridge University Press, Cambridge, 2009. Zbl 1184.93002
[7] S.A. Ashmanov, *Introduction to mathematical economics*, URSS, Moscow, 2022. Zbl 0569.90001 (1984)
[8] A. Pichler, J.D. Farmer, *Simultaneous supply and demand constraints in input–output networks: the case of Covid-19 in Germany, Italy, and Spain*, Economic Systems Research, **34**:3 (2021), 273–293.
[9] B. Bonadio, Zhen Huo, A.A. Levchenko, N. Pandalai-Nayar, *Global supply chains in the pandemic*, NBER Working Paper No. 27224, 2020.
[10] A. Pichler, M. Pangallo, et al., *Forecasting the propagation of pandemic shocks with a dynamic input-output model*, J. Econ. Dyn. Control, **144** (2022), Paper No. 104527.
[11] J. Bonet-Moron, D. Ricciulli-Marín, G.J. Perez-Valbuena, et al. *Regional economic impact of COVID-19 in Colombia: An input–output approach*, Reg. Sci. Policy Pract., **12**:6 (2020), 1123–1150.
[12] David Baqaee, Emmanuel Farhi, *Nonlinear production networks with an application to the Covid-19 crisis*, (May 2020), NBER Working Paper No. w27281.
[13] V.M. Carvalho, A. Tahbaz-Salehi, *Production networks: A primer*, Annual Review of Economics, **11** (2019), 635–663.
[14] A.A. Shananin, *Young duality and aggregation of balances*, Dokl. Math., **102**:1 (2020), 330–333. Zbl 1475.91131
[15] A.A. Shananin, *Problem of aggregating of an input-output model and duality*, Comput. Math. Math. Phys., **61**:1 (2021), 153–166. Zbl 1459.91071
[16] S. Boranbayev, N. Obrosova, A. Shananin, *Production network centrality in connection to economic development by the case of Kazakhstan statistics*, LNCS, **13078**, 2021, Springer, Cham, 321–335.
[17] N. Obrosova, A. Shananin, A. Spiridonov, *On the comparison of two approaches to intersectoral balance analysis*, J. Phys.: Conf. Ser., **2131**:2 (2021), Paper No. 022112.
[18] A.V. Rassokha, A.A. Shananin, *Inverse problems of analysis of input-output balances*, Math. Models Comput. Simul., **13**:6 (2021), 943–954.
[19] A.A. Shananin, A.V. Rassokha, *Inverse problems in analysis of input-output model in the class of CES functions*, J. Inverse Ill-Posed Probl., **29**:2 (2021), 305–316. Zbl 1460.91110
[20] A.A. Shananin, *Duality for generalized programming problems and variational principles in models of economic equilibrium*, Dokl. Math., **59**:3 (1999), 432–434. Zbl 0964.91018
[21] P.B. Dixon, D.W. Jorgenson (eds.), *Handbook of computable general equilibrium modeling*, Elsevier, Amsterdam, 2013. Zbl 1257.91001

Askar Boranbayev
Nazarbayev University,
53 Kabanbay Batyr Ave.,
010000, Astana, Kazakhstan
Email address: aboranbayev@nu.edu.kz

Nataliia Obrosova
Moscow Center for Fundamental and Applied Mathematics, Lomonosov Moscow State University,
GSP-1, Leninskie Gory,
119991, Moscow, Russian Federation,
Federal Research Center ≪Computer Science and Control≫ of Russian Academy of Sciences,
Vavilov Street 44/2,
119333, Moscow, Russian Federation,
Moscow Institute of Physics and Technology (State University),
Institutskiy per. 9,
141701, Dolgoprudny, Moscow region, Russian Federation
Email address: nobrosova@ya.ru

Alexander Shananin
Moscow Center for Fundamental and Applied Mathematics, Lomonosov Moscow State University,
GSP-1, Leninskie Gory,
119991, Moscow, Russian Federation,
Federal Research Center ≪Computer Science and Control≫ of Russian Academy of Sciences,
Vavilov Street 44/2,
119333, Moscow, Russian Federation,
Moscow Institute of Physics and Technology (State University),
Institutskiy per. 9,
141701, Dolgoprudny, Moscow region, Russian Federation
Email address: alexshan@ya.ru